The Asymptotic Form of the Sum $\sum_{i=0}^{n} i^p \binom{n+i}{i}$: Two Proofs

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Abstract

The asymptotic form of the sum $\sum_{i=0}^{n} i^p \binom{n+i}{i}$ is established in two quite different ways—by means of the longstanding Euler-Maclaurin summation formula, and then via a direct (and somewhat more contemporary) proof.

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1 Introduction

Suppose $p \geq 1$ is integral. In [1] properties of the sum $S_p(n) = \sum_{i=0}^{n} i^p \binom{n+i}{i}$ $= \sum_{i=1}^{n} i^p \binom{n+i}{i}$ were examined by the authors P.J.L. and E.J.F. In particular, the following result was established in a first principles type proof:

Theorem *Let $p \geq 1$ be integer. Then for n large $S_p(n) \sim 2n^p \binom{2n}{n}$.*

In this paper we offer two alternative—and evidently quite different—formulations for the asymptotic form of $S_p(n)$.

2 The Proofs

2.1 Proof I

2.1.1 Preamble

This utilises the so called Euler-Maclaurin summation formula whose derivation and/or application is to be found in texts such as those by de Bruijn [2, Sections 3.6-3.9, pp.40-46] and Knopp [3, Section 64, pp.518-535] (see also the shorter entries of, for instance, [4, Section 7.21, p.128] and [5, Section 5.9, (5.168b), p.331]), with an overview given by Apostol [6]. The version of the formula we employ originates with Euler (and an independent discovery by Maclaurin), and can be used in areas of numerical analysis, analytic number theory and asymptotic analysis, generalising also to the complex plane (as Darboux's formula). It is suitable for yielding the asymptotic form, for large n, of many sums of general type $\sum_{i=1}^{n} a_i(n)$ in which both the number of terms and the terms themselves may be dependent on n.

Briefly, if $n \geq m \geq 0$ are integers, real $x \in [m, n]$, then for any sufficiently 'smooth' (*i.e.*, sufficiently differentiable) function $f(x)$

$$\begin{aligned} I(m,n) &= \int_m^n f(x)\ dx \\ &\approx \frac{1}{2}f(m) + f(m+1) + \cdots + f(n-1) + \frac{1}{2}f(n) \\ &= S(m,n), \end{aligned} \tag{1}$$

say, by the well known approximating Trapezoidal Rule used in numerical analysis. Noting that the Bernoulli number sequence $\{B_n\}_0^\infty$ is generated by the exponential generating function

$$z/(e^z - 1) = \sum_{n=0}^{\infty} B_n z^n/n!, \tag{2}$$

the Euler-Maclaurin formula gives the difference between the integral and the sum as

$$S(m,n) - I(m,n) = \sum_{k=1}^{r} \frac{B_{2k}}{(2k)!}[f^{(2k-1)}(n) - f^{(2k-1)}(m)] + R_r(m,n) \quad (3)$$

in terms of (odd) derivatives d^{2k-1}/dx^{2k-1} of $f(x)$ and (even) Bernoulli numbers;[1] the remainder term is

$$R_r(m,n) = \frac{1}{(2r+1)!} \int_m^n f^{(2r+1)}(x) P_{2r+1}(x)\ dx \quad (4)$$

where, writing $B_n(x)$ for the usual general Bernoulli polynomial (with e.g.f. $ze^{xz}/(e^z-1) = \sum_{n=0}^{\infty} B_n(x) z^n/n!$, so that $B_n(0) = B_n$), $P_n(x) = B_n(x - \lfloor x \rfloor)$ is known as the corresponding *periodic* Bernoulli polynomial. Thus, since $\sum_{i=m}^{n} f(i) = S(m,n) + \frac{1}{2}[f(m) + f(n)]$, (3) can be rearranged in the familiar form

$$\begin{aligned} \sum_{i=m}^{n} f(i) &= I(m,n) + \frac{1}{2}[f(m) + f(n)] \\ &\quad + \sum_{k=1}^{r} \frac{B_{2k}}{(2k)!}[f^{(2k-1)}(n) - f^{(2k-1)}(m)] + R_r(m,n). \end{aligned} \quad (5)$$

For $p \geq 1$ we can write

$$S_p(n) = \sum_{i=1}^{n} i^p \binom{n+i}{i} = \sum_{i=1}^{n} \tau_i(n,p), \quad (6)$$

say, so that, with

$$f(x) = f(x;n,p) = \tau_x(n,p) = x^p \binom{n+x}{x} = \frac{x^p}{n!} \frac{\Gamma(x+n+1)}{\Gamma(x+1)}, \quad (7)$$

then (running out the right hand side sum of (5) (with r terms) as an infinite one under the (justified[2]) assumption that $\lim_{r\to\infty}\{R_r(1,n)\} = 0$) we work with the formula

$$S_p(n) = \sum_{i=1}^{n} f(i) = A_0(n;p) + A_1(n;p) + A_2(n;p) \quad (8)$$

[1]Excepting $B_1 = -1/2$, all odd Bernoulli numbers are zero, with $B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, *etc.* (see the resp. numerator and denominator On-Line Encyclopaedia of Integer Sequences A000367 and A002445).

[2]As a polynomial in x, $f(x) = x^p[x+n]_n/n!$ is degree $p+n$, with $d^k/dx^k\{f(x)\} = 0$ for $k > p+n$.

with constituent parts

$$
\begin{aligned}
A_0(n;p) &= \int_1^n f(x)\,dx, \\
A_1(n;p) &= \frac{1}{2}[f(1)+f(n)], \\
A_2(n;p) &= \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}[f^{(2k-1)}(n) - f^{(2k-1)}(1)], \qquad (9)
\end{aligned}
$$

seeking the large n asymptotic form of $S_p(n)$ which we denote by $S_p(n_{\rightarrow})$. We require each of $A_0(n_{\rightarrow};p)$, $A_1(n_{\rightarrow};p)$ and $A_2(n_{\rightarrow};p)$, which will combine by (8) to yield our desired result $S_p(n_{\rightarrow}) = 2\tau_n(n,p)$.

