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Inertial Krasnosel′skiĭ-Mann iterative algorithm with step-size parameters involving nonexpansive mappings with applications to solve image restoration problems

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ABSTRACT. In this work, we propose and study an inertial Krasnosel′skiĭ-Mann iterative algorithm with step-size parameters involving nonexpansive mapping to find a solution of a fixed point problem of a nonexpansive mapping in the frame work of Hilbert spaces. Strong convergence of the new proposed algorithm is proved under some useful properties of a nonexpansive mapping and inequalities on real Hilbert spaces together with the appropriate conditions of scalar controls without relying on the concept of viscosity methods. For the applications, we employ the obtained results to find a zero point of some monotone inclusion problems and then apply to solve image restoration problems. For representing the advantage of the new algorithm, the signal to noise ratio (SNR) with various types of blurring operators and some numerical experiments are presented to compare and illustrate the behavior of the new algorithm with numerical results of some previous existing algorithms.

1. INTRODUCTION

Throughout this work, the notations $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^k$ and I stand for the set of all natural numbers, the set of all real numbers, the Euclidean space $\mathbb{R}^k$ ($k \in \mathbb{N}$) and the identity mapping, respectively. Let H be a real Hilbert space with its inner product $\langle \cdot, \cdot \rangle$ which induces its norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.

Let H be a real Hilbert space and let T be a mapping with domain $D(T)$ and range $R(T)$ in H (i.e., $T : D(T)(\subseteq H) \to R(T)(\subseteq H)$). Then, the set of all fixed points of T is denoted by $F(T) := \{u \in D(T) : Tu = u\}$. A mapping $T : D(T)(\subseteq H) \to R(T)(\subseteq H)$ is said to be:

(i) a *Lipshitzian mapping* if there exists a real constant $L \geq 0$ such that

$$\|Tu - Tv\| \leq L \|u - v\|, \quad \forall u, v \in D(T).$$

(ii) a *strict pseudo-contractive mapping* if there exists $\kappa \in (-\infty, 1)$ such that

$$\|Tu - Tv\|^2 \leq \|u - v\|^2 + \kappa \|(I - T)u - (I - T)v\|^2, \quad \forall u, v \in D(T). \tag{1.1}$$

In particular, it is well known that if $\kappa = -1$ in (1.1), then the mapping T in (1.1) is said to be a *firmly nonexpansive mapping* and the equivalent form of this mapping can be written as $\|Tu - Tv\|^2 \leq \langle u - v, Tu - Tv \rangle, \quad \forall u, v \in H$. If $\kappa = 0$ in (1.1), then the mapping T in (1.1) is said to be a *nonexpasive mapping*, that is, $\|Tu - Tv\| \leq \|u - v\|, \quad \forall u, v \in H$. Furthermore, if κ is outside $(-\infty, 1)$ and $\kappa = 1$, then T is called a *pseudo-contractive mapping*, see [9, 16, 37] for more details.

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Image restoration is an interesting field of image processing which assists in the recovery of images that have been degraded by a variety of factors under varied conditions. The goal of image restoration mathematically is to retrieve the original image x from a degraded image y. One can assume the below mathematical model that relates x and y as follows:

$$y = \Gamma x + w,$$

where Γ is the blur operator and w is noise. To obtain the reconstructed image, one can solve the following least-squares problem, that is, $\min_{x\in\mathbb{R}^k} \left\{ \frac{1}{2} \|\Gamma x - y\|_2^2 + \tau\phi(x) \right\}$, where $\tau > 0$ is the regularization parameter and $\phi(\cdot)$ is the regularization functional. A well-known regularization functional that is used to remove noise in the restoration problem is the l_1 norm. The problem above can be written in the form of the following problem as:

$$\text{find } x \in \operatorname*{arg\,min}_{x\in\mathbb{R}^k} \left\{ \frac{1}{2} \|\Gamma x - y\|_2^2 + \tau\|x\|_1 \right\}, \tag{1.2}$$

where $\|\cdot\|_1$ is the l_1 norm, $\|\cdot\|_2$ is the usual norm. Let $f, L : \mathbb{R}^k \to \mathbb{R}$ be defined by $f(x) = \tau\|x\|_1$ and $L(x) = \frac{1}{2} \|Lx - y\|_2^2$, respectively. Then, solving (1.2) is equivalent to solve the following monotone inclusion problem:

$$\text{find } \hat{x} \in \mathbb{R}^k \text{ such that } 0 \in (\Phi + \Pi)\hat{x}, \tag{1.3}$$

where $\Phi(x) = \partial f(x) = \{z \in H : f(y) \geq f(x) + \langle z, y - x\rangle, \quad \forall y \in H\}$ is the subdifferential of f and $\Pi = \nabla L = \nabla \left(\frac{1}{2} \|\Gamma(\cdot) - y\|_2^2 \right) = \Gamma^*(\Gamma(\cdot) - y)$ is the gradient of L where Γ^* is the transpose of Γ. Moreover, from (1.3) it can be observed that

$$\hat{x} \in \left\{ x \in \mathbb{R}^k \;\middle|\; 0 \in (\Phi + \Pi)x \right\} =: \operatorname{zer}(\Phi + \Pi),$$

where $\operatorname{zer}(\Phi + \Pi)$ is the set of all zeros of $\Phi + \Pi$. Furthermore, it can be seen from Section 4 that $\operatorname{zer}(\Phi + \Pi) = F(T)$ where T is nonexpansive which in the form of $T := J_{\eta\Phi} \circ (I - \eta\Pi)$ where $J_{\eta\Phi}$ is the resolvent operator of $\eta\Phi$ ($\eta > 0$) (see Section 4 for more details). These show that image restoration problems, monotone inclusion problems and fixed point problems have a strong relationship among them.

Recall that if C is a nonempty closed and convex subset of H, then for each $w \in H$ there is a unique $\hat{x} \in C$ such that $\|w - \hat{x}\| = \inf_{x\in C} \|w - x\|$. Then, the mapping $P_C : H \to C$ which is defined by $P_C(w) = \hat{x}$ for all $w \in H$ is said to be the **metric projection** (or the nearest point projection) of H onto C, see [6, 33, 34] for more details.

Fixed Point Problem: The fixed point problem for the mapping T, which is typically expressed by:

$$\text{find } x \in H \text{ such that } x = Tx. \tag{1.4}$$

The construction of fixed points for nonexpansive mappings is a significant topic within the field of nonexpansive mappings and has wide-ranging applications in various practical domains such as image restoration, image recovery, and signal processing (see, e.g. [4, 9, 10, 12, 29]). However, creating a tool to find a solution of (1.4) for a nonexpansive mapping T in the form of a simple iterative algorithm such as Picard iteration [26], i.e. $x_{n+1.} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$ where x_0 is any arbitrary point in H may fail to converge to a solution of (1.4). For example, if we let $T : [0,1] \to [0,1]$ defined by $Tx = 1 - x$ for every $x \in [0,1]$ it is not hard to verify that T is a nonexpansive mapping with $F(T) = \{\frac{1}{2}\}$ and if starting with $x_0 = \frac{1}{3}$ then it generates $x_1 = 1 - x_0 = \frac{2}{3}$ and $x_2 = 1 - x_1 = \frac{1}{3}$ which goes on and on and then it produces an alternating sequence, that is, $(x_n)_{n\geq 0} = (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \ldots)$ which is a sequence that does not converge to the desired answer that is $\frac{1}{2}$. In order to overcome and solve problems which are disadvantages like this, the Krasnosel'skiĭ-Mann algorithm [20], which is widely recognized and significant for solving (1.4), was introduced in the following manner:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 0, \end{cases} \tag{1.5}$$

where $(\alpha_n)_{n\geq 0}$ is some appropriate sequence chosen from $[0,1]$. Reich [31] proved that if T in (1.5) is nonexpansive with a fixed point and $(\alpha_n)_{n\geq 0}$ in accordance with certain favorable conditions, then (1.5) converges weakly to a fixed point of T. Subsequently, Ishikawa [16] drew inspiration from Krasnosel'skiĭ-Mann's concepts and put forward the subsequent iterative approach for a Lipschitzian pseudo-contractive mapping T in Hilbert spaces, as outlined below:

$$\begin{cases} x_0 \in C, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, \quad \forall n \geq 0. \end{cases} \tag{1.6}$$

In specific circumstances involving appropriate choices of C, $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$, he successfully demonstrated that (1.6) converges strongly to a fixed point of T. In another way, Halpern [15] introduced a novel method for solving a fixed point problem (1.4) of a nonexpansive mapping T. This approach deviates not much from Krasnosel'ski-Mann's technique by requiring that one element of the vector remains unchanged, as follows:

$$\begin{cases} u, x_0 \in C, \\ x_{n+1} = (1-\alpha_n)u + \alpha_n T x_n, \quad \forall n \geq 0, \end{cases} \tag{1.7}$$

where $(\alpha_n)_{n\geq 0}$ is taken from $[0,1]$. He showed that (1.7) converges strongly to a fixed point of T under some suitable conditions. Moudafi [21] later devised an improved iterative technique that ensures strong convergence, which subsequently became known as the viscosity approximation method. This approach was developed by merging the Halpern iterative method with the concept of contraction mapping. Subsequently, numerous scholars explored the idea of the viscosity approximation method and further developed it in various directions. For more comprehensive information, please refer to references [3, 11, 25, 27, 28, 35].

