Generalised Catalan Polynomials and their Properties

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Abstract

We introduce a new type of polynomial, termed a *generalised* Catalan polynomial. We list essential mathematical properties and give two associated combinatorial interpretations.

1 Introduction

Let the general $(n+1)$th term of the Catalan sequence $\{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{1, 1, 2, 5, 14, \ldots\}$ be c_n, with closed form

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n = 0, 1, 2, \ldots \tag{1}$$

The ordinary generating function (o.g.f.) $C(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{i=0}^{\infty} c_i x^i$ satisfies the quadratic $0 = xC^2(x) - C(x) + 1$. In [1] the authors explored the concept of (linearly convergent) so called iterated generating functions, resulting in the appearance of Catalan polynomials whose role in generating finite subsequences of Catalan numbers was identified and formalised as a theorem. A suite of non-linear identities for these polynomials was also developed in [2] based on an algebraic implementation of numeric root finding schemes taken from a general formulation due to Householder (and delivering the well known Newton-Raphson and Halley algorithms as particular instances) in the context of which the link between these polynomials and Padé approximants of the Catalan sequence o.g.f. was examined.

The Catalan polynomial $P_n(x)$ is defined for $n \geq 0$ as the binomial sum

$$P_n(x) = \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i} (-x)^i \tag{2}$$

or, hypergeometrically, as

$$P_n(x) = {}_2F_1\left(\left. \begin{matrix} -\frac{1}{2}n, -\frac{1}{2}(n-1) \\ -n \end{matrix} \right| 4x \right), \tag{3}$$

or indeed via matrices as

$$P_n(x) = (\sqrt{x})^n (1, 1/\sqrt{x}) \begin{pmatrix} 0 & -1 \\ 1 & \frac{1}{\sqrt{x}} \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4}$$

or

$$P_n(x) = (1, 0) \begin{pmatrix} 1 & x \\ -1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5}$$

Forms (2)-(4) are first seen in [1] (see (67),(68),(76), pp.16,17,19, resp.; also equations (78),(91) (pp.20,23) therein for others). Basic characteristics were set out in some depth in [1, Section 5, pp.16-29], which included a combinatorial interpretation of associated continued fractions in terms of Dyck paths. The equivalence of (4) and (5) was shown in the Appendix of [3], where further Catalan polynomial identities were given.

In this paper we generalise the notion of the Catalan polynomial, detailing some properties accordingly and providing for an associated 'limiting function' interpretations in the context of Dyck paths once more, and the historical problem of polygon decomposition.

2 A Generalised Polynomial

The notion of the Catalan polynomial can be naturally extended to the formulation of a *generalised* Catalan polynomial $P_n(x; m)$, say, defined for fixed integer $m \geq 1$ as

$$P_n(x;m) = \sum_{i=0}^{\lfloor \frac{n}{m+1} \rfloor} \binom{n-mi}{i} (-x)^i, \qquad n \geq 0, \tag{6}$$

for which $P_n(x;1) = P_n(x)$ is a special case. Expressed hypergeometrically, we have

$$\begin{aligned}
P_n(x;2) &= {}_3F_2\left(\begin{matrix} -\frac{1}{3}n, -\frac{1}{3}(n-1), -\frac{1}{3}(n-2) \\ -\frac{1}{2}n, -\frac{1}{2}(n-1) \end{matrix} \,\middle|\, \frac{27x}{4} \right), \\
P_n(x;3) &= {}_4F_3\left(\begin{matrix} -\frac{1}{4}n, -\frac{1}{4}(n-1), -\frac{1}{4}(n-2), -\frac{1}{4}(n-3) \\ -\frac{1}{3}n, -\frac{1}{3}(n-1), -\frac{1}{3}(n-2) \end{matrix} \,\middle|\, \frac{256x}{27} \right), \\
P_n(x;4) &= \\
{}_5F_4&\left(\begin{matrix} -\frac{1}{5}n, -\frac{1}{5}(n-1), -\frac{1}{5}(n-2), -\frac{1}{5}(n-3), -\frac{1}{5}(n-4) \\ -\frac{1}{4}n, -\frac{1}{4}(n-1), -\frac{1}{4}(n-2), -\frac{1}{4}(n-3) \end{matrix} \,\middle|\, \frac{3125x}{256} \right), \qquad (7)
\end{aligned}$$

and so on, with a general form

$$P_n(x;m) = {}_{m+1}F_m\left(\begin{matrix} -\alpha_1, -\alpha_2, \ldots, -\alpha_{m+1} \\ -\beta_1, -\beta_2, \ldots, -\beta_m \end{matrix} \,\middle|\, \frac{(m+1)^{m+1}x}{m^m} \right), \tag{8}$$

where $\alpha_1 = \frac{1}{m+1}n$, $\alpha_2 = \frac{1}{m+1}(n-1), \ldots, \alpha_{m+1} = \frac{1}{m+1}(n-m)$, $\beta_1 = \frac{1}{m}n$, $\beta_2 = \frac{1}{m}(n-1), \ldots, \beta_m = \frac{1}{m}(n-(m-1))$.