2.1.2 Detail

The Term $A_1(n;p)$

Dealing with the term $A_1(n_{\rightarrow};p)$ is elementary, for $f(1) = \tau_1(n,p) = n+1 = O(n)$ for large n, whilst

$$f(n) \;=\; \tau_n(n,p) \;=\; n^p \binom{2n}{n} \;\sim\; (4^n/\sqrt{\pi})\,n^{p-\frac{1}{2}} \qquad \text{(I.1)}$$

using Stirling's approximation

$$n! \sim \sqrt{2\pi n}\,(n/e)^n; \qquad \text{(I.2)}$$

clearly, then, $\tau_n(n,p) \gg \tau_1(n,p)$ for large n, and

$$A_1(n_{\rightarrow};p) = \frac{1}{2}\tau_n(n,p). \qquad \text{(I.3)}$$

The Term $A_2(n;p)$

The Polygamma function is defined generally as

$$\psi_n(z) = \frac{d^{n+1}}{dz^{n+1}}\left\{\ln[\Gamma(z)]\right\}, \qquad n \geq 0, \qquad \text{(I.4)}$$

of which the Digamma (or Psi) function

$$\psi_0(z) \;=\; \frac{d}{dz}\{\ln[\Gamma(z)]\} \;=\; \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{(I.5)}$$

is the $n=0$ instance. It is, noting that $\frac{d}{dz}\{\Gamma(z+c)\} = \Gamma(z+c)\psi_0(z+c)$, easy to show that

$$\frac{d}{dx}\left\{x^p \frac{\Gamma(x+a)}{\Gamma(x+b)}\right\} = x^{p-1}\frac{\Gamma(x+a)}{\Gamma(x+b)}\{x[\psi_0(x+a) - \psi_0(x+b)] + p\}, \quad \text{(I.6)}$$

giving immediately

$$f'(x) = \alpha(x) f(x), \tag{I.7}$$

where

$$\alpha(x) \quad = \quad \alpha(x;n,p) \quad = \quad \psi_0(x+n+1) - \psi_0(x+1) + p/x. \tag{I.8}$$

Let us consider, initially,

$$\begin{aligned} f'(n) - f'(1) &= \alpha(n) f(n) - \alpha(1) f(1) \\ &= [\psi_0(2n+1) - \psi_0(n+1) + p/n]\tau_n(n,p) \\ &\qquad - [\psi_0(n+2) - \psi_0(2) + p]\tau_1(n,p) \\ &= [\psi_0(2n+1) - \psi_0(n+1) + p/n]\tau_n(n,p) \\ &\quad - [\psi_0(n+1) + 1/(n+1) - (1-\gamma) + p]\tau_1(n,p), \end{aligned} \tag{I.9}$$

using the identity $\psi_0(z+1) = \psi_0(z) + 1/z$ (with the Euler-Mascheroni constant γ known to be $\gamma = -\psi_0(1)$). Now, with H_n the Harmonic series $H_n = \sum_{k=1}^{n} \frac{1}{k}$, it is known that for large n

$$H_n \sim \ln(n) + \gamma + \frac{1}{2} n^{-1} - \frac{1}{12} n^{-2} + \frac{1}{120} n^{-4} - \cdots, \tag{I.10}$$

so that, from the relation (for s integer)

$$\psi_0(s) = -\gamma + H_{s-1}, \tag{I.11}$$

an examination of the right hand side of (I.9) allows us to identify the relative size of all terms therein and write

$$\begin{aligned} f'(n) - f'(1) &\sim [\psi_0(2n+1) - \psi_0(n+1) + p/n]\tau_n(n,p) \\ &= \alpha(n)\tau_n(n,p). \end{aligned} \tag{I.12}$$

Further, employing (I.10),(I.11),

$$\begin{aligned} \alpha(n) &= H_{2n} - H_n + p/n \\ &\sim \ln(2) + \left(p - \frac{1}{4}\right) n^{-1} + \frac{1}{16} n^{-2} - \frac{1}{128} n^{-4} + \cdots \\ &= \ln(2) + O(1/n), \end{aligned} \tag{I.13}$$

whence, for large n, we arrive at

$$f'(n) - f'(1) \sim [\ln(2) + O(1/n)]\tau_n(n,p). \tag{I.14}$$

Next, after differentiating (I.7) to give $f''(x) = [\alpha^2(x) + \alpha'(x)]f(x)$, we consider

$$f''(n) - f''(1) = [\alpha^2(n) + \alpha'(n)]\tau_n(n,p) - [\alpha^2(1) + \alpha'(1)]\tau_1(n,p). \tag{I.15}$$

Since it is known that

$$\psi_0'(x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2}, \tag{I.16}$$

it is a straightforward matter to obtain

$$\alpha'(x) = -\left(\frac{p}{x^2} + \sum_{k=0}^{n-1} \frac{1}{(x+k+1)^2}\right), \tag{I.17}$$

with

$$\alpha'(n) = -\left(\frac{p}{n^2} + \sum_{k=0}^{n-1} \frac{1}{(n+k+1)^2}\right) \tag{I.18}$$

and in turn

$$\alpha'(n) \quad \sim \quad O\left(\frac{1}{n^2}\right) + n\,O\left(\frac{1}{n^2}\right) \quad = \quad O\left(\frac{1}{n}\right) \tag{I.19}$$

for n large. From (I.15), therefore, we have

$$\begin{aligned} f''(n) - f''(1) &\sim [\alpha^2(n) + \alpha'(n)]\tau_n(n,p) \\ &\sim \{[\ln(2) + O(1/n)]^2 + O(1/n)\}\tau_n(n,p) \\ &= [\ln^2(2) + O(1/n)]\tau_n(n,p). \end{aligned} \tag{I.20}$$

We omit the details here but, based on the derivative $f'''(x) = [\alpha^3(x) + 3\alpha(x)\alpha'(x) + \alpha''(x)]f(x)$, a similar analysis yields[3] $f'''(n) - f'''(1) \sim [\ln^3(2) + O(1/n)]\tau_n(n,p)$ and the general result is (see Remark 2)

$$f^{(k)}(n) - f^{(k)}(1) \sim [\ln^k(2) + O(1/n)]\tau_n(n,p), \qquad \text{uniform in } k \geq 1. \tag{I.21}$$