In 2009, Yao et al. [39] proposed a modified Krasnosel'skiĭ-Mann iterative algorithm for non-expansive mappings by employing some step-size parameters. They showed that the new algorithm converges strongly to a fixed point of a nonexpansive mapping in Hilbert spaces. Their algorithm was defined as follows:

$$\begin{cases} x_0 \in H, \\ y_n = (1-\alpha_n)\, x_n, \\ x_{n+1} = (1-\beta_n)\, y_n + \beta_n T y_n, \quad \forall n \geq 0, \end{cases} \tag{1.8}$$

where $x_0 \in H$ and $(\alpha_n)_{n\geq 0}, (\beta_n)_{n\geq 0}$ are some appropriate sequences in $[0,1]$.

In 2019, Bot et al. [8] studied and improved (1.5) to achieve strong convergence to a fixed point of a nonexpansive mapping. Their method is the following procedure:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\lambda_n)\sigma_n x_n + \lambda_n T \sigma_n x_n, \quad \forall n \geq 0, \end{cases} \tag{1.9}$$

when $(\lambda_n)_{n\geq 0}$ and $(\sigma_n)_{n\geq 0}$ are sequences chosen from $(0,1]$. Moreover, they proved under some suitable assumptions on $(\lambda_n)_{n\geq 0}$ and $(\sigma_n)_{n\geq 0}$ that the sequence generated by (1.9) converges strongly to the fixed point $\hat{x}$ of T which is the nearest point of zero, that is, $\hat{x} = P_{F(T)}(0)$.

In 1964, Polyak [30] put up a number of ideas to increase the convergence speed of iterative algorithms. These approaches involve adjustments to traditional iterative procedures, such as incorporating variable relaxation parameters and employing acceleration methods that utilize an inertial extrapolation term, that is, $\theta_n(x_n - x_{n-1})$ where (θ_n) is a sequence that meets some suitable assumptions. After that, inertial extrapolation has garnered significant interest and been the subject of investigation by numerous researchers, see [1, 2, 4, 5, 7, 18, 22, 24, 23] for more details. In 2019, Shehu [32] introduced an algorithm that integrates the inertial technique, the Halpern method, and error components to find a solution for a fixed point of a nonexpansive mapping. Subsequently, Kitkuan et al. [17] applied the concept of inertial extrapolation terms to the viscosity approximation method for solving some monotone inclusion problems and then utilized their methods to address issues related to image restoration. Artsawang and Ungchittrakool [4] proposed and studied the inertial Mann-type iterative scheme, which was inspired by Bot et al. [8], for solving some fixed point problems of a nonexpansive mapping. They then applied it to address problems related to monotone inclusion and image restoration. Their procedure can be expressed as follows:

$$(\mathbf{AU2020}) \begin{cases} x_0, x_1 \in H, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \sigma_n y_n + \alpha_n(T\sigma_n y_n - \sigma_n y_n) + \varepsilon_n, \quad \forall n \geq 1, \end{cases}$$

where $(\theta_n)_{n\geq1}, (\alpha_n)_{n\geq1}, (\sigma_n)_{n\geq1}$ are sequences chosen from $[0,1]$ which satisfy certain desirable properties.

Motivated by above research works in this direction, it inspired us to come up with the idea to propose the following iterative algorithm:

$$(\textbf{Algorithm 1}) \begin{cases} x_0, x_1 \in H, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n \ = \ (1 - a_n)Ty_n, \\ x_{n+1} = \sigma_n z_n + b_n(T\sigma_n z_n - \sigma_n z_n) + \varepsilon_n, \quad \forall n \geq 1, \end{cases}$$

where $T : H \to H$ is nonexpansive, $(\theta_n)_{n\geq1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, $(a_n)_{n\geq1}, (b_n)_{n\geq1} \subseteq [0, 1)$, $(\sigma_n)_{n\geq1} \subseteq [0, 1]$, and $(\varepsilon_n)_{n\geq1} \subseteq H$ which correspond to certain favorable conditions. Moreover, we can show in Section 3 that the defined by Algorithm 1 above converges strongly to $\hat{x} \in F(T)$ which is the nearest point of zero, that is, $\hat{x} = P_{F(T)}(0)$.

Inclusion Problem of the Sum of Three Monotone Operators: The inclusion problem of the sum of three monotone operators can be stated by:

$$(\mathbf{IPSTMO}) \quad \text{find } x \in H \text{ such that } 0 \in \Phi x + \Psi x + \Pi x$$

where $\Phi, \Psi : H \to 2^H$ and $\Pi : H \to H$ are some monotone operators (see Section 4 for more details).

In the part of applications by utilizing Algorithm 1, it is important to note that the considered nonexpansive mapping T can be constructed by the three monotone operators Φ, Ψ, Π. In addition, it was found that $\operatorname{zer}(\Phi + \Psi + \Pi) = J_{\eta\Psi}(F(T))$, where $\operatorname{zer}(\Phi + \Psi + \Pi)$ is the set of all zeros of $\Phi + \Psi + \Pi$ and $J_{\eta\Psi}$ is the resolvent operator of $\eta\Psi$ ($\eta > 0$). With this direction, we can apply Algorithm 1 to solve IPSTMO (see Section 4 for more details).

Finally, to demonstrate the usefulness and numerical advantages of the newly invented tool in the form of Algorithm 1, we can create a new algorithm that is a product of Algorithm 1 to solve the image restoration problems. In addition, we can validate the efficacy of our new algorithm by showcasing numerical results in different circumstances. These findings undeniably demonstrate that our algorithm surpasses its previous version, as indicated by the superior performance observed in the numerical analys is (see Section 5 for more details).

2. Preliminaries

In this section, some related and useful tools that play an important role in proving the main theorem in the framework of real Hilbert spaces are collected for use in the next section.

Lemma 2.1 ([33, 34]). *Let H be a real Hilbert space. Then, the following equality and inequality hold.*

(1) $\|u+v\|^2 \leq \|u\|^2 + 2\langle u+v, v\rangle, \quad \forall u, v \in H;$

(2) $\|\alpha u + (1-\alpha)v\|^2 = \alpha\|u\|^2 + (1-\alpha)\|v\|^2 - \alpha(1-\alpha)\|u-v\|^2, \quad \forall \alpha \in \mathbb{R}$ *and* $u, v \in H$.

Lemma 2.2 ([38, Lemma 2.5], [19, Lemma 3.1]). *Let $(c_n)_{n\geq 0}, (\epsilon_n)_{n\geq 0} \subseteq [0, +\infty)$, $(\rho_n)_{n\geq 0} \subseteq [0, 1]$ and $(\lambda_n)_{n\geq 0} \subseteq \mathbb{R}$ be sequences such that*

$$c_{n+1} \leq (1-\rho_n)c_n + \rho_n\lambda_n + \epsilon_n, \quad \forall n \geq 0.$$

Assume that $\sum_{n=0}^{\infty} \epsilon_n < +\infty$. Then, the statements below are true.

(1) If $\rho_n\lambda_n \leq k\rho_n$ (where $k \geq 0$), then $(c_n)_{n\geq 0}$ is bounded.

(2) If $\sum_{n=0}^{\infty} \rho_n = +\infty$ and $\limsup_{n\to\infty} \lambda_n \leq 0$, then $\lim_{n\to\infty} c_n = 0$.

Proposition 2.1 ([14, Theorem 1.]). *Let $T : H \to H$ be nonexpansive such that $F(T) \neq \varnothing$. Then, $F(T)$ is closed and convex.*

Throughout this study, the symbols "$\to$" and "$\rightharpoonup$" stand for strong and weak convergence, respectively.

