Journal of Global Optimization manuscript No.
(will be inserted by the editor)

Unifying local-global type properties in vector optimization

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Received: date / Accepted: date

Abstract It is well-known that all local minimum points of a semistrictly quasiconvex real-valued function are global minimum points. Also, any local maximum point of an explicitly quasiconvex real-valued function is a global minimum point, provided that it belongs to the intrinsic core of the function's domain. The aim of this paper is to show that these "local min - global min" and "local max - global min" type properties can be extended and unified by a single general local-global extremality principle for certain generalized convex vector-valued functions with respect to two proper subsets of the outcome space. For particular choices of these two sets, we recover and refine several local-global properties known in the literature, concerning unified vector optimization (where optimality is defined with respect to an arbitrary set, not necessarily a convex cone) and, in particular, classical vector/multicriteria optimization.

Keywords Unified vector optimization · Algebraic local extremal point · Topological local extremal point · Generalized convexity

Mathematics Subject Classification (2000) 90C26 · 90C29

1 Introduction

Generalized convex functions play an important role in optimization, variational inequalities, equilibrium problems, game theory, and other variational problems (see, e.g., Luc [23], Göpfert *et al.* [16], Cambini and Martein [8] and Jahn [20]).

Among many types of generalized convex functions known in the literature, the semistrictly quasiconvex functions and, in particular, the explicitly quasiconvex

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ones are of special interest, as they preserve some fundamental properties of convex functions. For real-valued functions it is known that:

• the semistrict quasiconvexity ensures the "*local min - global min*" property, i.e., every local minimum point is a global minimum point (see, e.g., Ponstein [25]);

• the explicit quasiconvexity ensures a "*local max - global min*" property, namely every local maximum point is a global minimum point if it belongs to the intrinsic core of the function's domain (see, e.g., Bagdasar and Popovici [2]).

The "local min - global min" property has been extended for different classes of generalized convex vector-valued functions, with respect to a convex cone C of the outcome space Y, in the context of vector optimization (see, e.g., Jahn and Sachs [21], Luc and Schaible [24], Cambini and Martein [8], Jahn [20], Bagdasar and Popovici [3]). A more general "local min - global min" type property was obtained by Flores-Bazán and Vera [15] for semistrictly (K)-quasiconvex vector functions in the framework of unified vector optimization, where the optimality is defined with respect to any proper subset K of Y (not necessarily a cone).

Some vectorial counterparts of the "local max - global min" property have been obtained by us in [3] for componentwise explicitly quasiconvex functions taking values in the Euclidean space $\mathbb{R}^m$, partially ordered by the standard cone $C = \mathbb{R}^m_+$, in terms of (ideal; weak) minimality. A natural question is whether the "local max - global min" property can be extended for certain classes of generalized convex functions taking values in a real topological linear space Y, with respect to a convex cone C or an arbitrary set K.

In this paper we give a positive answer to this question and, even more, we show that all "local min - global min" type and "local max - global min" type properties can be unified into a single general local-global extremality principle. To this aim, we consider algebraic (and, in particular, topological) local extremal points and we introduce novel concepts of generalized convexity for vector-valued functions, with respect to two arbitrary sets $K_1, K_2 \subseteq Y$. In particular, when $K_1 = \pm K_2 = K$, we recover or refine several local-global extremal properties known in the literature.

The paper is organized as follows. In Section 2, after recalling some notions and results of convex analysis, we introduce the new concept of algebraic local K-extremality for vector functions with respect to a set K, which represents an algebraic counterpart of a similar (topological) concept introduced by Flores-Bazán and Hernández [13] within unified vector optimization. Then we review several important classes of generalized convex scalar or vector functions, known to possess "local min - global min" and/or "local max - global min" type properties in scalar optimization (Propositions 2 and 3), in unified vector optimization with respect to a set K (Proposition 4), and in vector/multicriteria optimization with respect to a convex cone C (Propositions 5, 6, 7 and 8).

In order to generalize and unify these local-global extremality properties, in Section 3 we introduce new concepts of generalized convexity for vector-valued functions with respect to two arbitrary sets $K_1, K_2 \subseteq Y$, namely: the semistrict and explicit (K_1, K_2)-quasiconvexity, the unidirectional (K_1, K_2)-quasiconvexity (in particular the unidirectional (K)-quasiconvexity, obtained for $K = K_1 = -K_2$), and the bidirectional (K_1, K_2)-quasiconvexity. Among other results, we establish relationships between these new concepts and also illustrate how are they related to those presented in the previous section or to other known concepts, as for instance the (P, Q)-quasiconvexity proposed by Cambini, Luc and Martein [7].

Section 4 contains our main results. Theorem 9 represents a general local-global extremality principle for bidirectional (K_1, K_2)-quasiconvex vector-valued functions, which encompasses both "local min - global min" type properties (when $K_1 = K_2$) and "local max - global min" type properties (when $K_1 = -K_2$). Two of its consequences, namely Theorems 10 and 11, also provide general local-global extremality properties, for unidirectional (K_1, K_2)-quasiconvex and semistrictly (K_1, K_2)-quasiconvex functions, respectively. These general results can be applied to unified vector optimization, letting $K_1 = \pm K_2 = K$. Indeed, we derive "local min - global min" type properties (Theorems 12 and 13) and "local max - global min" type properties (Theorems 14 and 15). In particular, when Y is partially ordered by a solid convex cone, C, which is not a linear subspace, by letting $K \in \{-C^c, C \setminus \ell(C), \operatorname{cor} C\}$ we obtain new "local min - global min" type properties (Corollaries 6 and 7) and "local max - global min" type properties (Corollaries 8, 9, 10 and 11).

Section 5 contains concluding remarks concerning further possible extensions of our results to appropriate classes of generalized convex set-valued functions with respect to variable ordering structures.

2 Preliminaries

2.1 Generalized interiority concepts

Given a real topological linear space E, we represent its origin by 0_E. For any points $u, v \in E$, we denote $[u, v] = \{u + t(v - u) \mid t \in [0, 1]\}$, $]u, v] = [v, u[= [u, v] \setminus \{u\}$ and $]u, v[=]u, v] \setminus \{v\}$. Also, for any sets $A, B \subseteq E$ and $T \subseteq \mathbb{R}$, we use the notations $A \pm B = \{x \pm y \mid (x, y) \in A \times B\}$, $v \pm A = \{v\} \pm A$, $T \cdot A = \{tx \mid (t, x) \in T \times A\}$, $T \cdot v = T \cdot \{v\}$ and $-A = \{-1\} \cdot A$. By A^c, $\operatorname{span} A$ and $\operatorname{int} A$ we denote the complement (w.r.t. E), the linear hull and the (topological) interior of A, respectively. Recall that the core (algebraic interior) and the intrinsic core (relative algebraic interior) of A are defined by (see, e.g., Holmes [19]):

$$\operatorname{cor} A = \{x \in A \mid \forall\, y \in E,\ \exists\, \delta > 0\ :\ [x,\, x + \delta y] \subseteq A\},$$
$$\operatorname{icr} A = \{x \in A \mid \forall\, y \in \operatorname{span}(A - A),\ \exists\, \delta > 0\ :\ [x,\, x + \delta y] \subseteq A\}.$$

Following Adán and Novo [1], we say that a set A is solid if $\operatorname{cor} A \neq \emptyset$, and relatively solid if $\operatorname{icr} A \neq \emptyset$, respectively.

Remark 1 a) We always have

$$\operatorname{int} A \subseteq \operatorname{cor} A \subseteq \operatorname{icr} A.$$

b) If A is solid, then $\operatorname{span}(A - A) = E$ hence $\operatorname{cor} A = \operatorname{icr} A$ (cf. Zălinescu [28, p. 3]).

Whenever A is convex, i.e., $[x, y] \subseteq A$ for all $x, y \in A$, more interesting properties hold. We recall two of them in the next result.

Proposition 1 **(*Support Theorem* in [19] and Thm. 1.1.2 (v) in [28])** *Let $A \subseteq E$ be a convex set. The following assertions hold:*

1° *If A is relatively solid and $v \in (\operatorname{icr} A)^c$, then there exists a linear functional $\ell : E \to \mathbb{R}$ such that $\ell(u) > \ell(v)$ for all $u \in \operatorname{icr} A$.*

2° *If $\operatorname{int} A \neq \emptyset$, then $\operatorname{int} A = \operatorname{cor} A$.*

For any point $x \in E$, we denote the family of its neighborhoods by

$$\mathcal{V}(x) = \{V \subseteq E \mid x \in \operatorname{int} V\}.$$

Usually in both scalar and vector optimization, local optimal solutions are defined by means of neighborhoods. However, it is also possible to define local optimality by an algebraic approach (see, e.g., Jahn [20]). In order to adapt this approach for unified vector optimization, it will be convenient to consider the family

$$\mathcal{U}(x) = \{U \subseteq E \mid x \in \operatorname{cor} U\}.$$

By a classical argument in functional analysis we have

$$\mathcal{V}(x) = \{x + V \mid V \in \mathcal{V}(0_E)\} \subseteq \mathcal{U}(x). \tag{1}$$

Also, it is easily seen that

$$U_1 \cap U_2 \in \mathcal{U}(x), \ \forall U_1, U_2 \in \mathcal{U}(x). \tag{2}$$

2.2 K-extremal points

A very interesting topic in vector optimization is to unify several classical concepts of optimality (usually defined by means of an ordering cone) via a general notion of optimality (see, e.g., Flores-Bazán and Hernández [13], Gutiérrez, Jiménez and Novo [17], and Gutiérrez *et al.* [18]).

Let X and Y be two real topological linear spaces, $Y \neq \{0_Y\}$. Following the unifying approach proposed by Flores-Bazán and Hernández [13], we introduce some concepts of generalized optimality for functions defined on a nonempty set $D \subseteq X$ with values in Y, with respect to a proper set $K \subseteq Y$, i.e., $\emptyset \neq K \neq Y$.

Definition 1 Given a function $f : D \to Y$, for any nonempty set $S \subseteq D$ we denote

$$K\text{-Ext}(f \mid S) = \{x^0 \in S \mid \nexists x \in S \setminus \{x^0\} \, : \, f(x^0) - f(x) \in K\}. \tag{3}$$

An element $x^0 \in D$ is called:

- *global K-extremal point of f*, if $x^0 \in K\text{-Ext}(f \mid D)$;
- *topological local K-extremal point of f*, if there exists a neighborhood $V \in \mathcal{V}(x^0)$ such that $x^0 \in K\text{-Ext}(f \mid V \cap D)$;
- *algebraic local K-extremal point of f*, if there exists a set $U \in \mathcal{U}(x^0)$ such that $x^0 \in K\text{-Ext}(f \mid U \cap D)$.