With (I.21) established, we are now able to derive $A_2(n_{\rightarrow};p)$ with little difficulty, for we write (recalling that $B_3 = B_5 = B_7 = \cdots = 0$, and employing (2))

$$\begin{aligned} A_2(n_{\rightarrow};p) &= \frac{\tau_n(n,p)}{\ln(2)} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \ln^{2k}(2) \\ &= \frac{\tau_n(n,p)}{\ln(2)} \left(\sum_{k=0}^{\infty} \frac{B_k}{k!} \ln^k(2) - \frac{B_0}{0!} \ln^0(2) - \frac{B_1}{1!} \ln^1(2) \right) \\ &= \frac{\tau_n(n,p)}{\ln(2)} \left(\sum_{k=0}^{\infty} \frac{B_k}{k!} \ln^k(2) + \frac{1}{2} \ln(2) - 1 \right) \end{aligned}$$

[3] We see, trivially, that $\alpha''(n) \sim O(1/n^2)$.

$$
\begin{aligned}
&= \frac{\tau_n(n,p)}{\ln(2)}\left(\frac{\ln(2)}{e^{\ln(2)}-1}+\frac{1}{2}\ln(2)-1\right)\\
&= \frac{\tau_n(n,p)}{\ln(2)}\left(\frac{3}{2}\ln(2)-1\right). \qquad \text{(I.22)}
\end{aligned}
$$

The Term $A_0(n;p)$

We first state and prove a Lemma on which the formulation of $A_0(n_{\rightarrow};p)$ is based (a generalised form of the result is given at the conclusion of Appendix B).

Lemma *For real $\alpha \in (\frac{1}{2},1)$*

$$\int_1^n f(x)\,dx \sim \int_{\alpha n}^n f(x)\,dx$$

for n large.

Proof For integer $n \geq 2$, suppose real $\alpha \in (\frac{1}{2},1)$ (so that $\alpha n > 1$) and consider

$$\int_1^n f(x)\,dx = I_1(n,\alpha;p) + I_2(n,\alpha;p), \qquad \text{(L1)}$$

where

$$I_1(n,\alpha;p) = \int_1^{\alpha n} f(x)\,dx, \qquad I_2(n,\alpha;p) = \int_{\alpha n}^n f(x)\,dx. \qquad \text{(L2)}$$

Since $f(x)$ is a (strictly increasing) monotonic function over $[1,n]$ we can write

$$
\begin{aligned}
I_1(n,\alpha;p) &< (\alpha n-1)f(\alpha n) < \alpha n f(\alpha n)\\
&= (\alpha n)^{p+1}\frac{\Gamma((\alpha+1)n+1)}{\Gamma(n+1)\Gamma(\alpha n+1)}, \qquad \text{(L3)}
\end{aligned}
$$

whilst

$$
\begin{aligned}
I_2(n,\alpha;p) &= \int_{\alpha n}^n x^p\binom{n+x}{x}\,dx > \int_{\alpha n}^n (\alpha n)^p\binom{n+x}{x}\,dx\\
&= \frac{(\alpha n)^p}{\Gamma(n+1)}\int_{\alpha n}^n \frac{\Gamma(x+n+1)}{\Gamma(x+1)}\,dx, \qquad \text{(L4)}
\end{aligned}
$$

giving an inequality for the ratio $I_1(n,\alpha;p)/I_2(n,\alpha;p)$ as

$$0 < \frac{I_1(n,\alpha;p)}{I_2(n,\alpha;p)} < \frac{\alpha n\,\Gamma((\alpha+1)n+1)}{\Gamma(\alpha n+1)}\frac{1}{\int_{\alpha n}^n \frac{\Gamma(x+n+1)}{\Gamma(x+1)}\,dx}. \qquad \text{(L5)}$$

We now choose real β such that $\alpha < \beta < 1$ whence, writing $g(x) = g(x; n) = \Gamma(x + n + 1)/\Gamma(x + 1)$, we observe that

$$\begin{aligned} \int_{\alpha n}^{n} g(x)\,dx &> \int_{\beta n}^{n} g(x)\,dx > (n - \beta n) g(\beta n) \\ &= (1-\beta) n \frac{\Gamma((\beta+1)n+1)}{\Gamma(\beta n + 1)}. \end{aligned} \tag{L6}$$

Thus we have a suitable inequality for the integral in the upper bound of (L5), which latter becomes

$$0 < \frac{I_1(n,\alpha;p)}{I_2(n,\alpha;p)} < \frac{\beta}{1-\beta} \frac{\Gamma((\alpha+1)n+1)}{\Gamma(\alpha n)} \frac{\Gamma(\beta n)}{\Gamma((\beta+1)n+1)} \tag{L7}$$

on employing the well known identity

$$z\Gamma(z) = \Gamma(z+1). \tag{L8}$$

Repeated use of (L8) also gives

$$\frac{\Gamma(z+n+1)}{\Gamma(z)} = z(z+1)(z+2)\cdots(z+n), \tag{L9}$$

application of which to (L7) (with $z = \alpha n, \beta n$) yields

$$\begin{aligned} 0 < \frac{I_1(n,\alpha;p)}{I_2(n,\alpha;p)} &< \frac{\alpha}{1-\beta} \prod_{i=1}^{n} \left(\frac{\alpha n + i}{\beta n + i} \right) \\ &\leq \frac{\alpha}{1-\beta} \prod_{i=1}^{n} \left(\frac{\alpha n + n}{\beta n + n} \right) \\ &= \frac{\alpha}{1-\beta} \left(\frac{\alpha+1}{\beta+1} \right)^{n} \\ &\to 0^{+} \end{aligned} \tag{L10}$$

for large n, since $\frac{\alpha+1}{\beta+1} < 1$. By (L1),(L10) the Lemma is now immediate, for

$$\begin{aligned} \frac{\int_1^n f(x)\,dx}{\int_{\alpha n}^n f(x)\,dx} &= \frac{I_1(n,\alpha;p) + I_2(n,\alpha;p)}{I_2(n,\alpha;p)} \\ &= \frac{I_1(n,\alpha;p)}{I_2(n,\alpha;p)} + 1 \\ &\to 1^{+} \end{aligned} \tag{L11}$$

for large n.□

Remark 1 In arriving at (L10) we have used the fact that, for $i = 1, \ldots, n$, $(\alpha n + i)/(\beta n + i) \leq (\alpha n + n)/(\beta n + n)$. This is not, perhaps, intuitively obvious, but follows from the inequality $(n - i)\alpha n \leq (n - i)\beta n$ (this is re-arranged to read $\alpha n^2 + i\beta n \leq \beta n^2 + i\alpha n \Rightarrow (\alpha\beta n^2 + in) + \alpha n^2 + i\beta n \leq (\alpha\beta n^2 + in) + \beta n^2 + i\alpha n$, and so on, which delivers the result trivially).