Lemma 2.3 (Demi-closed principle [6]). *Let $T : H \to H$ be nonexpansive, $(u_n)_{n\geq 0} \subseteq H$. Then $I - T$ is demi-closed at 0, that is, if $u_n \rightharpoonup u \in H$ as $n \to \infty$ and $\|u_n - Tu_n\| \to 0$ as $n \to \infty$, then $(I-T)u = 0$ or equivalently to say that $u \in F(T)$.*

There are some nice characteristics of the metric projection that will be useful as a tool for proving the main theorem in the next section, which can be stated as follows:

Lemma 2.4 ([6, 33, 34]). *Let C be a nonempty closed convex subset of H. Then for every $w \in H$ and $\hat{x} \in C$,*

$$\hat{x} = P_C(w) \text{ if and only if } \langle w - \hat{x}, v - \hat{x}\rangle \leq 0, \quad \forall v \in C.$$

3. Main results

In order to analyze the convergence of Algorithm 1 in this section, we present the control condition for the convergence of Algorithm 1 to the solution of the considered fixed point problem.

Condition 3.1. *Let $(a_n)_{n\geq 0}, (b_n)_{n\geq 0} \subseteq [0, 1)$, $(\sigma_n)_{n\geq 0} \subseteq [0, 1]$, and $(\varepsilon_n)_{n\geq 0} \subseteq H$ satisfy the following conditions:*

(1) $\sum_{n=0}^{+\infty} a_n < +\infty$.

(2) $\limsup_{n\to\infty} \; b_n < 1$.

(3) $\sum\limits_{n=1}^{+\infty} |b_n - b_{n-1}| < +\infty$.

(4) (a) $\lim\limits_{n\to\infty} \sigma_n = 1$.

(b) $\sum\limits_{n=0}^{\infty} (1-\sigma_n) = +\infty$, and $\sum\limits_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < +\infty$.

(5) $\sum\limits_{n=0}^{\infty} \|\varepsilon_n\| < +\infty$.

We can create some examples of simple sequences that satisfy the conditions in Condition 3.1 as follows:

Remark 3.1. Let $v \in H$. We set $a_n = \frac{1}{2^n},\ b_n = \frac{1}{2} + \frac{1}{n+1},\ \sigma_n = 1 - \frac{1}{n+3}$, and $\varepsilon_n = \frac{v}{3^n}$ for all $n \geq 0$. It is not hard to verify that all of the sequences above satisfy Condition 3.1.

Lemma 3.5. *Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $(x_n)_{n\geq 0}$ be defined by Algorithm 1. Let $(\theta_n)_{n\geq 1} \subseteq [0,\theta]$ with $\theta \in [0,1)$ such that $\sum\limits_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$. Assume that Condition 3.1 holds. Then $(x_n)_{n\geq 0}$ is bounded.*

Proof. Let $n \in \mathbb{N}$ and $u \in F(T)$. Then, let us consider

$$\|y_n - u\| = \|x_n + \theta_n(x_n - x_{n-1}) - u\| \leq \|x_n - u\| + \theta_n \|x_n - x_{n-1}\|. \tag{3.10}$$

By using (3.10), we get that

$$\begin{aligned}\|z_n - u\| &= \|(1-a_n)Ty_n - u\| = \|(1-a_n)(Ty_n - u) - a_n u\| \\ &\leq (1-a_n)\|y_n - u\| + a_n\|u\| \\ &\leq (1-a_n)(\|x_n - u\| + \theta_n\|x_n - x_{n-1}\|) + a_n\|u\| \\ &\leq (1-a_n)\|x_n - u\| + \theta_n\|x_n - x_{n-1}\| + a_n\|u\|. \end{aligned}\tag{3.11}$$

By using (3.11), it can be observed that

$$\begin{aligned}&\|\sigma_n z_n - u\| \\ &= \|\sigma_n(z_n - u) + (\sigma_n - 1)u\| \leq \sigma_n\|z_n - u\| + (1-\sigma_n)\|u\| \\ &\leq \sigma_n(1-a_n)\|x_n - u\| + \sigma_n\theta_n\|x_n - x_{n-1}\| + \sigma_n a_n\|u\| + (1-\sigma_n)\|u\| \\ &\leq (1-(1-\sigma_n(1-a_n)))\|x_n - u\| + (1-\sigma_n(1-a_n))\|u\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned}\tag{3.12}$$

Using (3.12) for connecting, we will have

$$\begin{aligned}&\|x_{n+1} - u\| \\ &= \|\sigma_n z_n + b_n(T\sigma_n z_n - \sigma_n z_n) + \varepsilon_n - u\| \\ &= \|(1-b_n)(\sigma_n z_n - u) + b_n(T\sigma_n z_n - u) + \varepsilon_n\| \\ &\leq (1-b_n)\|\sigma_n z_n - u\| + b_n\|T\sigma_n z_n - u\| + \|\varepsilon_n\| \leq \|\sigma_n z_n - u\| + \|\varepsilon_n\| \\ &\leq (1-(1-\sigma_n(1-a_n)))\|x_n - u\| + (1-\sigma_n(1-a_n))\|u\| \\ &\quad + \theta_n\|x_n - x_{n-1}\| + \|\varepsilon_n\|. \end{aligned}\tag{3.13}$$

Applying Lemma 2.2 (1) to (3.13) via setting $1 - \sigma_n(1-a_n) = \rho_n$, $\|x_n - u\| = c_n$, $\|u\| = \lambda_n = k$, and $\theta_n\|x_n - x_{n-1}\| + \|\varepsilon_n\| = \epsilon_n$, then it allows $(x_n)_{n\geq 0}$ to be a bounded sequence. □

Lemma 3.6. *Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $(x_n)_{n\geq 0}$ be defined by Algorithm 1. Let $(\theta_n)_{n\geq 1} \subseteq [0,\theta]$ with $\theta \in [0,1)$ such that $\sum\limits_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$. Assume that Condition 3.1 holds. Then $\|x_{n+1} - x_n\| \to 0$ as $n \to +\infty$.*

Proof. We first note that

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|x_n - x_{n-1} + \theta_n(x_n - x_{n-1}) - \theta_{n-1}(x_{n-1} - x_{n-2})\| \\ &\leq \|x_n - x_{n-1}\| + \theta_n \|x_n - x_{n-1}\| + \theta_{n-1} \|x_{n-1} - x_{n-2}\| . \end{aligned} \tag{3.14}$$

Using (3.14), we get that

$$\begin{aligned} &\|z_n - z_{n-1}\| \\ &= \|(1 - a_n)Ty_n - (1 - a_{n-1})Ty_{n-1}\| \\ &= \|(1 - a_n)(Ty_n - Ty_{n-1}) - (a_n - a_{n-1})Ty_{n-1}\| \\ &\leq (1 - a_n) \|Ty_n - Ty_{n-1}\| + |a_n - a_{n-1}| \|Ty_{n-1}\| \\ &\leq \|y_n - y_{n-1}\| + |a_n - a_{n-1}| M_1 \\ &\leq \|x_n - x_{n-1}\| + \theta_n \|x_n - x_{n-1}\| + \theta_{n-1} \|x_{n-1} - x_{n-2}\| + |a_n - a_{n-1}| M_1, \end{aligned} \tag{3.15}$$

where $M_1 := \sup\{ \|Ty_{n-1}\| \mid n \in \mathbb{N}\}$. By connecting (3.15) with the inequality below, we obtain that

$$\begin{aligned} &\|\sigma_n z_n - \sigma_{n-1}z_{n-1}\| \\ &= \|\sigma_n(z_n - z_{n-1}) + (\sigma_n - \sigma_{n-1})z_{n-1}\| \\ &\leq \sigma_n \|z_n - z_{n-1}\| + |\sigma_n - \sigma_{n-1}| \|z_{n-1}\| \\ &\leq \sigma_n \|x_n - x_{n-1}\| + \theta_n \|x_n - x_{n-1}\| + \theta_{n-1} \|x_{n-1} - x_{n-2}\| \\ &\quad + |a_n - a_{n-1}| M_1 + |\sigma_n - \sigma_{n-1}| M_2, \end{aligned} \tag{3.16}$$

where $M_2 := \sup\{ \|z_{n-1}\| \mid n \in \mathbb{N}\}$. By employing (3.16), we have the following:

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \|(1 - b_n)\sigma_n z_n + b_n T\sigma_n z_n + \varepsilon_n - ((1 - b_{n-1})\sigma_{n-1}z_{n-1} + b_{n-1}T\sigma_{n-1}z_{n-1} + \varepsilon_{n-1})\| \\ &= \|(1 - b_n)(\sigma_n z_n - \sigma_{n-1}z_{n-1}) - (b_n - b_{n-1})\sigma_{n-1}z_{n-1} + b_n(T\sigma_n z_n - T\sigma_{n-1}z_{n-1}) \\ &\quad +(b_n - b_{n-1})T\sigma_{n-1}z_{n-1} + (\varepsilon_n - \varepsilon_{n-1})\| \\ &\leq (1 - b_n) \|\sigma_n z_n - \sigma_{n-1}z_{n-1}\| + |b_n - b_{n-1}| \|\sigma_{n-1}z_{n-1}\| + b_n \|T\sigma_n z_n - T\sigma_{n-1}z_{n-1}\| \\ &\quad + |b_n - b_{n-1}| \|T\sigma_{n-1}z_{n-1}\| + \|\varepsilon_n - \varepsilon_{n-1}\| \\ &\leq \|\sigma_n z_n - \sigma_{n-1}z_{n-1}\| + |b_n - b_{n-1}| M_3 + \|\varepsilon_n - \varepsilon_{n-1}\| \\ &\leq \sigma_n \|x_n - x_{n-1}\| + \theta_n \|x_n - x_{n-1}\| + \theta_{n-1} \|x_{n-1} - x_{n-2}\| + |a_n - a_{n-1}| M_1 \\ &\quad + |\sigma_n - \sigma_{n-1}| M_2 + |b_n - b_{n-1}| M_3 + \|\varepsilon_n - \varepsilon_{n-1}\| \\ &= (1 - (1 - \sigma_n)) \|x_n - x_{n-1}\| + (1 - \sigma_n)0 + \zeta_n, \end{aligned} \tag{3.17}$$

where $M_3 := \sup\{ \|\sigma_{n-1}z_{n-1}\| + \|T\sigma_{n-1}z_{n-1}\| \mid n \in \mathbb{N}\}$ and

$$\begin{aligned}\zeta_n := \theta_n \|x_n - x_{n-1}\| + \theta_{n-1} \|x_{n-1} - x_{n-2}\| + |a_n - a_{n-1}| M_1 + |\sigma_n - \sigma_{n-1}| M_2 \\ + |b_n - b_{n-1}| M_3 + \|\varepsilon_n - \varepsilon_{n-1}\| .\end{aligned}$$

By applying Lemma 2.2 (2) and the Condition 3.1 to (3.17), we can conclude that $\|x_{n+1} - x_n\| \to 0$ as $n \to +\infty$. □

Theorem 3.2. *Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $(x_n)_{n\geq 0}$ be defined by Algorithm 1. Let $(\theta_n)_{n\geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$ such that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$. Assume that Condition 3.1 holds. Then, the sequence $(x_n)_{n\geq 0}$ converges strongly to $\hat{x} := P_{F(T)}(0)$.*

Proof. From Lemma 3.5, we have $(x_n)_{n\geq 0}$ is bounded. Since $F(T)\neq\emptyset$, $\mathrm{y}_n = x_n + \theta_n(x_n - x_{n-1})$ and $z_n = (1-a_n)Ty_n$, so $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ are also bounded. Let $\hat{x} := P_{F(T)}(0)$. Then $\hat{x}\in F(T)$. By using Lemma 2.1 (1), we get that

$$\begin{aligned}
\|y_n-\hat{x}\|^2 &= \|x_n-\hat{x}+\theta_n(x_n-x_{n-1})\|^2 \leq \|x_n-\hat{x}\|^2 + 2\theta_n\langle y_n-\hat{x}, x_n-x_{n-1}\rangle \\
&\leq \|x_n-\hat{x}\|^2 + \theta_n\|x_n-x_{n-1}\|L_1, \qquad (3.18)
\end{aligned}$$

where $L_1 := \sup\{2\|y_n-\hat{x}\| \mid n\in\mathbb{N}\}$. Therefore, using (3.18), we get that

$$\begin{aligned}
\|z_n-\hat{x}\|^2 &= \|(1-a_n)Ty_n-\hat{x}\|^2 = \|(Ty_n-\hat{x})-a_nTy_n\|^2 \\
&= \left\|(1-a_n)\left(\frac{1}{(1-a_n)}(Ty_n-\hat{x}) + \frac{-a_n}{(1-a_n)}Ty_n\right)\right\|^2 \\
&= (1-a_n)^2\left\|\frac{1}{(1-a_n)}(Ty_n-\hat{x}) + \frac{-a_n}{(1-a_n)}Ty_n\right\|^2 \\
&= (1-a_n)^2\left(\frac{1}{(1-a_n)}\|Ty_n-\hat{x}\|^2 + \frac{-a_n}{(1-a_n)}\|Ty_n\|^2 + \frac{a_n}{(1-a_n)^2}\|\hat{x}\|^2\right) \\
&= (1-a_n)\|Ty_n-\hat{x}\|^2 - a_n(1-a_n)\|Ty_n\|^2 + a_n\|\hat{x}\|^2 \\
&\leq \|y_n-\hat{x}\|^2 + a_n\|\hat{x}\|^2 \\
&\leq \|x_n-\hat{x}\|^2 + \theta_n\|x_n-x_{n-1}\|L_1 + a_n\|\hat{x}\|^2. \qquad (3.19)
\end{aligned}$$

Employing (3.19) we obtain that

$$\begin{aligned}
\|\sigma_nz_n-\hat{x}\|^2 &= \|\sigma_n(z_n-\hat{x})+(\sigma_n-1)\hat{x}\|^2 \\
&= \sigma_n^2\|z_n-\hat{x}\|^2 + 2\sigma_n(1-\sigma_n)\langle -\hat{x}, z_n-\hat{x}\rangle + (1-\sigma_n)^2\|\hat{x}\|^2 \\
&\leq \sigma_n\left(\|x_n-\hat{x}\|^2 + \theta_n\|x_n-x_{n-1}\|L_1 + a_n\|\hat{x}\|^2\right) \\
&\quad + (1-\sigma_n)\left(2\sigma_n\langle -\hat{x}, z_n-\hat{x}\rangle + (1-\sigma_n)\|\hat{x}\|^2\right) \\
&\leq (1-(1-\sigma_n))\|x_n-\hat{x}\|^2 + (1-\sigma_n)\left(2\sigma_n\langle -\hat{x}, z_n-\hat{x}\rangle + (1-\sigma_n)\|\hat{x}\|^2\right) \\
&\quad + \theta_n\|x_n-x_{n-1}\|L_1 + a_n\|\hat{x}\|^2. \qquad (3.20)
\end{aligned}$$

By using (3.20), it can be observed that

$$\begin{aligned}
\|x_{n+1}-\hat{x}\|^2 &= \|\sigma_nz_n + b_n(T\sigma_nz_n-\sigma_nz_n)+\varepsilon_n-\hat{x}\|^2 \\
&= \|(1-b_n)(\sigma_nz_n-\hat{x}) + b_n(T\sigma_nz_n-\hat{x})+\varepsilon_n\|^2 \\
&\leq \|(1-b_n)(\sigma_nz_n-\hat{x}) + b_n(T\sigma_nz_n-\hat{x})\|^2 + 2\langle x_{n+1}-\hat{x}, \varepsilon_n\rangle \\
&\leq (1-b_n)\|\sigma_nz_n-\hat{x}\|^2 + b_n\|T\sigma_nz_n-\hat{x}\|^2 + \|\varepsilon_n\|L_2 \\
&\leq \|\sigma_nz_n-\hat{x}\|^2 + \|\varepsilon_n\|L_2 \\
&\leq (1-(1-\sigma_n))\|x_n-\hat{x}\|^2 + (1-\sigma_n)\left(2\sigma_n\langle -\hat{x}, z_n-\hat{x}\rangle + (1-\sigma_n)\|\hat{x}\|^2\right) \\
&\quad + \theta_n\|x_n-x_{n-1}\|L_1 + a_n\|\hat{x}\|^2 + \|\varepsilon_n\|L_2, \qquad (3.21)
\end{aligned}$$

where $L_2 := \sup\{2\|x_{n+1}-\hat{x}\| \mid n\in\mathbb{N}\}$. Next, we will prove that $\|T\sigma_nz_n-\sigma_nz_n\|\to 0$ as $n\to+\infty$. We observe that

$$\begin{aligned}
\|Tx_{n+1}-z_n\| &= \|Tx_{n+1}-(1-a_n)Ty_n\| = \|Tx_{n+1}-Ty_n+a_nTy_n\| \\
&\leq \|Tx_{n+1}-Ty_n\| + a_n\|Ty_n\| \leq \|x_{n+1}-y_n\| + a_nM_1
\end{aligned}$$

$$
\begin{aligned}
&= \|x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\| + a_n M_1 \\
&\le \|x_{n+1} - x_n\| + \theta_n \|x_n - x_{n-1}\| + a_n M_1. \qquad (3.22)
\end{aligned}
$$