Remark 2 a) The terminology proposed by us in Definition 1 is slightly different from the one introduced by Flores-Bazán and Hernández in [13]. More precisely, since (3) can be rewritten as

$$K\text{-Ext}(f \mid S) = \{x^0 \in S \mid \ f(x) - f(x^0) \in -K^c, \forall x \in S \setminus \{x^0\}\},$$

the global K-extremal points and topological local K-extremal points correspond to the "global $-K^c$-minimizers" and "local $-K^c$-minimizers", respectively, while algebraic local K-extremal points have no correspondent in [13].

b) If $0_Y \notin K$, then (3) reduces to

$$\begin{aligned} K\text{-Ext}(f \mid S) &= \{x^0 \in S \mid f(x) - f(x^0) \in -K^c,\, \forall\, x \in S\}, \\ &= \{x^0 \in S \mid \nexists\, x \in S \,:\, f(x^0) - f(x) \in K\}. \end{aligned}$$

c) If $S \subseteq \widetilde{S} \subseteq D$ and $K \subseteq \widetilde{K} \subseteq Y$ are nonempty sets, with $\widetilde{K} \neq Y$, then we have

$$\widetilde{K}\text{-Ext}(f \mid \widetilde{S}) \cap S \subseteq K\text{-Ext}(f \mid S).$$

d) If K_1 and K_2 are proper subsets of Y, such that $K_1 \cup K_2 \neq Y$, then

$$K_1\text{-Ext}(f \mid S) \cap K_2\text{-Ext}(f \mid S) = (K_1 \cup K_2)\text{-Ext}(f \mid S).$$

In particular, if $K_1 \cup K_2 = Y \setminus \{0_Y\}$, i.e., $K_1^c \cap K_2^c = \{0_Y\}$, it follows by b) that for any $x^0 \in S$ the following equivalence holds true:

$$x^0 \in K_1\text{-Ext}(f \mid S) \cap K_2\text{-Ext}(f \mid S) \;\Leftrightarrow\; f \text{ is constant on } S.$$

The following two examples illustrate classical optimality concepts which can be recovered from Definition 1 for particular choices of K, in scalar and vector optimization, respectively.

Example 1 In the framework of scalar optimization, where $Y = \mathbb{R}$, we obtain the classical notions of optimality as follows. Consider a function $f : D \to \mathbb{R}$.

a) If $K = \mathbb{R}_+^* =]0, \infty[$, then for any set $S \subseteq D$ relation (3) becomes

$$K\text{-Ext}(f \mid S) = \operatorname*{argmin}_{x \in S} f(x) := \{x^0 \in S \mid f(x^0) \leq f(x),\, \forall\, x \in S\},$$

hence an element $x^0 \in D$ is a (global, topological/algebraic local) K-extremal point if and only if x^0 is a (global, topological local, algebraic local) minimum point.

b) If $K = \mathbb{R}_-^* =]-\infty, 0[$, then for any set $S \subseteq D$ we have

$$K\text{-Ext}(f \mid S) = \operatorname*{argmax}_{x \in S} f(x) := \{x^0 \in S \mid f(x^0) \geq f(x),\, \forall\, x \in S\},$$

hence $x^0 \in D$ is a (global, topological/algebraic local) K-extremal point if and only if x^0 is a (global, topological local, algebraic local) maximum point.

Example 2 In the framework of vector optimization (see, e.g., Luc [23] and Jahn [20]), the real topological linear space Y is partially ordered by a convex cone $C \subseteq Y$ (i.e., $0_Y \in C = \mathbb{R}_+ \cdot C = C + C$). More precisely, C induces on Y an order relation $\leqq_C$ defined for any $y', y'' \in Y$ by

$$y' \leqq_C y'' \;\Leftrightarrow\; y'' - y' \in C.$$

When $C \neq \ell(C) := C \cap (-C)$, we also consider the binary relation $\leq_C$ given by

$$y' \leq_C y'' \;\Leftrightarrow\; y'' - y' \in C \setminus \ell(C).$$

In particular, if $C \neq \ell(C)$ is solid, a third binary relation $<_C$ is usually defined by

$$y' <_C y'' \;\Leftrightarrow\; y'' - y' \in \operatorname{cor} C.$$

Consider a vector-valued function $f : D \to Y$.

a) By choosing $K \in \{C^c, (-C) \setminus \ell(C), -\operatorname{cor} C\}$, we recover from K-extremality three well-known concepts of minimality, as follows.

- Letting $K = (-C)^c$, for any $S \subseteq D$ relation (3) becomes

$$K\text{-Ext}(f \mid S) = \{x^0 \in S \mid f(x^0) \leqq_C f(x),\ \forall\, x \in S\},$$

hence $x^0 \in D$ is a (global, topological/algebraic local) K-extremal point of f if and only if it is a (global, topological/algebraic local) ideal C-minimal point of f.

- Assume that $C \neq \ell(C)$ and let $K = C \setminus \ell(C)$. Then, for any $S \subseteq D$, we have

$$K\text{-Ext}(f \mid S) = \{x^0 \in S \mid \nexists\, x \in S : f(x) \leq_C f(x^0)\},$$

hence $x^0 \in D$ is a (global, topological/algebraic local) K-extremal point of f if and only if it is a (global, topological/algebraic local) C-minimal point of f.

- Assume that $C \neq \ell(C)$ is solid and let $K = \operatorname{cor} C$. Then, for any $S \subseteq D$,

$$K\text{-Ext}(f \mid S) = \{x^0 \in S \mid \nexists\, x \in S : f(x) <_C f(x^0)\},$$

hence $x^0 \in D$ is a (global, topological/algebraic local) K-extremal point of f if and only if it is (global, topological/algebraic local) weakly C-minimal point of f.

b) By choosing $K \in \{C^c, (-C)\setminus\ell(C), -\operatorname{cor} C\}$, i.e., by letting $-C$ in the role of C in a), we recover the corresponding concepts of ideal C-maximality, C-maximality, and weak C-maximality, currently used in vector optimization.

Remark 3 a) Besides the concepts of C-minimality and C-maximality presented in Example 2, some other optimality notions can be also defined for suitable choices of K, as shown by Flores-Bazán and Hernández [13], Gutiérrez, Jiménez and Novo [17], and Gutiérrez *et al.* [18].

b) Weakly C-minimal points, as well as weakly C-maximal points, may also be defined by means of $\operatorname{icr} C$ instead of $\operatorname{cor} C$, when C is relatively solid but not solid (see, e.g., Adán and Novo [1]) or by other generalized interiors (see, e.g., Borwein and Lewis [5], or Boţ and Csetnek [6]).

Remark 4 a) Global K-extremality implies topological local K-extremality. Note also that, in the particular case when X is endowed with the indiscrete topology $\{\emptyset, X\}$, the concepts of topological local K-extremality and global K-extremality actually coincide.

b) By (1), topological local K-extremality implies algebraic local K-extremality. However, the converse is not true, as shown by the following example.

Example 3 Let $X = \mathbb{R}^2$ be endowed with the Euclidean topology, let $Y = \mathbb{R}$ be equipped with the usual topology and $K = \mathbb{R}^*_+$. Define $f : D = \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x) = \begin{cases} 1 \text{ if } x \in U \\ 0 \text{ if } x \notin U, \end{cases}$$

i.e., the indicator function of the set

$$U = \mathbb{R}^2 \setminus \big\{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0,\ (x_1)^2 > x_2 > 0\big\}.$$

Observe that $x^0 = (0, 0) \in \operatorname{argmin}_{x \in U \cap D} f(x)$. Since $x^0 \in \operatorname{cor} U$, i.e., $U \in \mathcal{U}(x^0)$, it follows that x^0 is an algebraic local minimum point of f. However, for every $V \in \mathcal{V}(x^0)$ we have $\min_{x \in V \cap D} f(x) = 0 < f(x^0)$, hence $x^0 \notin \operatorname{argmin}_{x \in V \cap D} f(x)$. Therefore, x^0 is not a topological local minimum point of f.

We end this section with a technical result, which will be used in Section 4.

Lemma 1 *Let K_1 and K_2 be proper subsets of Y, such that $K_1 \cup K_2 \neq Y$. For any $x^0 \in D$ the following assertions are equivalent:*

1° *x^0 is an algebraic local K_i-extremal point of f, for both $i \in \{1, 2\}$.*

2° *x^0 is an algebraic local $(K_1 \cup K_2)$-extremal point of f.*

In particular, when $K_1 \cup K_2 = Y \setminus \{0_Y\}$, these assertions are also equivalent to:

3° *There exists $U \in \mathcal{U}(x^0)$ such that f is constant on $U \cap D$.*

Proof Assuming that 1° holds, we can find two sets $U_1, U_2 \in \mathcal{U}(x^0)$ such that $x^0 \in K_1\text{-Ext}(f \mid U_1 \cap D) \cap K_2\text{-Ext}(f \mid U_2 \cap D)$. Letting $U := U_1 \cap U_2$, by (2) we have $U \in \mathcal{U}(x^0)$. Since $x^0 \in U \cap D \subseteq U_i \cap D$, we can deduce by Remark 2 c) and d) that $x^0 \in (K_1 \cup K_2)\text{-Ext}(f \mid U \cap D)$, hence x^0 is an algebraic local $(K_1 \cup K_2)$-extremal point of f. Thus 1° implies 2°.

Conversely, assume that 2° holds. Then, there exists a set $U \in \mathcal{U}(x^0)$ such that $x^0 \in (K_1 \cup K_2)\text{-Ext}(f \mid U \cap D)$. Since $K_i \subseteq K_1 \cup K_2$, it follows by Remark 2 c), that $x^0 \in K_i\text{-Ext}(f \mid U \cap D)$ for any $i \in \{1, 2\}$, hence 1° holds.

Now, consider the particular case when $K_1 \cup K_2 = Y \setminus \{0_Y\}$.

If 2° holds, then there is $U \in \mathcal{U}(x^0)$ such that $x^0 \in (K_1 \cup K_2)\text{-Ext}(f \mid U \cap D)$, hence f is constant on $U \cap D$, by Remark 2 d). Thus 2° implies 3°.

Conversely, if 3° holds, then there is $U \in \mathcal{U}(x^0)$ such that $f(x) = f(x^0)$, i.e., $f(x^0) - f(x) = 0_Y \notin K_1 \cup K_2$, for any $x \in U \cap D$. Hence $x^0 \in (K_1 \cup K_2)\text{-Ext}(f \mid U \cap D)$, and therefore 2° holds. □

2.3 Semistrict/explicit quasiconvex real-valued functions

Let X be a real topological linear space and $D \subseteq X$ a nonempty set. The following definition recalls classical concepts of convex analysis.