The Lemma shows that 'almost all' of the area under the function $f(x)$ can be made to reside in an arbitrarily small (proportionately) region at the upper end of the integral range by choosing a sufficiently large value for n. To emphasise the role of α in it, let us re-label the integral $I_2(n, \alpha; p)$ as $I_\alpha(n; p)$. It remains—in view of the Lemma—but to find

$$I_\alpha(n_\rightarrow; p) = A_0(n_\rightarrow; p). \tag{I.23}$$

We begin by noting that the $p = 0$ instance of (I.6) affords a form of $f(x)$, and so $I_\alpha(n; p)$, thus:

$$n! I_\alpha(n; p) = \int_{\alpha n}^{n} \frac{x^p}{\psi_0(x + n + 1) - \psi_0(x + 1)} \frac{d}{dx} \left\{ \frac{\Gamma(x + n + 1)}{\Gamma(x + 1)} \right\} dx. \tag{I.24}$$

If α is chosen to be close enough to 1 the range of integration in $I_\alpha(n; p)$ shrinks in proportion. However, for large enough n then $\psi_0(x + n + 1) - \psi_0(x + 1) \sim \ln(2)$ and in this case (I.24) reduces to (see Remark 2)

$$\begin{aligned} \ln(2) I_\alpha(n; p) &\sim \frac{1}{n!} \int_{\alpha n}^{n} x^p \frac{d}{dx} \left\{ \frac{\Gamma(x + n + 1)}{\Gamma(x + 1)} \right\} dx \\ &= E_1(n; p) - E_2(n, \alpha; p) - E_3(n, \alpha; p), \end{aligned} \tag{I.25}$$

via Integration by Parts, where

$$\begin{aligned} E_1(n; p) &= n^p \binom{2n}{n} = \tau_n(n, p), \\ E_2(n, \alpha; p) &= \frac{(\alpha n)^p}{n!} \frac{\Gamma((\alpha + 1)n + 1)}{\Gamma(\alpha n + 1)}, \\ E_3(n, \alpha; p) &= \frac{p}{n!} \int_{\alpha n}^{n} x^{p-1} \frac{\Gamma(x + n + 1)}{\Gamma(x + 1)} dx, \end{aligned} \tag{I.26}$$

with each of $E_1(n; p), E_2(n, \alpha; p), E_3(n, \alpha; p) > 0$. Now, using (L8),(L9), we see that the ratio

$$\begin{aligned} 0 < \frac{E_2(n, \alpha; p)}{E_1(n; p)} &= \alpha^p \frac{\Gamma((\alpha + 1)n + 1)}{\Gamma(\alpha n + 1)} \frac{n!}{(2n)!} \\ &= \alpha^p \prod_{i=1}^{n} \left(\frac{\alpha n + i}{n + i} \right) \end{aligned}$$

$$
\begin{aligned}
&\le \quad \alpha^p \left(\frac{\alpha+1}{2}\right)^n \\
&\to \quad 0^+ \qquad \text{(I.27)}
\end{aligned}
$$

for increasing n. Next, we find that (simple reader exercise)

$$0 \quad < \quad \frac{E_3(n,\alpha;p)}{I_\alpha(n;p)} \quad < \quad p/(\alpha n) \quad \to \quad 0^+, \qquad \text{(I.28)}$$

whence, noting clearly $E_1(n;p) > \ln(2) I_\alpha(n;p)$ from (I.25), we infer

$$0 \quad < \quad \frac{E_3(n,\alpha;p)}{E_1(n;p)} \quad < \quad \frac{1}{\ln(2)} \frac{E_3(n,\alpha;p)}{I_\alpha(n;p)} \quad \to \quad 0^+ \qquad \text{(I.29)}$$

similarly. Thus, in the limit, (I.27),(I.29) give (I.25) as

$$
\begin{aligned}
0 \quad < \quad \frac{\ln(2) I_\alpha(n_\to;p)}{E_1(n_\to;p)} \quad &= \quad \lim_{n\to\infty} \left\{1 - \frac{E_2(n,\alpha;p)}{E_1(n;p)} - \frac{E_3(n,\alpha;p)}{E_1(n;p)}\right\} \\
&= \quad 1, \qquad \text{(I.30)}
\end{aligned}
$$

and (see (I.23)) we have identified $A_0(n_\to;p)$:

$$A_0(n_\to;p) \quad = \quad I_\alpha(n_\to;p) \quad = \quad \frac{1}{\ln(2)} E_1(n_\to;p) \quad = \quad \frac{1}{\ln(2)} \tau_n(n,p). \qquad \text{(I.31)}$$

The proof is, finally, completed trivially, for combining (I.3),(I.22) and (I.31) then (8) reads, asymptotically,

$$
\begin{aligned}
S_p(n_\to) \quad &= \quad A_0(n_\to;p) + A_1(n_\to;p) + A_2(n_\to;p) \\
&= \quad \frac{1}{\ln(2)}\tau_n(n,p) + \frac{1}{2}\tau_n(n,p) + \frac{1}{\ln(2)}\left(\frac{3}{2}\ln(2) - 1\right)\tau_n(n,p) \\
&= \quad 2\tau_n(n,p), \qquad \text{(I.32)}
\end{aligned}
$$

as required.□

It should be noted that of course we may only use (8) asymptotically in the manner shown here since the ratio of any pair of the three terms $A_0(n_\to;p), A_1(n_\to;p), A_2(n_\to;p)$ is a positive constant (recall that, in general, if functions $Q_a(n), Q_b(n) \sim q_a(n), q_b(n)$ (resp.) for large n, then it does not necessarily follow that $Q_a(n) + Q_b(n) \sim q_a(n) + q_b(n)$).