By using (3.22), we get that

$$
\begin{aligned}
\|T\sigma_n z_n - \sigma_n z_n\| &= \|T\sigma_n z_n - Tx_{n+1} + Tx_{n+1} - \sigma_n z_n\| \\
&\le \|T\sigma_n z_n - Tx_{n+1}\| + \|Tx_{n+1} - \sigma_n z_n\| \\
&\le \|\sigma_n z_n - x_{n+1}\| + \|(1-\sigma_n)Tx_{n+1} + \sigma_n(Tx_{n+1} - z_n)\| \\
&\le \|\sigma_n z_n - (\sigma_n z_n + b_n(T\sigma_n z_n - \sigma_n z_n) + \varepsilon_n)\| \\
&\quad + (1-\sigma_n)\|Tx_{n+1}\| + \sigma_n \|Tx_{n+1} - z_n\| \\
&\le b_n \|T\sigma_n z_n - \sigma_n z_n\| + \|\varepsilon_n\| + (1-\sigma_n)L_3 \\
&\quad + \|x_{n+1} - x_n\| + \theta_n \|x_n - x_{n-1}\| + a_n M_1, \qquad (3.23)
\end{aligned}
$$

where $L_3 := \sup\{\|Tx_{n+1}\| \mid n \in \mathbb{N}\}$. It follows from (3.23), Condition 3.1 and Lemma 3.6 that

$$
\begin{aligned}
&\|T\sigma_n z_n - \sigma_n z_n\| \\
&\le \frac{1}{(1-b_n)}\left(\|\varepsilon_n\| + (1-\sigma_n)L_3 + \|x_{n+1} - x_n\| + \theta_n \|x_n - x_{n-1}\| + a_n M_1\right) \\
&\quad \to 0 \text{ as } n \to \infty. \qquad (3.24)
\end{aligned}
$$

From (3.24), we can conclude that

$$\lim_{n\to\infty} \|T\sigma_n z_n - \sigma_n z_n\| = 0. \tag{3.25}$$

Next, we expect that the sequence $(x_n)_{n\ge 0}$ converges strongly to $\hat{x}$ which it is enough to show that

$$\limsup_{n\to+\infty} \langle -\hat{x}, z_n - \hat{x}\rangle \le 0. \tag{3.26}$$

Let us assume on the contrary that (3.26) does not hold. Then, there exists a real number $r > 0$ and a subsequence $(z_{n_m})_{m\ge 1} \subseteq (z_n)_{n\ge 1}$ such that

$$\langle -\hat{x}, z_{n_m} - \hat{x}\rangle \ge r > 0, \quad \forall m \ge 1.$$

The boundedness of $(z_{n_m})_{m\ge 1}$ implies that there is a subsequence $(z_{n_{m_l}})_{l\ge 1}$ of $(z_{n_m})_{m\ge 1}$ such that $z_{n_{m_l}} \rightharpoonup z \in H$ as $l \to +\infty$. Therefore,

$$0 < r \le \lim_{l\to+\infty} \langle -\hat{x}, z_{n_{m_l}} - \hat{x}\rangle = \langle -\hat{x}, z - \hat{x}\rangle. \tag{3.27}$$

Since $\lim_{n\to+\infty} \sigma_n = 1$, so we get that

$$\sigma_{n_{m_l}} z_{n_{m_l}} \rightharpoonup z \text{ as } n \to \infty. \tag{3.28}$$

By (3.25), (3.28) and Lemma 2.3, it implies that $z \in F(T)$. Due to the assurances of Proposition 2.1 and Lemma 2.4, the inequality $\langle -\hat{x}, z - \hat{x}\rangle = \langle 0 - \hat{x}, z - \hat{x}\rangle \le 0$ is valid which causes a contradiction with (3.27). Therefore, it leads to the conclusion that (3.26) is true. And then, Condition 3.1 (4.1) ensures that

$$\limsup_{n\to+\infty} \left(2\sigma_n \langle -\hat{x}, z_n - \hat{x}\rangle + (1-\sigma_n)\|\hat{x}\|^2\right) \le 0.$$

Finally, (3.21) and Lemma 2.2 (2) give us the desired result, that is, $\lim_{n\to+\infty} x_n = \hat{x}$. The proof is complete. □

Remark 3.2. *Let* $\{\xi_n\}_{n\geq 1} \subseteq [0, +\infty)$ *be the sequence such that* $\sum_{n=1}^{\infty} \xi_n < +\infty$. *Then, we define*

$$\tilde{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\xi_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise,} \end{cases}$$

where $(x_n)_{n\geq 0}$ *and* θ *are taken from Theorem 3.2. Next, if* $(\theta_n)_{n\geq 1}$ *is chosen from* $[0, \tilde{\theta}_n]$ *for all* $n \in \mathbb{N}$, *then it is not hard to verify that* $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$.

4. Applications to monotone inclusion problems

In this section, we will focus on applying Algorithm 1 to find a zero point of some monotone inclusion problems of three operators on the framework of real Hilbert spaces.

Let $\Psi : H \to 2^H$ be a multivalued operator, where 2^H stands for the power set of H. Then, the set of all zero points of Ψ is defined by $\mathrm{zer}(\Psi) := \{z \in H \mid 0 \in \Psi z\}$ and the graph of Ψ is denoted by $\mathrm{G}(\Psi) := \{(u, v) \in H \times H \mid v \in \Psi u\}$, respectively. Then, the multivalued operator Ψ is said to be:

(A) *monotone* if
$\langle u - v, \tilde{u} - \tilde{v}\rangle \geq 0$ for all $(u, \tilde{u}), (v, \tilde{v}) \in \mathrm{G}(\Psi)$.
(B) γ-*strongly monotone* if there is $\gamma > 0$ such that
$\langle u - v, \tilde{u} - \tilde{v}\rangle \geq \gamma\|u - v\|^2$ for all $(u, \tilde{u}), (v, \tilde{v}) \in \mathrm{G}(\Psi)$.
(C) λ-*cocoercive* (or λ-*inverse strongly monotone*) if there is $\lambda > 0$ such that
$\langle u - v, \tilde{u} - \tilde{v}\rangle \geq \lambda\|\tilde{u} - \tilde{v}\|^2$ for all $(u, \tilde{u}), (v, \tilde{v}) \in \mathrm{G}(\Psi)$.
(D) *maximal monotone* if Ψ is monotone and $\mathrm{G}(\Psi)$ is not properly contained in any graph of other multivalued monotone operator, that is, if $\Phi : H \to 2^H$ is a multivalued monotone operator such that $G(\Psi) \subseteq G(\Phi)$, then $G(\Psi) = G(\Phi)$.

In particular, if $\Psi : H \to H$ is a single-value operator, then the inequalities in (A), (B) and (C) reduce to:

(a) *monotone* if
$\langle u - v, \Psi u - \Psi v\rangle \geq 0$ for all $u, v \in H$.
(b) γ-*strongly monotone* if there is $\gamma > 0$ such that
$\langle u - v, \Psi u - \Psi v\rangle \geq \gamma\|u - v\|^2$ for all $u, v \in H$.
(c) λ-*cocoercive* (or λ-*inverse strongly monotone*) if there is $\lambda > 0$ such that
$\langle u - v, \Psi u - \Psi v\rangle \geq \lambda\|\Psi u - \Psi v\|^2$ for all $u, v \in H$.

Recall that for a multivalued operator $\Psi : H \to 2^H$, if we define $J_\Psi := (I + \Psi)^{-1}$, then $J_\Psi : H \to 2^H$ is said to be the *resolvent* operator of Ψ. It is well known that if $\Psi : H \to 2^H$ is maximal monotone and $\eta > 0$, then $J_{\eta\Psi}$ is single-valued and firmly nonexpansive.

We focus on the monotone inclusion problem of three operators as follows:

$$\text{find } x \in H \text{ such that } 0 \in \Phi x + \Psi x + \Pi x \tag{4.29}$$

where $\Phi, \Psi : H \to 2^H$ are maximal monotone operators and $\Pi : H \to H$ is a λ-cocoercive operator with $\lambda > 0$.