Definition 2 Assuming that D is convex, a scalar function $f : D \to \mathbb{R}$ is called:

- *convex*, if $f(x' + t(x'' - x')) \leq f(x') + t[f(x'') - f(x')]$ for all $x', x'' \in D$, $t \in [0,1]$;
- *quasiconvex*, if $f(x) \leq \max\{f(x'), f(x'')\}$ for any $x', x'' \in D$ and $x \in [x', x'']$;
- *semistrictly quasiconvex*, if $f(x) < \max\{f(x'), f(x'')\}$ for any $x', x'' \in D$ with $f(x') \neq f(x'')$ and $x \in]x', x''[$.
- *explicitly quasiconvex*, if f is both quasiconvex and semistrictly quasiconvex.

Remark 5 The following properties concern scalar functions:

a) f is quasiconvex if and only if for any distinct points $x', x'' \in D$ such that $f(x') \leq f(x'')$ and any $x \in]x', x''[$ we have $f(x) \leq f(x'')$.

b) f is semistrictly quasiconvex if and only if for any distinct points $x', x'' \in D$ such that $f(x') < f(x'')$ and any $x \in]x', x''[$ we have $f(x) < f(x'')$.

c) Convex functions, as well as lower semicontinuous semistrictly quasiconvex functions are explicitly quasiconvex.

d) There are quasiconvex functions which are not semistrictly quasiconvex (see Example 4). Also, there are semistrictly quasiconvex functions which are not quasiconvex (see Example 5).

Example 4 Consider the function $f : D = \mathbb{R} \to \mathbb{R}$, defined for any $x \in \mathbb{R}$ by

$$f(x) = \min\{x, 0\}.$$

This function is quasiconvex, as being monotone. However, it is not semistrictly quasiconvex, since for $x' = -1 < x = 0 < x'' = 1$ we have $f(x') < f(x'')$, but $f(x) \geq f(x'')$.

Remark 6 It is easily seen that every point $x^0 > 0$ is a topological local minimum point for function f defined in Example 4, but is not a global minimum point. This happens because f is not semistrictly quasiconvex, as shown by the following classical "local min - global min" property, due to Ponstein [25].

Proposition 2 (Thm. 2 in [25]) *Let $f : D \to \mathbb{R}$ be a semistrictly quasiconvex function. A point $x^0 \in D$ is a topological local minimum point of f if and only if it is a global minimum point.*

Example 5 Let $f : D = \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0. \end{cases}$$

Function f is semistrictly quasiconvex, since whenever $f(x') < f(x'')$ we actually have $x' \neq 0$ and $x'' = 0$, hence $f(x) < f(x'')$ for any $x \in]x', x''[$. However, f is not quasiconvex, because $f(0) \not\leq \max\{f(-1), f(1)\}$.

Remark 7 It is easily seen that $x^0 = 0$ is a global maximum point for each of the functions considered in Examples 4 and 5 (which are not explicitly quasiconvex!), but is not a local minimum point. This phenomenon is explained by the following "local max - global min" property, established by us in [2].

Proposition 3 (Thm. 3.1 in [2]) *Assume that D is convex. Let $f : D \to \mathbb{R}$ be an explicitly quasiconvex function and let $x^0 \in \operatorname{icr} D$. If x^0 is a topological local maximum point of f, then x^0 is a global minimum point.*

2.4 Semistrictly/explicitly (K)-quasiconvex vector functions

Let X and Y be real topological linear spaces and let $D \subseteq X$ and $K \subseteq Y$ be nonempty sets.

The following notions were proposed by Flores-Bazán and Vera [15] (where X and Y were assumed to be real normed spaces) and also by Flores-Bazán in [12] (in finite-dimensional spaces).

Definition 3 (Defs. 4.1 and 4.3 in [15]) Assume that D is convex and let $x^0 \in D$. A function $f : D \to Y$ is called:

- *semistrictly (K)-quasiconvex at x^0* if for any $x' \in D \setminus \{x^0\}$ with $f(x^0) - f(x') \in K$, we have $f(x^0) - f(x) \in K$ for all $x \in]x^0, x'[$;
- *explicitly (K)-quasiconvex at x^0* if f is both semistrictly (K)-quasiconvex at x^0 and semistrictly $(-K^c)$-quasiconvex at x^0;
- *semistrictly/explicitly (K)-quasiconvex on D* if function f is semistrictly/explicitly (K)-quasiconvex at every point of D.

Remark 8 Since $(-K)^c = -K^c$, a function f is explicitly (K)-quasiconvex at x^0 if and only if it is explicitly $(-K^c)$-quasiconvex at x^0.

Example 6 a) When K is a convex cone, the semistrict (K)-quasiconvexity reduces to the quasiconvexity (w.r.t. K) in the sense of Borwein [4, Def. 9] (see also Jahn and Sachs [21, Def. 2.1]).

b) As shown by Flores-Bazán [12], for $Y = \mathbb{R}$, the semistrict (K)-quasiconvexity reduces to classical notions of generalized convexity for real-valued functions (see Definition 2), namely quasiconvexity when $K = \mathbb{R}_+$, and semistrict quasiconvexity when $K = \mathbb{R}_+^*$. On the other hand, the explicit (K)-quasiconvexity coincides with the classical notion of explicit quasiconvexity for $K = \mathbb{R}_+^*$ as well as for $K = \mathbb{R}_+$.

Definition 4 Assume that D is convex. A function $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$ is called *componentwise convex* (*quasiconvex*; *semistrictly quasiconvex*; *explicitly quasiconvex*) if $f_1, \ldots, f_m$ are convex (quasiconvex; semistrictly quasiconvex; explicitly quasiconvex) in the sense of Definition 2.

In what follows we point out the relationship between the componentwise semistrict/explicit quasiconvexity and the semistrict/explicit (K)-quasiconvexity.

Lemma 2 *Assume that D is convex. If function $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$ is componentwise semistrictly quasiconvex, then it is semistrictly (K)-quasiconvex on D for $K \in \{(-\mathbb{R}_+^m)^c, \operatorname{int} \mathbb{R}_+^m\}$.*

Proof When $K = \operatorname{int} \mathbb{R}_+^m$, the result was obtained by Flores-Bazán [12, p. 142] for $X = \mathbb{R}^n$, the proof being still valid for any real topological linear space X.

Now let $K = (-\mathbb{R}_+^m)^c$. Consider any point $x^0 \in D$. Let $x' \in D \setminus \{x^0\}$ with $f(x^0) - f(x') \in (-\mathbb{R}_+^m)^c$ and let $x \in]x^0, x'[$. Then, there is $i \in \{1, \ldots, m\}$ such that $f_i(x') < f_i(x^0)$. As f_i is semistrictly quasiconvex, we have $f_i(x) < f_i(x^0)$, hence $f(x^0) - f(x) \in (-\mathbb{R}_+^m)^c$. Thus f is semistrictly (K)-quasiconvex at x^0. □

Remark 9 For $m \geq 2$ the converse of Lemma 2 does not hold, as shown by the following two examples.

Example 7 Let $f = (f_1, f_2) : D = [0, 1] \to \mathbb{R}^2$ be defined by

$$f_1(x) = \begin{cases} x \text{ if } x \in [0, 1[\\ 0 \text{ if } x = 1, \end{cases} \qquad f_2(x) = \begin{cases} -x \text{ if } x \in [0, 1[\\ 1 \quad \text{if } x = 1. \end{cases}$$

It is a simple exercise to show that function f is semistrictly (K)-quasiconvex on D for $K = (-\mathbb{R}_+^2)^c$. However, f is not componentwise semistrictly quasiconvex, because f_1 is not semistrictly quasiconvex.

Example 8 Let the function $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) = (x, \min\{0, -x\}).$$

It was noticed by Flores-Bazán in [12, p. 142], that f is semistrictly (K)-quasiconvex on D for $K = \operatorname{int} \mathbb{R}_+^2$. However, f is not componentwise semistrictly quasiconvex, since f_2 is not semistrictly quasiconvex.

Remark 10 For $K = \mathbb{R}^m_+ \setminus \{0\}$ and $m \geq 2$, the semistrict (K)-quasiconvexity on D and the componentwise semistrict quasiconvexity do not imply each other.

Indeed, it is easily seen that the function defined in Example 8 is semistrictly $(\mathbb{R}^2_+ \setminus \{0\})$-quasiconvex on D, but not componentwise semistrictly quasiconvex. A function which is componentwise semistrictly quasiconvex but not semistrictly $(\mathbb{R}^2_+ \setminus \{0\})$-quasiconvex on D is provided in the example below.

Example 9 Let $f = (f_1, f_2) : D = \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f_1(x) = x, \quad f_2(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ 1 \text{ if } x = 0. \end{cases}$$

Clearly, f is componentwise semistrictly quasiconvex. However, f is not semistrictly $(\mathbb{R}^2_+ \setminus \{0\})$-quasiconvex at $x^0 = 1$. Indeed, for $x' = -1 \in D \setminus \{x^0\}$ and $x = 0 \in]x^0, x'[$, we have $f(x^0) - f(x') = (2, 0) \in \mathbb{R}^2_+ \setminus \{0\}$ while $f(x^0) - f(x) = (1, -1) \notin \mathbb{R}^2_+ \setminus \{0\}$.

Remark 11 A characterization of componentwise explicitly quasiconvex functions was obtained by Popovici [27, Thm. 3.1] by means of an alternative concept of explicit quasiconvexity for vector functions, which is not considered in our paper.

Theorem 1 *Assume that D is convex. If function $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$ is componentwise explicitly quasiconvex, then it is explicitly (K)-quasiconvex on D for every $K \in \{(-\mathbb{R}^m_+)^c, \mathbb{R}^m_+ \setminus \{0\}, \operatorname{int} \mathbb{R}^m_+\}$.*

Proof Assume that f is componentwise explicitly quasiconvex and let $x^0 \in D$.

For $K = (-\mathbb{R}^m_+)^c$ the semistrict (K)-quasiconvexity of f at x^0 follows from Lemma 2. In order to prove that f is semistrictly $(-K^c)$-quasiconvex at x^0, observe that $(-K)^c = \mathbb{R}^m_+$ and let $x' \in D \setminus \{x^0\}$ with $f(x^0) - f(x') \in \mathbb{R}^m_+$ and $x \in]x^0, x'[$. Then, for any $i \in \{1, \ldots, m\}$ we have $f_i(x^0) \geq f_i(x')$, which yields $f_i(x^0) \geq f_i(x)$, by quasiconvexity of f_i. This shows that $f(x^0) - f(x) \in \mathbb{R}^m_+$. Hence, f is semistrictly $(-K^c)$-quasiconvex at x^0.