Remark 2: Some Comments on the Euler-Maclaurin Summation Formula and Additional Rigour Our application of the Euler-Maclaurin summation formula to the treatment of a particular infinite series—whilst undoubtedly lengthy—nevertheless reminds us of its usefulness in delivering the correct result, and it does so with a point of interest on which it is pertinent to remark. One would have expected, in advance, that the asymptotic form $2\tau_n(n,p)$ would have comprised the contributions of $A_0(n_{\rightarrow};p)$ and $A_1(n_{\rightarrow};p)$. In this case, however, we see that $\frac{1}{\ln(2)}+\frac{1}{2}\approx 1.9427$, with the remaining decimal part of 2 wrapped up in the (often negligible) term we have as $A_2(n_{\rightarrow};p)$; all three terms—each being a multiple of $\tau_n(n,p)$—are of commensurate magnitude, and in this aspect our formulation here provides a variation in the typical execution of the Euler-Maclaurin formula in relation to such a series. With this in mind there can be no doubt that the Euler-Maclaurin summation formula delivers the desired asymptotic form of the Sum $S_p(n)$ correctly, and our presentation contains its salient mathematical features. However (with regard to the two references to this remark at the appropriate places in the proof), for a *completely rigorous* argument to describe the large n behaviour of the terms $A_2(n;p), A_0(n;p)$ extra detailed and non-trivial analysis becomes a necessity; this is met fully in Appendices A,B, to which the interested reader is referred (this material is set down in appendices so as not to clutter Proof I).

2.2 Proof II

This second proof contrasts with Proof I by its compact and direct nature, reliant only on the algebraic manipulation of binomial coefficient sums and an appropriate starting point; what emerges is a rather pleasing and succinct route to the desired result. Note that the proof is eased by taking the lower summing index of $S_p(n)$ from zero.

Let, for $p,j\geq 0$, $\mathbf{s}(p,j)$ be the Stirling number of the 2nd kind often delineated in (lower triangular) formation

$$\begin{matrix} 1 & & & & & \cdots \\ 0 & 1 & & & & \cdots \\ 0 & 1 & 1 & & & \cdots \\ 0 & 1 & 3 & 1 & & \cdots \\ 0 & 1 & 7 & 6 & 1 & \cdots \\ \vdots & & & & & \ddots \end{matrix} \quad = $$

$$\begin{matrix} \mathbf{s}(0,0) & & & & & \cdots \\ \mathbf{s}(1,0) & \mathbf{s}(1,1) & & & & \cdots \\ \mathbf{s}(2,0) & \mathbf{s}(2,1) & \mathbf{s}(2,2) & & & \cdots \\ \mathbf{s}(3,0) & \mathbf{s}(3,1) & \mathbf{s}(3,2) & \mathbf{s}(3,3) & & \cdots \\ \mathbf{s}(4,0) & \mathbf{s}(4,1) & \mathbf{s}(4,2) & \mathbf{s}(4,3) & \mathbf{s}(4,4) & \cdots \\ \vdots & & & & & \ddots \end{matrix} \qquad \text{(II.1)}$$

with $\mathbf{s}(p,j) = 0$ when $j > p$. It is well known that for integral p the monomials x^p can be expressed in terms of these Stirling numbers as

$$x^p = \sum_{j=0}^{p} \mathbf{s}(p,j)[x]_j, \qquad p \geq 0, \tag{II.2}$$

where $[x]_j$ denotes (with $[x]_0 = 1$) the usual falling factorial function

$$[x]_j = x(x-1)(x-2)\cdots(x-j+1). \tag{II.3}$$

This offers an immediate starting point for Proof II, for we write, assuming $p \geq 1$,

$$\begin{aligned} S_p(n) &= \sum_{k=0}^{n} k^p \binom{n+k}{k} \\ &= \sum_{j=0}^{p} \mathbf{s}(p,j) \sum_{k=0}^{n} \binom{n+k}{k} [k]_j \\ &= \frac{1}{n!} \sum_{j=0}^{p} \mathbf{s}(p,j) \sum_{k=0}^{n} [n+k]_n [k]_j. \end{aligned} \tag{II.4}$$

Now, since $[k]_j = 0$ for $k = 0, \ldots, j-1$, then

$$\begin{aligned} \sum_{k=0}^{n} [n+k]_n [k]_j &= \left(\sum_{k=0}^{j-1} + \sum_{k=j}^{n} \right) [n+k]_n [k]_j \\ &= \sum_{k=j}^{n} [n+k]_n [k]_j \\ &= \sum_{k=j}^{n} [n+k]_{n+j} \\ &= (n+j)! \sum_{k=j}^{n} \binom{n+k}{n+j}, \end{aligned} \tag{II.5}$$

and (II.4) reads

$$\begin{aligned} S_p(n) &= \frac{1}{n!} \sum_{j=0}^{p} \mathbf{s}(p,j)(n+j)! \sum_{k=j}^{n} \binom{n+k}{n+j} \\ &= \frac{1}{n!} \sum_{j=0}^{p} \mathbf{s}(p,j)(n+j)! \sum_{k=0}^{n-j} \binom{k+n+j}{n+j} \\ &= \frac{1}{n!} \sum_{j=0}^{p} \mathbf{s}(p,j)(n+j)! \binom{2n+1}{n+j+1} \end{aligned} \tag{II.6}$$

upon evaluating the sum in k (this is available, for instance, from Gould's result [7, Identity No. (1.48), p.6] $\sum_{k=0}^{m}\binom{k+x}{r} = \binom{m+x+1}{r+1} - \binom{x}{r+1}$ with $m = n-j$, $x = r = n+j$). We continue by noting that

$$\begin{aligned} \frac{1}{n!}(n+j)!\binom{2n+1}{n+j+1} &= \binom{2n+1}{n}\frac{[n+1]_{j+1}}{n+j+1} \\ &= (2n+1)\binom{2n}{n}\frac{[n]_j}{n+j+1}, \end{aligned} \tag{II.7}$$

whence

$$\begin{aligned} S_p(n) &= (2n+1)\binom{2n}{n}\sum_{j=0}^{p}\mathbf{s}(p,j)\frac{1}{n+j+1}[n]_j \\ &= \binom{2n}{n}\sum_{j=0}^{p}\mathbf{s}(p,j)\left(2-\frac{2j+1}{n+j+1}\right)[n]_j \\ &= \binom{2n}{n}\left(2\sum_{j=0}^{p}\mathbf{s}(p,j)[n]_j - \sum_{j=0}^{p}\mathbf{s}(p,j)\frac{2j+1}{n+j+1}[n]_j\right) \\ &= \binom{2n}{n}[2n^p - F(n,p)], \end{aligned} \tag{II.8}$$

where, writing $\alpha_j(p) = (2j+1)\mathbf{s}(p,j) \geq 0$,

$$F(n,p) = \sum_{j=0}^{p}\alpha_j(p)\frac{[n]_j}{n+j+1}. \tag{II.9}$$

We finish Proof II by observing that, for n large,

$$\begin{aligned} F(n,p) &\sim O(1/n) + O(n/n) + O(n^2/n) + \cdots \\ &\qquad \cdots + O(n^{p-1}/n) + O(n^p/n) \\ &= O(n^{p-1}) \end{aligned} \tag{II.10}$$

(in fact it is easy to identify the coefficient of this term as $\alpha_p(p) = (2p+1)\mathbf{s}(p,p) = 2p+1$), leaving a dominant lead term of $O(n^p)$ in the bracket of (II.8); in other words,

$$S_p(n_{\rightarrow}) = 2n^p\binom{2n}{n}.\square \tag{II.11}$$

It is of interest to know that (II.8) has already been used by Paris and Larcombe [8] to tease out the first few terms of a full asymptotic expansion for $S_p(n)$ (another approach in [8] sees the same result achieved by applying the so called method of steepest descents based on a loop integral representation of the rational function $\Gamma(n+i+1)/\Gamma(i+1)$ which gives a suitable integral form of $S_p(n)$ with which to work).