In order to solve (4.29) by using Algorithm 1, we would like to mention some of the key tools below:

Proposition 4.2 (Davis and Yin [13, Proposition 2.1]). *Let* $F_1, F_2 : H \to H$ *be two firmly nonexpansive operators and* $\Pi : H \to H$ *be a* λ*-cocoercive operator with* $\lambda > 0$. *Let* $\eta \in (0, 2\lambda)$. *Then operator*

$$T := F_1 \circ (2F_2 - I - \eta\Pi \circ F_2) + I - F_2$$

is τ-averaged with coefficient $\tau := \frac{2\lambda}{4\lambda-\eta} < 1$. In particular, the following inequality holds for all $u, v \in H$

$$\|Tu - Tv\|^2 \leq \|u-v\|^2 - \frac{(1-\tau)}{\tau}\|(I-T)u - (I-T)v\|^2.$$

The set $\operatorname{zer}(\Phi+\Psi+\Pi)$ can be changed the writing form relative to the set $F(T)$ where T is obtained from Proposition 4.2 as follows:

Lemma 4.7 (Fixed point encoding [13, Lemma 2.2])**.** *Let $\Phi, \Psi : H \to 2^H$ be maximal monotone operators and $\Pi : H \to H$ be an operator. Suppose that $\operatorname{zer}(\Phi+\Psi+\Pi) \neq \emptyset$. Then*

$$\operatorname{zer}(\Phi+\Psi+\Pi) = J_{\eta\Psi}(F(T)),$$

where $T = J_{\eta\Phi} \circ (2J_{\eta\Psi} - I - \eta\Pi \circ J_{\eta\Psi}) + (I - J_{\eta\Psi})$ and $\eta > 0$.

It can be observed that if $T = J_{\eta\Phi} \circ (2J_{\eta\Psi} - I - \eta\Pi \circ J_{\eta\Psi}) + (I - J_{\eta\Psi})$, then

$$\begin{aligned} z_n &= (1-\alpha_n)Ty_n \\ &= (1-\alpha_n)\left(J_{\eta\Phi} \circ (2J_{\eta\Psi} - I - \eta\Pi \circ J_{\eta\Psi}) + (I - J_{\eta\Psi})\right)y_n \\ &= J_{\eta\Phi}\left(2J_{\eta\Psi}(y_n) - y_n - \eta\Pi J_{\eta\Psi}(y_n)\right) + y_n - J_{\eta\Psi}(y_n). \end{aligned} \tag{4.30}$$

On the other hand, we observe that

$$\begin{aligned} & T\sigma_n z_n - \sigma_n z_n \\ &= \left(J_{\eta\Phi} \circ (2J_{\eta\Psi} - I - \eta\Pi \circ J_{\eta\Psi}) + (I - J_{\eta\Psi})\right)\sigma_n z_n - \sigma_n z_n \\ &= J_{\eta\Phi}\left(2J_{\eta\Psi}(\sigma_n z_n) - \sigma_n z_n - \eta\Pi J_{\eta\Psi}(\sigma_n z_n)\right) + \sigma_n z_n - J_{\eta\Psi}(\sigma_n z_n) - \sigma_n z_n \\ &= J_{\eta\Phi}\left(2J_{\eta\Psi}(\sigma_n z_n) - \sigma_n z_n - \eta\Pi J_{\eta\Psi}(\sigma_n z_n)\right) - J_{\eta\Psi}(\sigma_n z_n). \end{aligned} \tag{4.31}$$

Therefore, by employing Algorithm 1, the following algorithm can be constructed for solving (4.29) as follows:

(4.32)

$$(\textbf{Algorithm 2}) \quad \begin{cases} x_0, x_1 \in H, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1-a_n)\left(J_{\eta\Phi}\left(2J_{\eta\Psi}(y_n) - y_n - \eta\Pi J_{\eta\Psi}(y_n)\right) + y_n - J_{\eta\Psi}(y_n)\right), \\ x_{n+1} = \sigma_n z_n + b_n\left(J_{\eta\Phi}(2J_{\eta\Psi}(\sigma_n z_n) - \sigma_n z_n - \eta\Pi J_{\eta\Psi}(\sigma_n z_n)) - J_{\eta\Psi}(\sigma_n z_n)\right) + \varepsilon_n \end{cases}$$

for all $n \geq 1$, where $\eta \in (0, 2\lambda)$.

Theorem 4.3. *Let $\Phi, \Psi : H \to 2^H$ be two maximal monotone operators and $\Pi : H \to H$ be λ-cocoercive with $\lambda > 0$. Suppose that $\operatorname{zer}(\Phi+\Psi+\Pi) \neq \emptyset$. Let $(\theta_n)_{n\geq 1}$ be a sequence in $[0, \theta]$ with $\theta \in [0, 1)$ and $\eta \in (0, 2\lambda)$. Let $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ be generated by Algorithm 2. Assume that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$ and the Condition 3.1 hold. Then the following statements are true:*

(1) $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ converge strongly to $\hat{x} := P_{F(T)}(0)$, where $T := J_{\eta\Phi} \circ (2J_{\eta\Psi} - I - \eta\Pi \circ J_{\eta\Psi}) + (I - J_{\eta\Psi})$.

(2) $(J_{\eta\Psi}(y_n))_{n\geq 1}$ and $(J_{\eta\Psi}(\sigma_n z_n))_{n\geq 1}$ converge strongly to $J_{\eta\Psi}(\hat{x}) \in \operatorname{zer}(\Phi+\Psi+\Pi)$.

Proof. (1) Let $(x_n)_{n\geq 0}$ be generated by Algorithm 2. By Proposition 4.2, we get T is nonexpansive. By applying Theorem 3.2, we have the sequence $(x_n)_{n\geq 0}$ strongly converges to $\hat{x} := P_{F(T)}(0)$ as $n \to +\infty$. Since $x_n \to \hat{x}$ and $y_n = x_n + \theta_n(x_n - x_{n-1})$, so it is not hard to see that $y_n \to \hat{x}$. Finally, from $\alpha_n \to 0$, the continuity of T and $y_n \to \hat{x}$, we get that $z_n = (1-\alpha_n)Ty_n \to T\hat{x} = \hat{x}$.

(2) By the first part, we know that $y_n \to \hat{x}$ and since $\sigma_n \to 1$ and $z_n \to \hat{x}$, so $\sigma_n z_n \to \hat{x}$. Then, the continuity of $J_{\eta\Psi}$ and Lemma 4.7 allows us to get that $J_{\eta\Psi}(y_n), J_{\eta\Psi}(\sigma_n z_n) \to J_{\eta\Psi}(\hat{x}) \in J_{\eta\Psi}(F(T)) = zer(\Phi + \Psi + \Pi)$. □

If we put $\Psi \equiv 0$ in Theorem 4.3, then $J_{\eta\Psi}(x) = (I + \eta\Psi)^{-1}(x) = (I + 0)^{-1}(x) = I(x)$ for all $x \in H$ and

$$\begin{aligned} T &= J_{\eta\Phi} \circ (2J_{\eta\Psi} - I - \eta\Pi \circ J_{\eta\Psi}) + (I - J_{\eta\Psi}) \\ &= J_{\eta\Phi} \circ (2I - I - \eta\Pi \circ I) + (I - I) \\ &= J_{\eta\Phi} \circ (I - \eta\Pi) . \end{aligned}$$

By applying Lemma 4.7, it yields $\operatorname{zer}(\Phi + \Pi) = F(T)$. These will give rise to the following corollary.

Corollary 4.1. *Let $\Phi : H \to 2^H$ be a maximal monotone operator and $\Pi : H \to H$ be λ-cocoercive with $\lambda > 0$ and $\operatorname{zer}(\Phi + \Pi) \neq \emptyset$. Let $(x_n)_{n\geq 0}$ be generated by the following*

$$(\textbf{Algorithm 3}) \begin{cases} x_0, x_1 \in H, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1 - a_n)J_{\eta\Phi}\left(y_n - \eta\Pi y_n\right), \\ x_{n+1} = (1 - b_n)\sigma_n z_n + b_n J_{\eta\Phi}\left(\sigma_n z_n - \eta\Pi\sigma_n z_n\right) + \varepsilon_n, \end{cases}$$

for all $n \geq 1$, where $\eta \in (0, 2\lambda)$ and $(\theta_n)_{n\geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$. Assume that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$ and the Condition 3.1 hold. Then, the sequence $(x_n)_{n\geq 0}$ converges strongly to $P_{\operatorname{zer}(\Phi+\Pi)}(0)$.