For $K = \mathbb{R}^m_+ \setminus \{0\}$ one has $(-K)^c = (-\mathbb{R}^m_+)^c \cup \{0\}$. First we prove that f is semistrictly (K)-quasiconvex at x^0. Let $x' \in D \setminus \{x^0\}$ with $f(x^0) - f(x') \in K$ and let $x \in]x^0, x'[$. We have $f_i(x^0) \geq f_i(x')$ for all $i \in \{1, \ldots, m\}$ and $f_j(x^0) > f_j(x')$ for some $j \in \{1, \ldots, m\}$. As f_j is semistrictly quasiconvex we have $f_j(x^0) > f_j(x)$. Also, for all $i \in \{1, \ldots, m\}$ we have $f_i(x^0) \geq f_i(x)$ by quasiconvexity of f_i. Therefore we have $f(x^0) - f(x) \in K$. We conclude that f is semistrictly (K)-quasiconvex at x^0.

Now we show that function f is semistrictly $(-K^c)$-quasiconvex at x^0. Let $x' \in D \setminus \{x^0\}$ with $f(x^0) - f(x') \in (-\mathbb{R}^m_+)^c \cup \{0\}$ and let $x \in]x^0, x'[$. We distinguish two cases. First, if $f(x^0) - f(x') \in (-\mathbb{R}^m_+)^c$, then there is $i \in \{1, \ldots, m\}$ such that $f_i(x^0) > f_i(x')$. As f_i is semistrictly quasiconvex, we have $f_i(x^0) > f_i(x)$, so $f(x^0) - f(x) \in (-\mathbb{R}^m_+)^c \subseteq (-K)^c$. Second, if $f(x^0) - f(x') = 0$, then for any $i \in \{1, \ldots, m\}$ we have $f_i(x^0) = f_i(x')$, hence $f_i(x^0) \geq f_i(x)$ by quasiconvexity of f_i. This shows that $f(x^0) - f(x) \in \mathbb{R}^m_+ \subseteq (-\mathbb{R}^m_+)^c \cup \{0\} = (-K)^c$. We conclude that f is semistrictly $(-K^c)$-quasiconvex at x^0.

For $K = \operatorname{int} \mathbb{R}^m_+$ the result was established by Flores-Bazán when $X = \mathbb{R}^n$ in [12, Thm. 5.3 (3)], the proof being valid for any real topological linear space X. □

We now present a "local min - global min" type property for vector functions obtained by Flores-Bazán and Hernández [14]. Here we reformulate it in the spirit of Remark 2 a).

Proposition 4 (Prop. 3.4 in [14]) *Assume that D is convex. Let $x^0 \in D$ be a topological local K-extremal point of a function $f : D \to Y$. Then, x^0 is a global K-extremal point if and only if f is semistrictly (K)-quasiconvex at x^0.*

2.5 K-quasiconvex vector functions

An important concept of generalized convexity, which plays a key role in "local min - global min" type properties, can be found in the monograph of Jahn [20] (see also the earlier work of Jahn and Sachs [21]). We reformulate these properties by adopting the terminology used in Remark 2 a) and Example 2 a). Let X and Y be real topological linear spaces, and let $D \subseteq X$ and $K \subseteq Y$ be nonempty sets.

Definition 5 (Def. 7.11 in [20]) A vector-valued function $f : D \to Y$ is called *K-quasiconvex at* $x^0 \in D$ if for any $x' \in D \setminus \{x^0\}$ with $f(x^0) - f(x') \in K$ there is some $x'' \in D \setminus \{x^0\}$ satisfying the following two conditions: $]x^0, x''] \subseteq D$ and $f(x^0) - f(x) \in K$ for all $x \in]x^0, x'']$ (the first condition being superfluous, whenever D is convex). We say that f is *K-quasiconvex on* D if it is K-quasiconvex at every point of D.

Proposition 5 (Thm. 7.15 in [20]) *Let $C \subseteq Y$ be a convex cone, $C \neq \ell(C)$. Assume that $x^0 \in D$ is an algebraic local C-minimal point of a function $f : D \to Y$. The following assertions are equivalent:*
1° x^0 *is a global C-minimal point of f.*
2° f *is $C \setminus \ell(C)$-quasiconvex at x^0.*

Proposition 6 (Thm. 7.16 in [20]) *Let $C \subseteq Y$ be a solid convex cone. Assume that $x^0 \in D$ is an algebraic local weakly C-minimal point of a function $f : D \to Y$. The following assertions are equivalent:*
1° x^0 *is a global weakly C-minimal point of f.*
2° f *is* $\operatorname{cor} C$*-quasiconvex at x^0.*

For ideal C-minimal points, we have established the following result.

Proposition 7 (Prop. 4.3 in [3]) *Let $Y = \mathbb{R}^m$ be endowed with the usual ordering cone $C = \mathbb{R}^m_+$. Let $x^0 \in D$ be a topological local ideal C-minimal point of a function $f : D \to \mathbb{R}^m$. The following assertions are equivalent:*
1° x^0 *is a global ideal C-minimal point of f.*
2° f *is $-C^c$-quasiconvex at x^0.*

The following result summarizes three "local max - global min" type properties for vector functions.

Proposition 8 (Thms. 4.6, 4.12 and 4.17 in [3]) *Assume that D is convex. Let $Y = \mathbb{R}^m$ be endowed with the usual ordering cone $C = \mathbb{R}^m_+$. Consider a componentwise explicitly quasiconvex function $f : D \to Y$ and let $x^0 \in \operatorname{icr} D$. If x^0 is a topological local ideal C-maximal point of f (resp. topological local C-maximal, topological local weakly C-maximal point of f), then it is a global ideal C-minimal point (resp. global C-minimal, global weakly C-minimal point of f).*

3 New concepts of generalized convexity for vector functions

In this section we extend several notions of generalized convexity for vector-valued functions that are known to play an important role in vector optimization. Throughout we assume that X and Y are real topological linear spaces, while $D \subseteq X$ and $K, K_1, K_2 \subseteq Y$ are nonempty sets.

3.1 Semistrictly/explicitly (K_1, K_2)-quasiconvex vector functions

Definition 6 We say that a function $f : D \to Y$ is *semistrictly* (K_1, K_2)*-quasiconvex at a point* $x^0 \in D$ if its domain D is convex and for any $x' \in D \setminus \{x^0\}$ with $f(x^0) - f(x') \in K_1$, we have $f(x^0) - f(x) \in K_2$ for all $x \in]x^0, x'[$. If f satisfies this property for every $x^0 \in D$, then f is called *semistrictly* (K_1, K_2)*-quasiconvex on* D.

Remark 12 Letting $K_1 = K_2 := K$, the notion of semistrict (K_1, K_2)-quasiconvexity recovers the semistrict (K)-quasiconvexity in the sense of Definition 3.

Besides the semistrictly (K)-quasiconvex functions, another class of generalized convex vector functions known in the literature can be recovered from Definition 6, as shown below.

Example 10 Consider the particular case when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ is partially ordered by a solid closed convex cone C, such that $C \neq \ell(C)$. By choosing $K_1 := P$ and $K_2 := Q$ from among the sets C, $\operatorname{int} C$ and $C \setminus l(C)$, we recover the concept of (P, Q)-quasiconvexity (on D) introduced by Cambini, Luc and Martein [7].

Remark 13 Function f is semistrictly (K_1, K_2)-quasiconvex on D if and only if for any distinct points $x', x'' \in D$ and $x^0 \in]x', x''[$ we have

$$f(x') - f(x'') \in K_1 \Rightarrow f(x') - f(x^0) \in K_2.$$

Lemma 3 *Let K_1, K_2 and K_1', K_2' be nonempty subsets of Y such that $K_1 \subseteq K_1'$ and $K_2' \subseteq K_2$. If D is convex and $f : D \to Y$ is semistrictly (K_1', K_2')-quasiconvex at some point $x^0 \in D$, then f is also semistrictly (K_1, K_2)-quasiconvex at x^0.*

Proof Assume that f is semistrictly (K_1', K_2')-quasiconvex at x^0. Consider any point $x' \in D \setminus \{x^0\}$ such that $f(x^0) - f(x') \in K_1$. Then $f(x^0) - f(x') \in K_1'$, since $K_1 \subseteq K_1'$. As the function is semistrictly (K_1', K_2')-quasiconvex at x^0, we have $f(x^0) - f(x) \in K_2' \subseteq K_2$ for all $x \in]x^0, x'[$. Therefore, function f is semistrictly (K_1, K_2)-quasiconvex at x^0. □

Definition 7 Assume that D is convex and let $x^0 \in D$. We say that $f : D \to Y$ is *explicitly* (K_1, K_2)*-quasiconvex at* x^0 if f is both semistrictly (K_1, K_2)-quasiconvex and semistrictly $(-K_2^c, -K_1^c)$-quasiconvex at x^0. When this property holds for every $x^0 \in D$ we say that f is *explicitly* (K_1, K_2)*-quasiconvex on* D.

Remark 14 Clearly, a function f is explicitly (K_1, K_2)-quasiconvex at x^0 if and only if it is explicitly $(-K_2^c, -K_1^c)$-quasiconvex at x^0.

The next example shows that two well-known generalized convexity concepts can be seen as particular instances of Definition 7.

Example 11 a) By considering $K_1 = K_2 = K$ we recover the notion of explicit (K)-quasiconvexity (see Definition 3).

b) For $Y = \mathbb{R}$ and $K_1 = K_2 = \mathbb{R}_+$ (or alternatively, $K_1 = K_2 = \mathbb{R}_+^*$) we obtain the classical notion of explicit quasiconvexity of scalar functions (see Definition 2).