Remark 3 As alluded to in the Introduction, our assumption that p be a positive integer was entirely for the purpose of consistency with the earlier article [1]. In fact experimental computations would indicate strongly that not only does the Theorem hold for $p \in \mathbf{Z}$ (provided the summing index of $S_p(n)$ starts at 1), but more generally for $p \in \mathbf{R}$, a somewhat surprising characteristic we consider worth highlighting (we are grateful to the referee for pointing this out); one of the proofs in the asymptotic expansions of [8] is also valid for arbitrary finite values of p.

Remark 4 By way of completeness, and to make the paper as self-contained as possible, we add to the Introduction by turning briefly to the origins of the Euler-Maclaurin summation formula. According to Apostol [6], Leonhard Euler first obtained the simplest case of what came to be known at the time as Euler's summation formula—a powerful tool for estimating sums by integrals and evaluating integrals in terms of sums.[4] He published this result, and a generalised version (being the one we have deployed), in two 1736 papers in Vol. 8 of the highly respected and widely read journal *Commentarii Academiae Scientiarum Imperialis Petropolitanae* (resp., pp.3-9 & 147-158), with the latter discovered independently by Colin Maclaurin and seen in his 1742 text *A Treatise of Fluxions*. A footnote by Whittaker and Watson [4, p.127] embellishes the Euler-Maclurin formula's history a little, with regard to which they cite an informative 1905 paper by E.W. Barnes for fuller details ('The Maclaurin Sum-Formula', *Proc. Lond. Math. Soc.* (Series 2), **3**, pp.253-272).

3 Summary

We have presented two new and entirely different proofs of the asymptotic form of our sum $S_p(n)$—a lengthy one based on the apparently little used Euler-Maclaurin formula for infinite series, and another much shorter one using direct manipulation of the given sum; both contrast with that in [1].

[4]See the short article by Robertson and Osler [9] who term this "intermediate" result Euler's "little summation formula"; also, the 28 page document 'Excerpts on the Euler-Maclaurin Summation Formula, from *Institutiones Calculi Differentialis* by Leonhard Euler' (translated by David Pengelley (2000) and available from his homepage **http://math.nmsu.edu/∼davidp/**). Other discussions of the Euler-Maclaurin formula are given by Apostol in [10, Sections 3.3-3.7, pp.54-62] and [11, Section 7.10, pp.149-150].

One further, and more general, point relates to Proof I. It is worth noting that, retaining p as a general parameter, such an asymptotic form is difficult to identify using any one of the mainstream algebraic computation packages. Certain p-instances of the asymptotic form may, however, reveal themselves under computational interrogation and the potential to generalise from the particular (using such software as a vehicle) might explain why—as a tool for examining the type of infinite sum seen here—the Euler-Maclaurin summation formula would appear to have fallen out of fashion somewhat. Nonetheless, for the reason stated in Remark 2 it is felt that our presentation is of interest mathematically *per se*, besides which—in the words of Apostol [6, p.418]—exposure to the formula "... and its relation to Bernoulli numbers and polynomials provides a treasure trove of interesting enrichment material suitable for elementary calculus courses" in whatever context it is examined. Not only this, but it is evident from the literature that at research level the formula continues to be both analysed and connected to other constructs in different areas of mathematics in ways far beyond the scope of this paper.

Finally, it was mentioned in the Summary of [1] that the sum $S_p(n)$ does not crop up naturally in any common combinatorial setting (indeed examples of the sum are scarce within the literature). Identification of an enumerative context for $S_p(n)$ is, therefore, an obvious topic for future consideration.

4 Acknowledgement

The authors wish to acknowledge the referee of the forerunner article [1] for suggesting Proof I and providing an initial outline of the method. The referee of this paper is thanked for some useful suggestions to improve the presentation.

Appendix A

We begin by re-examining the kth derivative of $f(x)$ (7). Using Leibniz's rule for differentiation this is (for $k \geq 0$)

$$\begin{aligned} f^{(k)}(x) &= \frac{1}{n!}\sum_{j=0}^{k}\binom{k}{j}[p]_j\,x^{p-j}\,\frac{d^{k-j}}{dx^{k-j}}\{[x+n]_n\} \\ &= f(x)\sum_{j=0}^{k}\binom{k}{j}[p]_j\sum_{1\leq i_1<\cdots<i_{k-j}\leq n}\frac{(k-j)!}{x^j(x+i_1)\cdots(x+i_{k-j})}, \end{aligned} \tag{A1}$$

from which, stripping off the first ($j = 0$) term and setting $x = n$, we have

$$f^{(k)}(n) = \tau_n(n,p)[C_k(n) + r_k(n,p)], \qquad k \geq 1, \tag{A2}$$

where

$$\begin{aligned} r_k(n,p) &= \sum_{j=1}^{k} \binom{k}{j} [p]_j \frac{1}{n^j} \sum_{1 \leq i_1 < \cdots < i_{k-j} \leq n} \frac{(k-j)!}{(n+i_1)\cdots(n+i_{k-j})}, \\ C_k(n) &= k! \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{(n+i_1)\cdots(n+i_k)}. \end{aligned} \tag{A3}$$