5. Applications to image restoration problems and their numerical experiments

In this section, we utilize the proposed algorithm to address image restoration challenges, encompassing tasks such as image deblurring and denoising. Our primary focus lies in employing a degradation model that faithfully reflects real-world image restoration problems, or, at the very least, offers the most pertinent mathematical approximations for such problems.

$$y = \Gamma x + w, \tag{5.33}$$

where y represents the corrupted image, Γ stands for the degradation operator (or blurring operator), x denotes the pristine image, and w represents the noise operator.

To obtain the reconstructed image, we solve the following regularized least-squares problem:

$$\min_x \left\{ \frac{1}{2}\|\Gamma x - y\|_2^2 + \tau\phi(x) \right\}, \tag{5.34}$$

where the regularization parameter is represented by $\tau > 0$, and $\phi(\cdot)$ stands for the regularization function. A well-established regularization technique often used to reduce noise in restoration problems is the l_1 norm, commonly known as Tikhonov regularization [36]. The problem described in equation (5.34) can be formulated as follows:

$$\text{find } x \in \arg\min_{x\in\mathbb{R}^k} \left\{ \frac{1}{2}\|\Gamma x - y\|_2^2 + \tau\|x\|_1 \right\}, \tag{5.35}$$

where the variable y denotes the corrupted image, and Γ represents a bounded linear operator. It's worth highlighting that problem (5.35) can be viewed as a specific case of problem (1.3) when configured with the following settings: $\Phi = \partial f(\cdot)$, $\Psi = 0$, and $\Pi = \nabla L(\cdot)$. Here, $f(x) = \|x\|_1$, $\tau = 0.001$, and $L(x) = \frac{1}{2}\|\Gamma x - y\|_2^2$. With this setup, it can be deduced that $\Pi(x) = \nabla L(x) = \Gamma^*(\Gamma x - y)$, where Γ^* represents the transpose of Γ. To initiate the problem-solving process, we begin by selecting images and subjecting them to different blurring techniques. By employing the setup specified in Corollary 4.1, we utilize Algorithm 3 to address problem (5.35) under the following conditions: $\alpha_n = \frac{1}{(10n+1)^2}$, $\beta_n = 0.97 + \frac{1}{(n+100)^2}$, $\sigma_n = 1 - \frac{1}{100n+1}$, $\varepsilon_n = 0$ and θ_n is defined by

$$\theta_n = \begin{cases} \min\left\{\frac{70n-9}{100n}, \frac{1}{(n+1)^2\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{70n-9}{100n}, & \text{otherwise.} \end{cases} \tag{5.36}$$

We compare our proposed algorithm with the algorithm (AU2020) presented in [4, Algorithm in Corollary 2], and the algorithm (KKMS2019) introduced by Kitkuan et al. [17].

For the AU2020, we choose the following parameter values: $\alpha_n = 0.97 + \frac{1}{(n+100)^2}$, $\sigma_n = 1 - \frac{1}{100n+1}$, and $\lambda_n = 0.7$. Concerning the KKMS2019, we select the following parameter values: $\varsigma_n = \theta_n$, $\alpha_n = \frac{1}{10n+1}$, $\lambda_n = 0.7$, and $h(x) = \frac{x^2}{12}$. To assess the quality of the reconstructed image, we gauge it using the signal to noise ratio (**SNR**) for images, which is defined as follows:

$$\mathbf{SNR}(n) = 20 \log_{10} \frac{\|x\|_2^2}{\|x - x_n\|_2^2},$$

where x and x_n represent the original and the restored image at iteration n, respectively.

All experiments were conducted using MATLAB 9.19 (R2022b) and performed all computations on a MacBook Pro 14-inch 2021 with an Apple M1 Pro processor and 16 GB memory. The numerical results corresponding to the selections mentioned above are presented in the following figures.

(A) Historical Park

(B) Gaussian blur

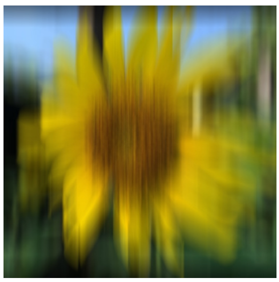
(C) KKMS2019

(D) AU2020

(E) Algorithm 3

FIGURE 1. **(A)** displays the original image 'Historical Park,' while **(B)** presents the images degraded by Gaussian blur. **(C)**, **(D)**, and **(E)** depict the reconstructed images obtained using KKMS2019, AU2020, and Algorithm 3, respectively.

(A) Sunflower



(B) Motion blur

(C) KKMS2019

(D) AU2020

(E) Algorithm 3

FIGURE 2. **(A)** displays the original image 'Sunflower,' while **(B)** presents the images degraded by motion blur. **(C)**, **(D)**, and **(E)** depict the reconstructed images obtained using KKMS2019, AU2020, and Algorithm 3, respectively.

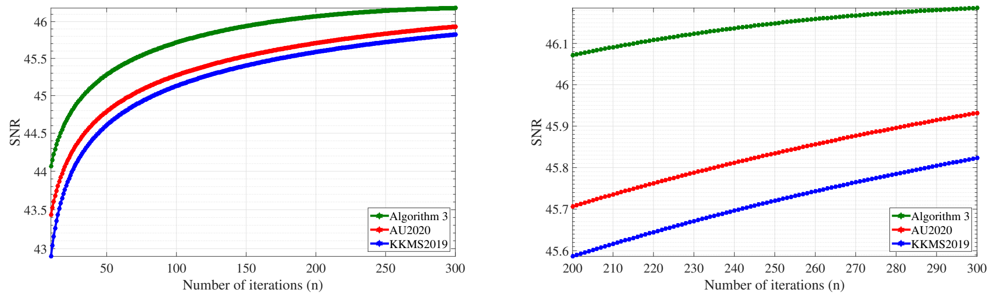


FIGURE 3. The figures illustrate the behavior of the signal to noise ratio (**SNR**) for three algorithms in Figure 1.

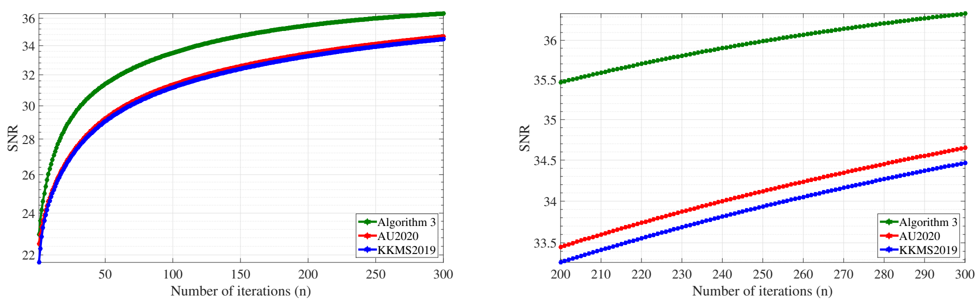


FIGURE 4. The figures illustrate the behavior of the signal to noise ratio (**SNR**) for three algorithms in Figure 2.

TABLE 1. The signal to noise ratio (**SNR**) is evaluated for two images to assess their performance.

n	**Historical Park**			**Sunflower**		
	KKMS2019	**AU2020**	**Algorithm 3**	**KKMS2019**	**AU2020**	**Algorithm 3**
1	36.5176	41.2997	39.4942	21.6616	22.5250	22.4580
10	42.9012	43.4347	43.6027	24.8251	24.9899	26.0268
20	43.7624	44.0617	44.2719	26.4078	26.5469	27.8841
50	44.6071	44.7894	44.9965	29.0489	29.2146	30.6414
100	45.1287	45.2720	45.4668	31.1685	31.3474	32.7318
200	45.5866	45.7068	45.8718	33.2680	33.4526	34.7679
300	45.8229	45.9319	46.0464	34.4659	34.6502	35.7912

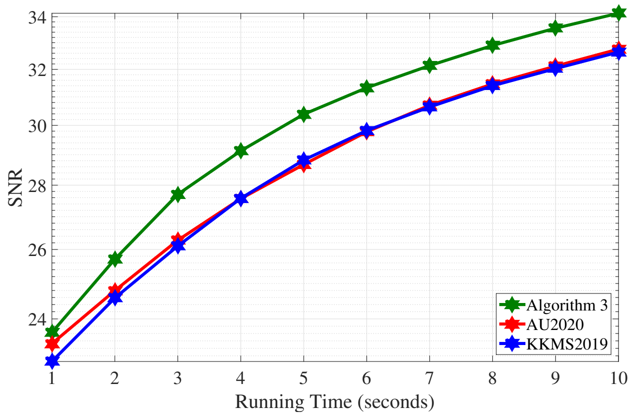


FIGURE 5. Behaviors of the behavior of the signal to noise ratio (**SNR**) for three algorithms in Figure 2.