Remark 15 Function f is semistrictly $(-K_2^c, -K_1^c)$-quasiconvex on D if and only if for any distinct points $x', x'' \in D$ and $x^0 \in]x', x''[$ we have

$$f(x^0) - f(x') \in K_1 \Rightarrow f(x'') - f(x') \in K_2.$$

Proposition 9 *Let $K_2 \subseteq K \subseteq K_1$ be nonempty subsets of Y. Assume that D is convex and let $f : D \to Y$. The following assertions hold:*

1° *If function f is semistrictly (K)-quasiconvex at x^0, then it is also semistrictly (K_2, K_1)-quasiconvex at x^0.*

2° *If function f is semistrictly (K_1, K_2)-quasiconvex at x^0, then it is semistrictly (K)-quasiconvex at x^0, hence semistrictly (K_2, K_1)-quasiconvex at x^0.*

3° *If f is explicitly (K_1, K_2)-quasiconvex at x^0, then it is explicitly (K)-quasiconvex at x^0, hence also explicitly (K_2, K_1)-quasiconvex at x^0.*

Proof 1° and 2° follow from Lemma 3, while the proof of 3° is based on the inclusions $-K_1^c \subseteq -K^c \subseteq -K_2^c$. □

Theorem 2 *Let $f : D \to Y$ be a function defined on the nonempty convex set D. Assume that one of the following three conditions is fulfilled:*

(A1) *f is explicitly (K_1, K_2)-quasiconvex and semistrictly (K_2)-quasiconvex on D;*

(A2) *$K_2 \subseteq K_1$ and f is explicitly (K_1, K_2)-quasiconvex on D;*

(A3) *$K_1 \subseteq K_2$ and f is explicitly K_2-quasiconvex on D.*

Then, for any distinct points $x', x'' \in D$ and $x^0 \in]x', x''[$ such that $f(x^0) - f(x') \in K_1$, the following two relations hold:

$$f(x^0) - f(u) \in K_2 \quad \text{for all} \quad u \in]x', x^0[;$$
$$f(x^0) - f(v) \in -K_2 \quad \text{for all} \quad v \in]x^0, x''[.$$

Proof First notice that (A2) ⇒ (A1) and (A3) ⇒ (A1) by Proposition 9. Therefore it suffices to assume that (A1) holds.

Let $x', x'' \in D$, $x' \neq x''$, and let $x^0 \in]x', x''[$ such that $f(x^0) - f(x') \in K_1$. Consider $u \in]x', x^0[$ and $v \in]x^0, x''[$. The relation $f(x^0) - f(u) \in K_2$ follows by the semistrict (K_1, K_2)-quasiconvexity of f at x^0. We then prove that

$$f(x') - f(v) \in -K_2. \tag{4}$$

Suppose to the contrary that this is not true, i.e., $f(x') - f(v) \in -K_2^c$. Since $x^0 \in]x', v[$ and f is semistrictly $(-K_2^c, -K_1^c)$-quasiconvex at x', it follows that $f(x') - f(x^0) \in -K_1^c$. Equivalently, this writes as $f(x^0) - f(x') \in K_1^c$, a contradiction. Hence, (4) holds, which means that $f(v) - f(x') \in K_2$. As function f is semistrictly (K_2)-quasiconvex at v and $x^0 \in]x', v[$, we conclude that $f(v) - f(x^0) \in K_2$, i.e., $f(x^0) - f(v) \in -K_2$. □

Corollary 1 *Let $f : D \to Y$ be a function defined on the nonempty convex set D. Assume that one of the following three conditions is fulfilled:*

(B1) *f is explicitly (K_1, K_2)-quasiconvex and semistrictly $(-K_1{}^c)$-quasiconvex on D;*

(B2) *$K_2 \subseteq K_1$ and f is explicitly (K_1, K_2)-quasiconvex on D;*

(B3) *$K_1 \subseteq K_2$ and f is explicitly (K_1)-quasiconvex on D.*

Then, for any distinct points $x', x'' \in D$ and $x^0 \in]x', x''[$ with $f(x^0) - f(x') \in -K_2^c$, the following two relations hold:

$$f(x^0) - f(u) \in -K_1^c \ \text{ for all } \ u \in]x', x^0[\,;$$
$$f(x^0) - f(v) \in K_1^c \ \text{ for all } \ v \in]x^0, x''[.$$

Proof Follows by Theorem 2, for the sets $-K_2^c$ and $-K_1^c$ in the role K_1 and K_2, in view of Remark 14. □

Corollary 2 *Assume that D is convex. If $f : D \to Y$ is explicitly (K)-quasiconvex on D, then for any distinct points $x', x'' \in D$ and $x^0 \in]x', x''[$ the following hold:*

1° *If $f(x^0) - f(x') \in K$, then we have*

$$f(x^0) - f(u) \in K \ \text{ for all } \ u \in]x', x^0[\,;$$
$$f(x^0) - f(v) \in -K \ \text{ for all } \ v \in]x^0, x''[.$$

2° *If $f(x^0) - f(x') \in -K^c$, then we have*

$$f(x^0) - f(u) \in -K^c \ \text{ for all } \ u \in]x', x^0[\,;$$
$$f(x^0) - f(v) \in K^c \ \text{ for all } \ v \in]x^0, x''[.$$

Proof Letting $K_1 = K_2 = K$, assertion 1° follows by Theorem 2, while assertion 2° follows by Corollary 1. □

Remark 16 Consider the particular framework where $Y = \mathbb{R}$ and $K = \mathbb{R}_+^*$. Let $f : D \to \mathbb{R}$ be explicitly quasiconvex and let x', x^0, x'' be distinct points in D with $x^0 \in]x', x''[$. For any $u \in]x', x^0[$ and $v \in]x^0, x''[$ the following implications hold

$$f(x') < f(x^0) \Rightarrow f(u) < f(x^0) < f(v),$$
$$f(x') = f(x^0) \Rightarrow f(u) \le f(x^0) \le f(v),$$

according to assertions 1° and 2° in Corollary 2, respectively.

3.2 Unidirectional (K_1, K_2)-quasiconvex vector functions

Definition 8 We say that function $f : D \to Y$ is *unidirectional* (K_1, K_2)*-quasiconvex at a point* $x^0 \in D$ if for any $x', x'' \in D \setminus \{x^0\}$ with $x^0 \in]x', x''[$, $f(x^0) - f(x') \in K_1$ entails $f(x^0) - f(x'') \in K_2$. If f satisfies this property for all $x^0 \in D$, then f is called *unidirectional* (K_1, K_2)*-quasiconvex on* D.

Remark 17 a) f is unidirectional (K_1, K_2)-quasiconvex at x^0 if and only if it is unidirectional (K_2^c, K_1^c)-quasiconvex at x^0. Indeed, for all distinct points $x', x'' \in D$ with $x^0 \in]x', x''[$, the implication $\left[f(x^0) - f(x') \in K_1 \Rightarrow f(x^0) - f(x'') \in K_2\right]$ is equivalent to the following one: $\left[f(x^0) - f(x'') \in K_2^c \Rightarrow f(x^0) - f(x') \in K_1^c\right]$.

b) f is unidirectional (K_1, K_2)-quasiconvex on D if and only if for any distinct points $x', x'' \in D$ and $x^0 \in]x', x''[$ we have

$$f(x^0) - f(x') \in K_1 \Rightarrow f(x^0) - f(x'') \in K_2.$$

Theorem 3 *Assume that D is convex and $K_2 \subseteq K_1$. If $f : D \to Y$ is explicitly (K_1, K_2)-quasiconvex on D, then it is unidirectional $(K_1, -K_2)$-quasiconvex on D.*

Proof Follows from Theorem 2 under the assumption (A2). Indeed, let $x^0 \in D$. Then, for any $x', x'' \in D \setminus \{x^0\}$ such that $x^0 \in]x', x''[$ and $f(x^0) - f(x') \in K_1$ we have $f(x^0) - f(v) \in -K_2$, for all $v \in]x^0, x'']$. In particular, for $v = x''$ we infer $f(x^0) - f(x'') \in -K_2$. □

Theorem 4 *Assume that D is convex and $K_2 + K_1 \subseteq K_2$. If a function $f : D \to Y$ is unidirectional $(K_1, -K_2)$-quasiconvex on D, then it is semistrictly $(-K_2^c, -K_1^c)$-quasiconvex on D. Also, if f is both semistrictly (K_1, K_2)-quasiconvex and unidirectional $(K_1, -K_2)$-quasiconvex on D, then it is explicitly (K_1, K_2)-quasiconvex on D.*

Proof Let $x', x'' \in D$ be distinct points and $x^0 \in]x', x''[$ with $f(x^0) - f(x') \in K_1$. By Remark 15, we just have to prove that $f(x'') - f(x') \in K_2$. To this end, in view of Remark 17 b), by the unidirectional $(K_1, -K_2)$-quasiconvexity of f at x^0, one obtains $f(x'') - f(x^0) \in K_2$. Since $K_2 + K_1 \subseteq K_2$, we deduce that

$$f(x'') - f(x') = \left[f(x'') - f(x^0)\right] + \left[f(x^0) - f(x')\right] \in K_2,$$

hence f is semistrictly $(-K_2^c, -K_1^c)$-quasiconvex. If in addition, f is also semistrictly (K_1, K_2)-quasiconvex, then it is explicitly (K_1, K_2)-quasiconvex. □

3.3 Unidirectional (K)-quasiconvex vector functions

Definition 9 A function $f : D \to Y$ is *unidirectional (K)-quasiconvex at a point* $x^0 \in D$ if it is unidirectional $(K, -K)$-quasiconvex at x^0, i.e., for any distinct points $x', x'' \in D$ such that $x^0 \in]x', x''[$, $f(x^0) - f(x') \in K$ entails $f(x^0) - f(x'') \in -K$. We call f *unidirectional (K)-quasiconvex on D* if this property holds for all $x^0 \in D$.

Theorem 5 *A function $f : D \to Y$ is unidirectional (K)-quasiconvex at x^0 if and only if it is unidirectional $(-K^c)$-quasiconvex at x^0.*

Proof Indeed, for any distinct points $x', x'' \in D$ with $x^0 \in]x', x''[$, the implication

$$f(x^0) - f(x') \in K \Rightarrow f(x^0) - f(x'') \in -K$$

is equivalent to the following one:

$$f(x^0) - f(x'') \in -K^c \Rightarrow f(x^0) - f(x') \in -(-K^c),$$

hence the conclusion is straightforward. □

Proposition 10 *Assume that D is convex and consider a function $f : D \to Y$. The following characterizations hold:*

1° *f is semistrictly (K)-quasiconvex on D if and only if for all distinct points $x', x'' \in D$ and $x^0 \in]x', x''[$ we have*

$$f(x') - f(x'') \in K \Rightarrow f(x') - f(x^0) \in K.$$

2° *f is semistrictly $(-K^c)$-quasiconvex on D if and only if for any distinct points $x', x'' \in D$ and $x^0 \in]x', x''[$ we have*

$$f(x^0) - f(x') \in K \Rightarrow f(x'') - f(x') \in K.$$

3° *f is unidirectional (K)-quasiconvex on D if and only if for any distinct points $x', x'' \in D$ and $x^0 \in]x', x''[$ we have*

$$f(x^0) - f(x') \in K \Rightarrow f(x'') - f(x^0) \in K.$$

Proof Assertion 1° actually states that f is semistrictly (K)-quasiconvex on D if and only if it is semistrictly (K)-quasiconvex at any point $x' \in D$.