A combinatorial consideration of the k-products (of strictly increasing natural numbers) occurring in the denominator of $C_k(n)$ allows us to write (reader exercise)

$$0 \leq C_k(n) = (H_{2n} - H_n)^k - \Sigma^* \tag{A4}$$

with

$$\Sigma^* = \sum \frac{1}{n_1 n_2 \cdots n_k}, \tag{A5}$$

where the sum of k-tuples runs over all n_i for which $n + 1 \leq n_i \leq 2n$ with at least two of them equal. Whence,

$$\begin{aligned} 0 \leq \Sigma^* &\leq \left(\sum_{r=n+1}^{2n} \frac{1}{r^2} \right) (H_{2n} - H_n)^{k-2} \\ &< \frac{1}{2n}(H_{2n} - H_n)^{k-2} \end{aligned} \tag{A6}$$

since $\sum_{r=n+1}^{2n} \frac{1}{r^2} < \int_n^{2n} \frac{1}{r^2}\, dr = 1/2n$ ($n \geq 1$). For n large we know from (I.13) that $H_{2n} - H_n \uparrow \ln(2)$ so that $(H_{2n} - H_n)^{-2} \downarrow \ln^{-2}(2)$, and we can always find (positive) reals b_1, b_2 such that $\ln^{-2}(2) < b_1 \leq (H_{2n} - H_n)^{-2} \leq b_2$. Thus (A6) reads

$$\begin{aligned} 0 \leq \Sigma^* &< \frac{1}{2n}(H_{2n} - H_n)^{k-2} \\ &= \frac{1}{2n}(H_{2n} - H_n)^{-2}(H_{2n} - H_n)^k \\ &\leq \frac{b_2}{2n}(H_{2n} - H_n)^k \\ &= (H_{2n} - H_n)^k O(1/n), \end{aligned} \tag{A7}$$

and in turn, by (A4),

$$0 \leq C_k(n) = (H_{2n} - H_n)^k [1 + O(1/n)], \qquad k \geq 1, \tag{A8}$$

with the $O(1/n)$ term (and beyond) *independent of* k. We now have a more precise version of (I.21), namely (for $k \geq 1$),

$$f^{(k)}(n) - f^{(k)}(1) = \{(H_{2n} - H_n)^k[1 + O(1/n)] + r_k(n,p)\}\tau_n(n,p), \quad \text{(A9)}$$

and, after looking at the term $r_k(n,p)$, it will be seen that $C_k(n)$ offers the main asymptotic contribution to $A_2(n;p)$.

First, we observe that

$$\begin{aligned} \sum_{1\leq i_1<\cdots<i_{k-j}\leq n} \frac{(k-j)!}{(n+i_1)\cdots(n+i_{k-j})} &= C_{k-j}(n) \\ &\leq (H_{2n} - H_n)^{k-j} \qquad \text{(A10)} \end{aligned}$$

since $C_k(n) \leq (H_{2n} - H_n)^k$ by (A4),(A5). Hence (with $[p]_j \leq p!$), we can bound $r_k(n,p)$ above according to

$$\begin{aligned} 0 \leq r_k(n,p) &\leq p! \sum_{j=1}^{k} \binom{k}{j} \frac{1}{n^j} (H_{2n} - H_n)^{k-j} \\ &= p![(H_{2n} - H_n + 1/n)^k - (H_{2n} - H_n)^k] \qquad \text{(A11)} \end{aligned}$$

after re-arranging a simple binomial expansion $(a+b)^k = \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j}$ with $a = 1/n$, $b = H_{2n} - H_n$. Equation (A9) now gives, taking $H_{2n} - H_n = \ln(2) + O(1/n)$,

$$\begin{aligned} A_2(n;p) &= \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}[f^{(2k-1)}(n) - f^{(2k-1)}(1)] \\ &\sim \Omega_1(n,p) + \Omega_2(n,p) \qquad \text{(A12)} \end{aligned}$$

for large n, where

$$\begin{aligned} \Omega_1(n,p) &= \frac{\tau_n(n,p)}{\ln(2)} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(H_{2n} - H_n)^{2k}, \\ \Omega_2(n,p) &= \tau_n(n,p) \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} r_{2k-1}(n,p). \qquad \text{(A13)} \end{aligned}$$

It is easy to show that (as in (I.22))

$$\Omega_1(n_{\rightarrow},p) = \frac{\tau_n(n,p)}{\ln(2)} \left(\frac{3}{2}\ln(2) - 1\right), \qquad \text{(A14)}$$

whilst a similar process yields, using the upper bound (A11),

$$\begin{aligned}\frac{\Omega_2(n,p)}{p!\tau_n(n,p)} &\leq \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left[(H_{2n}-H_n+1/n)^{2k-1}-(H_{2n}-H_n)^{2k-1}\right] \\ &= g_a(n)+g_b(n), \end{aligned} \tag{A15}$$

with (after some algebra)

$$\begin{aligned} g_a(n) &= \frac{1}{e^{H_{2n}-H_n+1/n}-1}-\frac{1}{e^{H_{2n}-H_n}-1} < 0, \\ g_b(n) &= \frac{1}{H_{2n}-H_n}-\frac{1}{H_{2n}-H_n+1/n} > 0, \end{aligned} \tag{A16}$$

both of which are $O(1/n)$ terms when examined. Thus, with $\Omega_2(n,p) \sim p!\tau_n(n,p)O(1/n)$, then by (A12)

$$A_2(n_{\rightarrow};p) = \Omega_1(n_{\rightarrow},p) = \frac{\tau_n(n,p)}{\ln(2)}\left(\frac{3}{2}\ln(2)-1\right), \tag{A17}$$

as in (I.22).

Appendix B

Here we provide the additional analysis in order to establish (I.31) rigorously.

We first note that, from the definition (I.5) of the Digamma function,

$$\begin{aligned}\psi_0(x+n+1)-\psi_0(x+1) &= \frac{d}{dx}\left\{\ln\left[\frac{\Gamma(x+n+1)}{\Gamma(x+1)}\right]\right\} \\ &= \frac{d}{dx}\left\{\ln\left[\prod_{i=1}^{n}(x+i)\right]\right\} \\ &= \frac{d}{dx}\left\{\sum_{i=1}^{n}\ln(x+i)\right\} \\ &= \sum_{i=1}^{n}\frac{1}{x+i}. \end{aligned} \tag{B1}$$