Figure 5 shows the behavior of the signal to noise ratio (**SNR**) with respect to the computational running time in seconds. It can be seen that Algorithm 3 consistently exhibits a higher **SNR** value compared to others within the same time frame. Our algorithm has showcased remarkable performance in image restoration, surpassing other algorithms, as substantiated by the experimental findings.

6. CONCLUSION

We proposed and studied an inertial Krasnosel'skiĭ-Mann iterative algorithm with stepsize parameters involving nonexpansive mapping as in Algorithm 1. We proved under weak scalar conditions that the newly developed tool in the form of Algorithm 1 converges strongly to the fixed point of a nonexpansive mappping T which is the nearest point to zero, that is, the fixed point in the form $\hat{x} = P_{F(T)}(0)$ (see Theorem 3.2). In order to see the advantages and benefits of using the newly invented tool in the form of Algorithm 1, we use Algorithm 2 (a product of Algorithm 1) to find a zero point of the monotone inclusion problem of three operators (4.29) (see Theorem 4.3). Additionally, the image restoration problem (5.35) can be solved by employing Algorithm 3 (a product of Algorithm 2) (see Corollary 4.1). Furthermore, by showing numerical outcomes under various conditions, we can confirm the positive effects of our new approach. These results clearly show that our method has greater advantages than its previous iteration, as shown by the superior performance displayed in the numerical analysis.

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REFERENCES

[1] Alvarez, F. Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space. *SIAM J. Optim.* **14** (2004), 773–782.

[2] Alvarez, F.; Attouch, H. An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. *Set-Valued Anal.* **9** (2001), 3–11.

[3] Artsawang, N. Accelerated preconditioning Krasnosel'skiĭ-Mann method for efficiently solving monotone inclusion problems. *AIMS Mathematics* **8** (2023), 28398–28412.

[4] Artsawang, N.; Ungchittrakool, K. Inertial Mann-type algorithm for a nonexpansive mapping to solve monotone inclusion and image restoration problems. *Symmetry* **12** (2020), no. 5, 750.

[5] Attouch, H.; Bolte, J.; Svaiter, B. F. Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forward-backward splitting, and regularized Gauss-Seidel methods. *Math. Program.* **137** (2009), 91–129.

[6] Bauschke, H. H.; Combettes, P. L. *Convex analysis and monotone operator theory in Hilbert spaces (2nd ed.) CMS Books in Mathematics*. Springer, New York, 2017.

[7] Beck, A.; Teboulle, M. Accelerated gradient methods for nonconvex optimization. *Math. Program.* **144** (2014), 1–35.

[8] Boţ, R. I.; Csetnek, E. R.; Meier, D. Inducing strong convergence into the asymptotic behaviour of proximal splitting algorithms in Hilbert spaces. *Optim. Methods Softw.* **34** (2019), 489–514.

[9] Browder, F. E.; Petryshyn, W. V. Construction of fixed points of nonlinear mappings in Hilbert spaces. *J. Math. Anal. Appl.* **20** (1967), 197–228.

[10] Byrne, C. A unified treatment of some iterative algorithms in signal processing and image reconstruction. *Inverse Probl.* **20** (2004), 103–120.

[11] Cholamjiak, P.; Suantai, S. Viscosity approximation methods for a nonexpansive semigroup in Banach spaces with gauge functions. *J. Glob. Optim.* **54** (2012), 185–197.

[12] Combettes, P. L. Constrained image recovery in a product space. *In Proceedings of the IEEE International Conference on Image Processing, Washington, DC, IEEE Computer Society Press, California* **2** (1995), 2025–2028.

[13] Davis, D.; Yin, W. A three-operator splitting scheme and its optimization applications. *Set-Valued Var. Anal.* **25** (2017), 829–858.

[14] Dotson W. G. Fixed points of quasi-nonexpansive mappings. *J. Austral. Math. Soc.* **13** (1972), no. 2, 167–170.

[15] Halpern, B. Fixed points of nonexpansive maps. *Bull. Amer. Math. Soc.* **73** (1967), 957–961.

[16] Ishikawa, S. Fixed points by a new iteration method. *Proc. Amer. Math. Soc.* **44** (1974), 147–150.

[17] Kitkuan, D.; Kumam, P.; Martínez-Moreno, J.; Sitthithakerngkiet, K. Inertial viscosity forward–backward splitting algorithm for monotone inclusions and its application to image restoration problems. *Int. J. Comput. Math.* **97** (2019), 482-497.

[18] Lorenz, D.; Pock, T. An inertial forward–backward algorithm for monotone inclusions. *J. Math. Imaging Vis.* **51** (2015), 311–325.

[19] Mainge, P. E. Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces. *J. Math. Anal. Appl.* **325** (2007), 469–479.

[20] Mann, W. R. Mean value methods in iteration. *Proc. Amer. Math. Soci.* **4** (1953), 506–510.

[21] Moudafi, A. Viscosity approximation methods for fixed points problems. *J. Math. Anal. Appl.* **241** (2000), 46–55.

[22] Moudafi, A.; Oliny, M. Convergence of a splitting inertial proximal method for monotone operators. *J. Comput. Appl. Math.* **155** (2003), 447–454.

[23] Nesterov, Y. A method for solving a convex programming problem with convergence rate $O(1/k^2)$. *Dokl. Akad. Nauk SSSR* **269** (1983), no. 3, 543–547.

[24] Nimana, N.; Artsawang, N. A strongly convergent simultaneous cutter method for finding the minimal norm solution to common fixed point problem. *Carpathian J. Math.* **40** (2024), no. 1, 155-171.

[25] Nimana, N.; Petrot, N. Viscosity Approximation Methods for Split Variational Inclusion and Fixed Point Problems in Hilbert Spaces. *In Proceedings of the International Multi-Conference of Engineers and Computer Scientists* **2** (2014), 12–14.

[26] Picard, E. Mémoire sur la théorie des équations aux dérivées partielles et la méthodedes approximations successives. *J. Amer. Math. Soc.* **6** (1890), 145–210.

[27] Plubtieng, S.; Punpaeng, R. A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces. *J. Math. Anal. Appl.* **336** (2007), 455–469.

[28] Plubtieng, S.; Ungchittrakool, K. Viscosity approximation methods for equilibrium problems and zeroes of an accretive operator in Hilbert spaces. *Int. Math. Forum.* **3** (2008), 1387–1400.

[29] Podilchuk, C. I.; Mammone, R. J. Image recovery by convex projections using a least-squares constraint. *J. Opt. Soc. Am.* **7** (1990), 517–521.

[30] Polyak, B. T. Some methods of speeding up the convergence of iterative methods. *Zh. Vychisl. Mat. Mat. Fiz.* **4** (1964), 1–17.
[31] Reich, S. Weak convergence theorems for nonexpansive mappings in Banach spaces. *J. Math. Anal. Appl.* **67** (1979), 274–276.
[32] Shehu, Y.; Iyiola, O. S.; Ogbuisi, F. U. Iterative method with inertial terms for nonexpansive mappings: applications to compressed sensing. *Numer. Algor.* **83** (2009), 1321–1347.
[33] Takahashi, W. *Nonlinear functional analysis*. Yokohama Publishers: Yokohama, 2000.
[34] Takahashi, W. *Introduction to nonlinear and convex analysis*. Yokohama Publishers: Yokohama, 2009.
[35] Takahashi, S.; Takahashi, W. Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. *J. Math. Anal. Appl.* **331** (2007), 506–515.
[36] Tikhonov, A. N.; Arsenin, V. Y. Solutions of ill–posed problems. *SIAM Rev.* **21** (1979), 266–267.
[37] Ungchittrakool, K. Existence and convergence of fixed points for a strict pseudo-contraction via an iterative shrinking projection technique. *J. Nonlinear Convex Anal.* **15** (2014), 693–710.
[38] Xu, H. K. Iterative algorithms for nonlinear operators. *J. London Math. Soc.* **66** (2002), 240–256.
[39] Yao, Y.; Zhou, H.; Liou, Y. C. Strong convergence of a modified Krasnosel'skiĭ-Mann iterative algorithm for non-expansive mappings. *J. Appl. Math. Comput.* **29** (2009), 383–389.

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