In order to prove 2° one can use 1° for $-K^c$ in the role of K and thereafter the logical equivalence $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$.

Assertion 3° follows from the definition. □

Corollary 3 *Assume that D is convex. If $f : D \to Y$ is explicitly (K)-quasiconvex on D, then it is unidirectional (K)-quasiconvex on D.*

Proof Letting $K_1 = K_2 := K$, the conclusion follows by Theorem 3, in view of Example 11 a). □

Remark 18 The converse of Corollary 3 is false, as the following example shows.

Example 12 Let $f : D = [0, 1] \to \mathbb{R}^2$ be the function defined in Example 7. Consider the ordering cone $K = \mathbb{R}^2_+$. Observe that f is unidirectional (K)-quasiconvex at any $x^0 \in D$. Indeed, if $x^0 \in \{0, 1\}$, then there are no distinct points $x', x'' \in D$ such that $x^0 \in]x', x''[$. Otherwise, if $x^0 \in]0, 1[$, then there is no $x' \in D \setminus \{x^0\}$ satisfying $f(x^0) - f(x') \in K$, hence the property holds trivially. However, f is not even semistrictly (K)-quasiconvex. Indeed, for $x^0 = 1$, $x' = 0$ and $x = 1/2 \in]x', x^0[$ we have $f(x^0) - f(x') = (0, 1) \in K$, but $f(x^0) - f(x) = (-1/2, 3/2) \notin K$.

The next result gives a characterization of real-valued explicitly quasiconvex functions in terms of unidirectional (K)-quasiconvexity.

Theorem 6 *Assume that D is convex and let $f : D \to \mathbb{R}$ be a function. The following assertions are equivalent:*

1° *f is explicitly quasiconvex.*

2° *f is unidirectional $(\mathbb{R}^*_+)$-quasiconvex on D.*

3° *f is unidirectional $(\mathbb{R}_+)$-quasiconvex on D.*

Proof First observe that 1° means that f is explicitly $(\mathbb{R}^*_+)$-quasiconvex on D, in view of Example 11 b). Therefore, 1° implies 2° by Corollary 3. The equivalence between 2° and 3° holds in view of Theorem 5.

To prove that 2° implies 1°, assume that f is unidirectional $(\mathbb{R}^*_+)$-quasiconvex on D. Suppose by the contrary that f is not quasiconvex. Then, by Remark 5 a), there exist distinct points $x', x'' \in D$ with $f(x') \leq f(x'')$ and $x^0 \in]x', x''[$, such that $f(x^0) > f(x'')$. Since f is unidirectional $(\mathbb{R}^*_+)$-quasiconvex at x^0, it follows that $f(x^0) < f(x')$. We deduce that $f(x^0) \leq f(x'')$, a contradiction.

Now suppose by the contrary that f is not semistrictly quasiconvex. Then, by Remark 5 b), there are $x', x'' \in D$ with $f(x') < f(x'')$ and $x^0 \in]x', x''[$ such that $f(x^0) \geq f(x'')$. Since 2° and 3° are equivalent, f is unidirectional $(\mathbb{R}_+)$-quasiconvex at x^0, hence $f(x^0) \leq f(x')$. We obtain $f(x') \geq f(x^0) \geq f(x'')$, a contradiction. □

Remark 19 There exist functions which are semistrictly/explicitly/unidirectional (K)-quasiconvex at certain points, but not at every point of their domain, as shown by the following example.

Example 13 Let $Y = \mathbb{R}$, $K = \mathbb{R}^*_+$ and $f : D = \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

It is easy to check that function f is semistrictly $(\mathbb{R}^*_+)$-quasiconvex, even explicitly $(\mathbb{R}^*_+)$-quasiconvex, and unidirectional $(\mathbb{R}^*_+)$-quasiconvex at the point $x^0 = 2$.

However, f has none of these properties at another point, namely $x^0 = -1/2$. Indeed, letting $x' = 0$ we have $f(x^0) - f(x') \in \mathbb{R}^*_+$, i.e., $f(x^0) > f(x')$. Clearly, for $x = -1/4 \in]x^0, x'[$ one obtains $f(x^0) - f(x) = -1/4 \notin \mathbb{R}^*_+$, hence f is not semistrictly (hence not explicitly) $(\mathbb{R}^*_+)$-quasiconvex at $x^0 = -1/2$. By setting $x'' = -2$, one has $x^0 \in]x'', x'[$ and $f(x^0) - f(x'') = 3/2 \notin -\mathbb{R}^*_+$, therefore f is not unidirectional $(\mathbb{R}^*_+)$-quasiconvex at $x^0 = -1/2$.

Remark 20 The explicit (K)-quasiconvexity and unidirectional (K)-quasiconvexity at a given point do not imply each other. Indeed, let $Y = \mathbb{R}$, $K = \mathbb{R}^*_+$ and $x^0 = 0$. The function f defined in Example 5 is explicitly (K)-quasiconvex at x^0, but not unidirectional (K)-quasiconvex at x^0. Also, the function f in Example 13 is unidirectional (K)-quasiconvex at x^0, but not explicitly (K)-quasiconvex at x^0.

Corollary 4 *Assume that D is convex and $K + K \subseteq K$. If function $f : D \to Y$ is unidirectional (K)-quasiconvex on D, then it is semistrictly $(-K^c)$-quasiconvex on D. Consequently, if function f is both semistrictly (K)-quasiconvex and unidirectional (K)-quasiconvex on D, then it is explicitly (K)-quasiconvex on D.*

Proof Follows by Theorem 4, for $K_1 = K_2 = K$. □

Remark 21 The notions of unidirectional (K)-quasiconvexity on D and semistrict $(-K^c)$-quasiconvexity on D are not identical, as the following example shows.

Example 14 Let $D = [0, 1] \subseteq X = \mathbb{R}$, let $K = [0, 1[\subseteq Y = \mathbb{R}$ and let $f : [0, 1] \to \mathbb{R}$ be defined for all $x \in [0, 1]$ by

$$f(x) = x.$$

Function f is unidirectional (K)-quasiconvex on D, by Proposition 10 (3°). Indeed, for any distinct points $x', x'' \in D$ and $x^0 \in]x', x''[$ such $f(x^0) - f(x') \in K$, we have $0 \leq x' < x^0 < x'' \leq 1$, hence $f(x'') - f(x^0) \in K$.

Moreover, function f is semistrictly (K)-quasiconvex on D. This follows by Proposition 10 (1°), since for any distinct points $x', x'' \in D$ and $x^0 \in]x', x''[$ such that $f(x') - f(x'') \in K$, we have $x'' < x^0 < x'$, hence $f(x') - f(x^0) \in K$.

However, function f is not semistrictly $(-K^c)$-quasiconvex on D, as shown by Proposition 10 (2°). Indeed, letting $x' = 0$, $x'' = 1$ and any $x^0 \in]x', x''[$ we have $f(x^0) - f(x') \in K$, but $f(x'') - f(x') = 1 \notin K$.

Notice that Corollary 4 does not apply in this example, because $K + K \nsubseteq K$.

3.4 Bidirectional (K_1, K_2)-quasiconvex vector functions

Definition 10 We say that a function $f : D \to Y$ is *bidirectional* (K_1, K_2)*-quasiconvex at a point* $x^0 \in D$ if for any $x' \in D \setminus \{x^0\}$ with $f(x^0) - f(x') \in K_1$ there exists $x'' \in D \setminus \{x^0\}$ such that $[x^0, x''] \subseteq D$ and $f(x^0) - f(x) \in K_2$ for all $x \in]x^0, x'']$. If f satisfies this property for every $x^0 \in D$, then f is said to be *bidirectional* (K_1, K_2)*-quasiconvex on* D.

Remark 22 Letting $K_1 = K_2 := K$, the bidirectional (K_1, K_2)-quasiconvexity at a point $x^0 \in D$ recovers the K-quasiconvexity at x^0 in the sense of Definition 5.

Theorem 7 *If* $f : D \to Y$ *is unidirectional* (K_1, K_2)*-quasiconvex at* $x^0 \in \operatorname{icr} D$*, then* f *is bidirectional* (K_1, K_2)*-quasiconvex at* x^0*.*

Proof Let $x' \in D \setminus \{x^0\}$ be such that $f(x^0) - f(x') \in K_1$. Since $x^0 \in \operatorname{icr} D$ and $x^0 - x' \in D - D \subseteq \operatorname{span}(D - D)$, there is $\delta > 0$ such that $[x^0, x^0 + \delta(x^0 - x')] \subseteq D$. Letting $x'' := x^0 + \delta(x^0 - x')$, we get $x'' \in D \setminus \{x^0\}$ and $[x^0, x''] \subseteq D$. Moreover, for any $x \in]x^0, x'']$ there is $t \in]0, 1]$ such that

$$x = x^0 + t(x'' - x^0) = x^0 + t\delta(x^0 - x'),$$

hence $x^0 = x' + (1 + t\delta)^{-1}(x - x') \in]x', x[$. As f is unidirectional (K_1, K_2)-quasiconvex at x^0, we infer that $f(x^0) - f(x) \in K_2$, which ends the proof. □

Theorem 8 *Assume that* D *is convex. If function* $f : D \to Y$ *is semistrictly* (K_1, K_2)*-quasiconvex at* $x^0 \in D$*, then it is bidirectional* (K_1, K_2)*-quasiconvex at* x^0*.*

Proof Follows by Definition 6 and Definition 10 (by choosing $x'' = x'$). □

Corollary 5 *Assume that* D *is convex. If* $f : D \to Y$ *is semistrictly* (K)*-quasiconvex at* $x^0 \in D$*, then it is* K*-quasiconvex at* x^0*.*

Proof It is a direct consequence of Theorem 8, in view of Remark 22. □

Remark 23 In particular, Corollary 5 shows that if f is semistrictly (K)-quasiconvex on D, then it is K-quasiconvex on D. However, the converse is not true, as shown by the following example.

Example 15 Consider $Y = \mathbb{R}$, $K = \mathbb{R}_+^*$ and define $f : D = \mathbb{R} \to Y$ by

$$f(x) = \begin{cases} 1 \text{ if } x \in \{0, 1\}, \\ 0 \text{ if } x \in \mathbb{R} \setminus \{0, 1\}. \end{cases}$$

Function f is $\mathbb{R}_+^*$-quasiconvex at every point $x^0 \in D$, but it is not semistrictly $(\mathbb{R}_+^*)$-quasiconvex on D.

Indeed, if x^0 and x' are two distinct points chosen such that $f(x^0) - f(x') \in \mathbb{R}_+^*$, then $x^0 \in \{0, 1\}$ and $x' \notin \{0, 1\}$. By choosing any $x'' \in [x^0 - 1/2, x^0 + 1/2]$, we have $f(x^0) - f(x) \in \mathbb{R}_+^*$ for all $x \in]x^0, x'']$. Hence, f is $\mathbb{R}_+^*$-quasiconvex at x^0.