For $x \in [\alpha n, n]$ then $x+i \in [\alpha n+i, n+i]$ and so $\frac{1}{n+i} \leq \frac{1}{x+i} \leq \frac{1}{\alpha n+i}$ $(i=1,\ldots,n)$, giving

$$\sum_{i=1}^{n}\frac{1}{n+i} \leq \sum_{i=1}^{n}\frac{1}{x+i} \leq \frac{1}{\alpha}\sum_{i=1}^{n}\frac{1}{n+i/\alpha} < \frac{1}{\alpha}\sum_{i=1}^{n}\frac{1}{n+i} \tag{B2}$$

since, with $\alpha < 1$, $\frac{1}{\alpha} > 1 \Rightarrow n + \frac{i}{\alpha} > n + i$. Thus we may write

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{n+i} - \ln(2) \quad &\leq \quad \sum_{i=1}^{n} \frac{1}{x+i} - \ln(2) \\ &< \quad \frac{1}{\alpha} \sum_{i=1}^{n} \frac{1}{n+i} - \ln(2) \\ &= \quad \frac{1}{\alpha} \left(\sum_{i=1}^{n} \frac{1}{n+i} - \ln(2) \right) + \left(\frac{1}{\alpha} - 1 \right) \ln(2). \qquad \text{(B3)}\end{aligned}$$

Now let $\varepsilon > 0$ and fix $\alpha_\varepsilon = \alpha(\varepsilon) = [\ln(4) + \varepsilon/2]/[\ln(4) + \varepsilon] \in (\frac{1}{2}, 1)$. Further, let N_1, N_2 be sufficiently large such that

$$\left| \sum_{i=1}^{n} \frac{1}{n+i} - \ln(2) \right| \; < \; \varepsilon \qquad (n \geq N_1) \qquad \text{(B4)}$$

and

$$\left| \frac{1}{\alpha_\varepsilon} \left(\sum_{i=1}^{n} \frac{1}{n+i} - \ln(2) \right) \right| \; < \; \varepsilon/2 \qquad (n \geq N_2). \qquad \text{(B5)}$$

It is easy (observing that $\alpha_\varepsilon > \ln(4)/[\ln(4) + \varepsilon]$) to deduce

$$\left(\frac{1}{\alpha_\varepsilon} - 1 \right) \ln(2) < \frac{\varepsilon}{2}, \qquad \text{(B6)}$$

whence, on setting $n = \max\{N_1, N_2\}$ the inequalities (B4),(B5) and (B6) give immediately (B3) as

$$\begin{aligned}-\varepsilon \quad < \quad \sum_{i=1}^{n} \frac{1}{n+i} - \ln(2) \quad &\leq \quad \sum_{i=1}^{n} \frac{1}{x+i} - \ln(2) \\ &< \quad \frac{1}{\alpha_\varepsilon} \left(\sum_{i=1}^{n} \frac{1}{n+i} - \ln(2) \right) + \left(\frac{1}{\alpha_\varepsilon} - 1 \right) \ln(2) \\ &< \quad \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \quad \varepsilon. \qquad \text{(B7)}\end{aligned}$$

In other words, we have a uniform bound

$$\left| \sum_{i=1}^{n} \frac{1}{x+i} - \ln(2) \right| \; = \; \left| \psi_0(x+n+1) - \psi_0(x+1) - \ln(2) \right| \; < \; \varepsilon \quad \text{(B8)}$$

which, for judiciously chosen α_ε, holds across the interval $[\alpha_\varepsilon n, n] = [\alpha n, n]$ on the integral $I_\alpha(n; p)$ seen in (I.24).

This, then, formalises the step made in moving from (I.24) to (I.25) with respect to the proof of (I.31) itself. We write trivially from (B8) that

$$\frac{1}{\ln(2)+\varepsilon} \quad < \quad \frac{1}{\psi_0(x+n+1)-\psi_0(x+1)} \quad < \quad \frac{1}{\ln(2)-\varepsilon} \tag{B9}$$

for large enough n, from which it follows that

$$\frac{1}{\ln(2)+\varepsilon} J_{\alpha_\varepsilon}(n;p) \quad < \quad I_{\alpha_\varepsilon}(n;p) \quad < \quad \frac{1}{\ln(2)-\varepsilon} J_{\alpha_\varepsilon}(n;p) \tag{B10}$$

where

$$J_{\alpha_\varepsilon}(n;p) = \frac{1}{n!}\int_{\alpha_\varepsilon n}^{n} x^p \frac{d}{dx}\left\{\frac{\Gamma(x+n+1)}{\Gamma(x+1)}\right\} dx. \tag{B11}$$

Expressing (B10) as

$$\frac{1}{\ln(2)+\varepsilon}\frac{J_{\alpha_\varepsilon}(n;p)}{\tau_n(n,p)} \quad < \quad \frac{I_{\alpha_\varepsilon}(n;p)}{A_0(n;p)}\frac{A_0(n;p)}{\tau_n(n,p)} \quad < \quad \frac{1}{\ln(2)-\varepsilon}\frac{J_{\alpha_\varepsilon}(n;p)}{\tau_n(n,p)}, \tag{B12}$$

and observing that $J_{\alpha_\varepsilon}(n;p)/E_1(n;p) = J_{\alpha_\varepsilon}(n;p)/\tau_n(n,p) \to 1^-$ (by (I.30)), whilst the Lemma gives $I_{\alpha_\varepsilon}(n;p)/A_0(n;p) \to 1^-$, then for large n

$$\frac{1}{\ln(2)+\varepsilon} \quad < \quad \frac{A_0(n;p)}{\tau_n(n,p)} \quad < \quad \frac{1}{\ln(2)-\varepsilon}; \tag{B13}$$

thus, with ε arbitrary we must have, in the limit,

$$\frac{A_0(n_\rightarrow;p)}{\tau_n(n,p)} \quad = \quad \frac{1}{\ln(2)} \tag{B14}$$

as required.

Remark As a point of interest we note that a generalised version of the Lemma used above is readily accessible; we state it here for completeness (omitting the proof as it parallels that of the Lemma in Proof I), noting that our function $f(x)$ (7) is but one instance of a class of functions to which this result applies:

Theorem *For $n = 2, 3, 4, \ldots$, let $f_n(x) : [l, \infty) \to [0, \infty)$ be a family of increasing (monotonic) functions such that*

(i) *Each $f_n(x)$ is integrable on $[l, n]$;*
(ii) *For $0 < \alpha < \beta < 1$, $\lim_{n\to\infty}\{f_n(\alpha n)/f_n(\beta n)\} = 0$.*

Then, for large n,

$$\int_l^n f_n(x)\, dx \sim \int_{\alpha n}^n f_n(x)\, dx.$$

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