However, f is not semistrictly $(\mathbb{R}_+^*)$-quasiconvex at $x^0 = 0$, since for $x' = 2$ and $x = 1$, we have $f(x^0) - f(x') \in \mathbb{R}_+^*$, but $f(x^0) - f(x) \notin \mathbb{R}_+^*$.

4 Local-global extremality properties

Throughout this section we assume that X and Y are real topological linear spaces and $f : D \to Y$ is a function defined on a nonempty set $D \subseteq X$.

4.1 A general local-global extremality principle

Let K_1 and K_2 be two proper subsets of Y. We first present a technical result.

Lemma 4 *Assume that $x^0 \in D$ is a global K_1-extremal point of f. Then we have:*
 1° *f is bidirectional (K_1, K_2)-quasiconvex at x^0.*
 2° *f is unidirectional (K_1, K_2)-quasiconvex at x^0.*
Moreover, when D is convex, we also have:
 3° *f is semistrictly (K_1, K_2)-quasiconvex at x^0.*

Proof Since x^0 is a global K_1-extremal point of f, i.e., $x^0 \in K_1\text{-Ext}(f \mid D)$, no point $x' \in D \setminus \{x^0\}$ satisfies $f(x^0) - f(x') \in K_1$, according to Definition 1. Therefore, assertions 1°, 2° and 3° follow by Definitions 10, 8 and 6, respectively. □

We now state the main result of this paper. This can be seen as a general local-global extremality principle, which encompasses all "local min - global min" type properties (when $K_1 = K_2$) and "local max - global min" type properties (when $K_1 = -K_2$), as we will see in Section 4.2.

Theorem 9 *If $x^0 \in D$ is an algebraic local K_2-extremal point of f, then the following assertions are equivalent:*
 1° *x^0 is a global K_1-extremal point of f.*
 2° *f is bidirectional (K_1, K_2)-quasiconvex at x^0.*

Proof Let $U \in \mathcal{U}(x^0)$ be such that $x^0 \in K_2\text{-Ext}(f \mid U \cap D)$.

If x^0 is a global K_1-extremal point of f, then function f is bidirectional (K_1, K_2)-quasiconvex at x^0, according to Lemma 4.

Conversely, assume that function f is bidirectional (K_1, K_2)-quasiconvex at x^0 and suppose to the contrary that $x^0 \notin K_1\text{-Ext}(f \mid D)$. Then there is $x' \in D \setminus \{x^0\}$ with $f(x^0) - f(x') \in K_1$. Since f is bidirectional (K_1, K_2)-quasiconvex at x^0, we can find $x'' \in D \setminus \{x^0\}$ such that $[x^0, x''] \subseteq D$ and

$$f(x^0) - f(x) \in K_2, \ \forall\, x \in]x^0, x'']. \tag{5}$$

As $x^0 \in \operatorname{cor} U$, there is $\delta > 0$ such that $[x^0,\, x^0 + \delta(x'' - x^0)] \subseteq U$. For $t = \min\{\delta, 1\}$ we obtain a point $x^* := x^0 + t(x'' - x^0) \in U \cap]x^0, x'']$, which in view of (5) satisfies

$$f(x^0) - f(x^*) \in K_2.$$

Since $x^* \in (U \cap D) \setminus \{x^0\}$, the above relation entails $x^0 \notin K_2\text{-Ext}(f \mid U \cap D)$, a contradiction. Thus $x^0 \in K_1\text{-Ext}(f \mid D)$. □

Theorem 10 *Let $x^0 \in D$ be an algebraic local K_2-extremal point of f. If $x^0 \in \operatorname{icr} D$, then the following assertions are equivalent:*
 1° *x^0 is a global K_1-extremal point of f.*
 2° *f is unidirectional (K_1, K_2)-quasiconvex at x^0.*

Proof If x^0 is a global K_1-extremal point of f, then function f is unidirectional (K_1, K_2)-quasiconvex at x^0, according to Lemma 4.

The converse follows by Theorems 7 and 9. □

Theorem 11 *Assume that D is convex. If $x^0 \in D$ is an algebraic local K_2-extremal point of f, then the following assertions are equivalent:*

1° *x^0 is a global K_1-extremal point of f.*

2° *f is semistrictly (K_1, K_2)-quasiconvex at x^0.*

Proof If x^0 is a global K_1-extremal point of f, then function f is semistrictly (K_1, K_2)-quasiconvex at x^0, by Lemma 4.

The converse follows by Theorem 8 and Theorem 9. □

Remark 24 Theorems 9, 10 and 11 show that under appropriate hypotheses, if function f possesses an algebraic local K_2-extremal point $x^0 \in D$, then x^0 is also a global (hence local) K_1-extremal point of f. In particular, when

$$K_1 \cup K_2 = Y \setminus \{0_Y\},$$

we can conclude by Lemma 1 that f is constant on $U \cap D$ for some set $U \in \mathcal{U}(x^0)$.

4.2 Local-global properties in unified vector optimization

In order to apply the general results from the previous section to unified vector optimization problems, in what follows we consider a proper subset K of Y.

We first present some "local min - global min" type properties for unified vector optimization.

Theorem 12 *If $x^0 \in D$ is an algebraic local K-extremal point of f, then the following assertions are equivalent:*

1° *x^0 is a global K-extremal point of f.*

2° *f is K-quasiconvex at x^0.*

Proof Follows by Theorem 9, for $K_1 = K_2 = K$, in view of Remark 22. □

Theorem 13 *Assume that D is convex. If $x^0 \in D$ is an algebraic local K-extremal point of f, then the following assertions are equivalent:*

1° *x^0 is a global K-extremal point of f.*

2° *f is semistrictly (K)-quasiconvex at x^0.*

Proof Follows by Theorem 11, for $K_1 = K_2 = K$, in view of Remark 12. □

Remark 25 Theorem 13 extends a well-known result obtained by Flores-Bazán and Hernández, reformulated by us in Proposition 4.

Next we present "local max - global min" type properties for unified vector optimization.

Theorem 14 *If $x^0 \in D$ is an algebraic local $-K$-extremal point of function f, such that $x^0 \in \operatorname{icr} D$, then the following assertions are equivalent:*

1° *x^0 is a global K-extremal point of f.*

2° *f is unidirectional (K)-quasiconvex at x^0.*

Proof Assume that x^0 is a global K-extremal point of f. By Lemma 4, applied for $K_1 = K$ and $K_2 = -K$, it follows that f is unidirectional $(K, -K)$-quasiconvex at x^0, i.e., f is unidirectional (K)-quasiconvex at x^0.

The converse can be deduced by Theorems 7 and 9. □

Theorem 15 *Assume that D is convex. If function f is explicitly (K)-quasiconvex on D, then its algebraic local $-K$-extremal points, located in the intrinsic core of D, are global K-extremal points.*

Proof Follows by Theorem 14 in view of Corollary 3. □

Remark 26 The "local max - global min" type properties given by Theorems 14 and 15 show that, under the corresponding generalized convexity assumptions, if function f has an algebraic local $-K$-extremal point $x^0 \in \operatorname{icr} D$, then x^0 is also a global (hence local) K-extremal point of f. In particular, when

$$K \cup (-K) = Y \setminus \{0_Y\}, \text{ i.e., } K^c \cap -K^c = \{0_Y\}, \tag{6}$$

by applying Lemma 1 for $K_1 = K$ and $K_2 = -K$, we conclude that there is a set $U \in \mathcal{U}(x^0)$ such that f is constant on $U \cap D$.

4.3 Local-global properties in vector optimization

Throughout this section we assume that

$$C \subseteq Y \text{ is a solid convex cone, such that } C \neq \ell(C). \tag{7}$$

Before stating our local-global properties for vector optimization, we present a technical result.

Lemma 5 *Under the assumption* (7) *the following assertions hold:*

1° $0_Y \notin \operatorname{cor} C$.

2° $\operatorname{cor} C \subseteq C \setminus \ell(C) \subseteq -C^c$.

3° *There exists a linear functional $\ell : Y \to \mathbb{R}$ such that $\ell(x) > 0$ for all $x \in \operatorname{cor} C$.*

Proof Assertion 1° holds as otherwise we have $C = Y$ (see, e.g., Jahn [20, p. 12]), which contradicts (7).

In order to prove assertion 2°, notice first that $C \setminus \ell(C) = C \setminus (-C)$. Now suppose by the contrary that $\operatorname{cor} C \not\subseteq C \setminus \ell(C)$, i.e., there exists $x \in (\operatorname{cor} C) \cap (-C)$. Then we should have (see, e.g., Jahn [20, Lemma 1.12])

$$0_Y = x + (-x) \in \operatorname{cor} C + C = \operatorname{cor} C,$$

which contradicts 1°. Thus the first inclusion in 2° holds true. The second one is obvious, since $C \setminus \ell(C) = C \setminus (-C) \subseteq Y \setminus (-C) = (-C)^c = -C^c$.

In what concerns assertion 3°, note that $\operatorname{cor} C = \operatorname{icr} C$, in view of Remark 1 b), since C is solid. Therefore assertion 1° reads as $0_Y \in (\operatorname{icr} C)^c$. By applying Proposition 1 for $E = Y$, $A = C$ and $v = 0_Y$, we infer the existence of a linear functional $\ell : Y \to \mathbb{R}$ such that $\ell(x) > 0$ for all $x \in \operatorname{cor} C$. □

By particularizing $K \in \{-C^c, C\setminus\ell(C), \operatorname{cor} C\}$ in Theorems 12 and 13, we derive the following two results, representing "local min - global min" type properties for vector optimization.

Corollary 6 *Let $x^0 \in D$. The following assertions hold:*

1° *If x^0 is an algebraic local ideal C-minimal point of f, then x^0 is a global ideal C-minimal point of f if and only if f is $-C^c$-quasiconvex at x^0.*

2° *If x^0 is an algebraic local C-minimal point of f, then x^0 is a global C-minimal point of f if and only if f is $C \setminus \ell(C)$-quasiconvex at x^0.*

3° *If x^0 is an algebraic local weakly C-minimal point of f, then x^0 is a global weakly C-minimal point of f if and only if f is $\operatorname{cor} C$-quasiconvex at x^0.*

Corollary 7 *Assume that D is convex and let $x^0 \in D$. The following assertions hold:*

1° *If x^0 is an algebraic local ideal C-minimal point of f, then x^0 is a global ideal C-minimal point of f if and only if f is semistrictly $(-C^c)$-quasiconvex at x^0.*

2° *If x^0 is an algebraic local C-minimal point of f, then x^0 is a global C-minimal point of f if and only if f is semistrictly $(C \setminus \ell(C))$-quasiconvex at x^0.*

3° *If x^0 is an algebraic local weakly C-minimal point of f, then x^0 is a global weakly C-minimal point of f if and only if f is semistrictly $(\operatorname{cor} C)$-quasiconvex at x^0.*

Remark 27 Assertion 1° of Corollary 6 extends Proposition 7, while assertions 2° and 3° recover two well-known results obtained by Jahn and Sachs, reformulated by us in Propositions 5 and 6.

The next three results represent "local max - global min" type properties for vector optimization, derived from Theorem 15 when $K \in \{-C^c, C \setminus \ell(C), \operatorname{cor} C\}$. Moreover, they show that, in contrast to scalar optimization (where a point is both local minimum and local maximum if and only if the objective function is locally constant), in vector optimization condition (6) pointed out in Remark 26 is rather restrictive, being fulfilled only for specific classes of convex cones.

Corollary 8 *Assume that f is explicitly $(-C^c)$-quasiconvex on the convex set D. If $x^0 \in D$ is an algebraic local ideal C-maximal point of f, such that $x^0 \in \operatorname{icr} D$, then x^0 is a global ideal C-minimal point. In addition, when C is pointed, i.e.,*

$$(-C^c) \cup C^c = Y \setminus \{0_Y\}, \tag{8}$$

there exists $U \in \mathcal{U}(x^0)$ such that f is constant on $U \cap D$.

Corollary 9 *Assume that f is explicitly $(C \setminus \ell(C))$-quasiconvex on the convex set D. If $x^0 \in D$ is an algebraic local C-maximal point of f, such that $x^0 \in \operatorname{icr} D$, then x^0 is a global C-minimal point. In addition, when C satisfies the property*

$$(C \setminus \ell(C)) \cup (-C \setminus \ell(C)) = Y \setminus \{0_Y\}, \tag{9}$$

there exists $U \in \mathcal{U}(x^0)$ such that f is constant on $U \cap D$.

Corollary 10 *Assume that f is explicitly $(\operatorname{cor} C)$-quasiconvex on the convex set D. If $x^0 \in D$ is an algebraic local weakly C-maximal point of f, such that $x^0 \in \operatorname{icr} D$, then x^0 is a global weakly C-minimal point. In addition, when C satisfies the property*

$$(\operatorname{cor} C) \cup (-\operatorname{cor} C) = Y \setminus \{0_Y\}, \tag{10}$$

there exists $U \in \mathcal{U}(x^0)$ such that f is constant on $U \cap D$.

Remark 28 a) In view of Lemma 5 (2°), the following implications hold:

$$(10) \Rightarrow (9) \Rightarrow (8).$$

Therefore, the least restrictive condition imposed on the cone in Corollaries 8, 9 and 10 is (8), which means that C is pointed.

b) It is easily seen that condition (9) implies

$$C \cup -C = Y, \tag{11}$$

which means that the ordering induced by C on Y is total, i.e., any two points of Y are comparable. Conversely, if C is pointed and satisfies (11), then (9) is fulfilled. This is the case of the lexicographic cone of $\mathbb{R}^m$ defined as the set of all vectors whose first nonzero coordinate (if any) is positive:

$$C_{\text{lex}} := \{0\} \cup \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid \exists\, i \in I : x_i > 0,\ \nexists\, j \in I, j < i : x_j \neq 0\},$$

where $I := \{1, \ldots, m\}$ (see, e.g., Popovici [26]). Notice that (9) and (11) are not equivalent in the absence of the pointedness assumption. For example, consider $C = \text{cl}(C_{\text{lex}}) = \mathbb{R}_+ \times \mathbb{R} \times \cdots \times \mathbb{R}$ in $Y = \mathbb{R}^m$, with $m \geq 2$.

c) The most restrictive condition on C is (10), which is satisfied only when $\dim Y = 1$. Actually, we can prove that for any given point $e \in \operatorname{cor} C$, we have

$$Y = \operatorname{span}\{e\}.$$

Indeed, by Lemma 5 (3°) there is a linear functional $\ell : Y \to \mathbb{R}$ with

$$\ell(x) > 0,\ \forall\, x \in \operatorname{cor} C. \tag{12}$$

Consider any $v \in \operatorname{cor} C$. Since ℓ is linear, the function $\varphi : [0,1] \to \mathbb{R}$, defined by

$$\varphi(t) := \ell((1-t)e + t(-v)),\ \forall\, t \in [0,1],$$

is affine, hence continuous. We also have $\varphi(0) = \ell(e) > 0$ and $\varphi(1) = -\ell(v) < 0$ by (12). By the Darboux property we infer the existence of $t_0 \in]0,1[$ such that $\varphi(t_0) = 0$. Consider the point

$$x_0 := (1-t_0)e + t_0(-v). \tag{13}$$

It is easily seen that $x_0 \in]e, -v[$ and $\ell(x_0) = \varphi(t_0) = 0$. By (12) it follows that $x_0 \notin (\operatorname{cor} C) \cup (-\operatorname{cor} C)$, which actually means that $x_0 = 0_Y$, in view of (10). Consequently, relation (13) shows that $v = \frac{1-t_0}{t_0} e \in \mathbb{R}_+^* \cdot e$. Since v was arbitrarily chosen in $\operatorname{cor} C$, we deduce that $\operatorname{cor} C \subseteq \mathbb{R}_+^* \cdot e$, hence $(\operatorname{cor} C) \cup (-\operatorname{cor} C) \subseteq \mathbb{R}^* \cdot e$. Finally, (10) yields $Y = \mathbb{R} \cdot e = \operatorname{span}\{e\}$.

We conclude this section with a result that summarizes “local min - global min” and “local max - global min” type properties for multicriteria optimization.

Corollary 11 *Assume that D is convex and let $f : D \to \mathbb{R}^m$ be a componentwise explicitly quasiconvex function. The following assertions hold:*

1° *If $x^0 \in D$ is an algebraic local ideal $\mathbb{R}^m_+$-minimal point of f (resp. an algebraic local $\mathbb{R}^m_+$-minimal point, algebraic local weakly $\mathbb{R}^m_+$-minimal point of f), then it is a global ideal $\mathbb{R}^m_+$-minimal point point of f (resp. a global $\mathbb{R}^m_+$-minimal point, global weakly $\mathbb{R}^m_+$-minimal point point of f).*

2° *If $x^0 \in D$ is an algebraic local ideal $\mathbb{R}^m_+$-maximal point of f (resp. an algebraic local $\mathbb{R}^m_+$-maximal point, algebraic local weakly $\mathbb{R}^m_+$-maximal point of f), such that $x^0 \in \operatorname{icr} D$, then it is a global ideal $\mathbb{R}^m_+$-minimal point point of f (resp. a global $\mathbb{R}^m_+$-minimal point, global weakly $\mathbb{R}^m_+$-minimal point of f). Actually, when x^0 is an algebraic local ideal $\mathbb{R}^m_+$-maximal point of f, there exists $U \in \mathcal{U}(x^0)$ such that f is constant on $U \cap D$.*

Proof 1° follows by Theorem 1 and Corollary 7, while 2° follows by Theorem 1 and Corollaries 8, 9 and 10, applied for $C = \mathbb{R}^m_+$. □

Remark 29 a) Corollary 11 (2°) extends Proposition 8.

b) When $m = 1$, Corollary 11 (2°) actually shows that every algebraic local maximum point $x^0 \in D$ of an explicitly quasiconvex function $f : D \to \mathbb{R}$, such that $x^0 \in \operatorname{icr} D$, is a global minimum point of f, hence there is $U \in \mathcal{U}(x^0)$ such that f is constant on $U \cap D$. Notice that for real-valued functions the three concepts of ideal maximality, maximality and weak maximality coincide.

c) In contrast to b), when $m \geq 2$ there exist componentwise explicitly quasiconvex (even linear) functions, which possess an algebraic local $\mathbb{R}^m_+$-maximal point $x^0 \in \operatorname{icr} D$, but are not constant on $U \cap D$ for any $U \in \mathcal{U}(x^0)$. For instance, the function $f : D = \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, -x)$ is componentwise explicitly quasiconvex and any $x^0 \in \mathbb{R} = \operatorname{icr} D$ is an algebraic local (even global) $\mathbb{R}^2_+$-maximal point of f. However, there is no $U \in \mathcal{U}(x^0)$ such that f is constant on $U \cap D$. This is because function f has no algebraic local ideal $\mathbb{R}^2_+$-maximal points.

d) Corollary 11 is relevant for multicriteria linear fractional optimization problems, which have many practical applications (see, e.g., Cambini and Martein [8], Göpfert *et al.* [16, Sect. 4.4]). Since the linear fractional functions as well as their opposites are explicitly quasiconvex, we can revert our "*local min - global min*" and "*local max - global min*" properties into new "*local max - global max*" and "*local min - global max*" type properties, as shown by us in [3, Sect. 5]. Following this approach, we can deduce that all $\mathbb{R}^m_+$-minimal points (resp. weakly $\mathbb{R}^m_+$-minimal points, ideal $\mathbb{R}^m_+$-minimal points) of a componentwise linear fractional function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, located in $\operatorname{icr} D$, coincide with the $\mathbb{R}^m_+$-maximal points (resp. weakly $\mathbb{R}^m_+$-maximal points, ideal $\mathbb{R}^m_+$-minimal points) of f, the objective function f being constant on D if and only if it possesses an ideal $\mathbb{R}^m_+$-minimal/maximal point in $\operatorname{icr} D$.

5 Conclusions

By introducing appropriate concepts of generalized convexity, we have established general local-global extremality properties for vector functions, with respect to two proper subsets of the outcome space ($K_1, K_2 \subseteq Y$). In particular, our Theorems 9, 10 and 11 are of special interest for unified vector optimization, since they encompass the "local min - global min" type properties (when $K_1 = K_2$), as well as the “local max - global min” type properties (when $K_1 = -K_2$). An interesting topic for further research would be to establish similar local-global extremality properties for set-valued functions with respect to variable ordering structures (see, e.g., Durea, Strugariu and Tammer [9], Eichfelder and Pilecka [10]-[11], and Köbis [22]).

Acknowledgements Nicolae Popovici's research was supported by a grant of the Romanian Ministry of Research and Innovation, CNCS-UEFISCDI, project number PN-III-P4-ID-PCE-2016-0190, within PNCDI III. The authors are grateful to the anonymous referees for the careful reading of the manuscript and especially for the constructive comments and suggestions.

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