Examples of this generalised polynomial are as follows:

$$\begin{aligned}
P_0(x;1) &= 1, \\
P_1(x;1) &= 1, \\
P_2(x;1) &= 1-x, \\
P_3(x;1) &= 1-2x, \\
P_4(x;1) &= 1-3x+x^2, \\
P_5(x;1) &= 1-4x+3x^2, \\
P_6(x;1) &= 1-5x+6x^2-x^3, \\
P_7(x;1) &= 1-6x+10x^2-4x^3, \\
P_8(x;1) &= 1-7x+15x^2-10x^3+x^4, \\
P_9(x;1) &= 1-8x+21x^2-20x^3+5x^4, \ldots,
\end{aligned}$$

$$P_0(x;2) = 1,$$

$$
\begin{aligned}
P_1(x;2) &= 1,\\
P_2(x;2) &= 1,\\
P_3(x;2) &= 1-x,\\
P_4(x;2) &= 1-2x,\\
P_5(x;2) &= 1-3x,\\
P_6(x;2) &= 1-4x+x^2,\\
P_7(x;2) &= 1-5x+3x^2,\\
P_8(x;2) &= 1-6x+6x^2,\\
P_9(x;2) &= 1-7x+10x^2-x^3,\ldots,\\
\\
P_0(x;3) &= 1,\\
P_1(x;3) &= 1,\\
P_2(x;3) &= 1,\\
P_3(x;3) &= 1,\\
P_4(x;3) &= 1-x,\\
P_5(x;3) &= 1-2x,\\
P_6(x;3) &= 1-3x,\\
P_7(x;3) &= 1-4x,\\
P_8(x;3) &= 1-5x+x^2,\\
P_9(x;3) &= 1-6x+3x^2,\ldots,\\
\\
P_0(x;4) &= 1,\\
P_1(x;4) &= 1,\\
P_2(x;4) &= 1,\\
P_3(x;4) &= 1,\\
P_4(x;4) &= 1,\\
P_5(x;4) &= 1-x,\\
P_6(x;4) &= 1-2x,\\
P_7(x;4) &= 1-3x,\\
P_8(x;4) &= 1-4x,\\
P_9(x;4) &= 1-5x,\ldots, \qquad (9)
\end{aligned}
$$

etc.

In the next section we give some properties of the generalised polynomial

$P_n(x;m)$ such as we have discovered.

Remark 1 Although we have not defined a generalised Catalan polynomial $P_n(x;0)$, we note that (I.1) (but not (I.2)) holds for $m = 0$, for which $P_n(x;0)$ has the immediate closed form $P_n(x;0) = (1-x)^n$ from (6). Equations (II.1)-(II.3) (see $\ell(x;m)$ defined later) also hold, with $\ell(x;0) = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

3 Mathematical Properties

Properties I: Repeated application of the fundamental second order linear recurrence [1, (69), p.17] satisfied by the Catalan polynomials results in one of order $m+1$ thus:

$$P_{n+m+1}(x;m) = P_{n+m}(x;m) - xP_n(x;m) \qquad n \geq 0, \qquad \text{(I.1)}$$

subject to initial values $P_0(x;m) = P_1(x;m) = \cdots = P_m(x;m) = 1$ (these values are immediate; over the range $n = 0, \ldots, m$, the ratio $n/(m+1) \in [0,1)$ and so here $P_n(x;m) = \binom{n}{0} = 1$). It is a straightforward, though tedious, matter to verify (I.1) using $P_n(x;m)$ as defined by (6), the recurrence leading to the matrix equation

$$\begin{bmatrix} P_n(x;m) \\ -P_{n-1}(x;m) \\ P_{n-2}(x;m) \\ -P_{n-3}(x;m) \\ \vdots \\ (-1)^m P_{n-m}(x;m) \end{bmatrix} = [\mathbf{P}_m(x)]^n \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{(I.2)}$$

with a little work, where (defining $P_{-n}(x;m) = 0$ for $n > 0$)

$$\mathbf{P}_m(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & (-1)^{m+1}x \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{bmatrix} \qquad \text{(I.3)}$$

is an $(m+1)$-square matrix containing a lower left $m \times m$ block entry $-\mathbf{I}_m$ ($\mathbf{I}_m$ being the order m identity matrix). Equation (I.2) holds for $n \geq 0$ being a non-recursive means of generating polynomials $P_n(x;m)$ sequentially in n for fixed $m \geq 1$ a special case of which is $n = 0$, whereby we see that

$$\begin{bmatrix} P_0(x;m) \\ -P_{-1}(x;m) \\ P_{-2}(x;m) \\ -P_{-3}(x;m) \\ \vdots \\ (-1)^m P_{-m}(x;m) \end{bmatrix} = \mathbf{I}_{m+1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{I.4}$$

whence $P_0(x;m) = 1$, $P_{-1}(x;m) = \cdots = P_{-m}(x;m) = 0$; each successive increase in n thereafter releases a newly defined polynomial.

We offer the reader a couple of additional non-trivial examples to illustrate the application of (I.2):

<u>Example 1: $m = 2$, $n = 4$</u>

$$\begin{aligned} \begin{bmatrix} P_4(x;2) \\ -P_3(x;2) \\ P_2(x;2) \end{bmatrix} &= [\mathbf{P}_2(x)]^4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -x \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1-2x & x & -x+x^2 \\ -1+x & -x & x \\ 1 & x & -x \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{I.5}$$

giving $P_2(x;2) = 1$, $P_3(x;2) = 1 - x$, $P_4(x;2) = 1 - 2x$ (see (9)).

<u>Example 2: $m = 4$, $n = 16$</u>

$$\begin{aligned} \begin{bmatrix} P_{16}(x;4) \\ -P_{15}(x;4) \\ P_{14}(x;4) \\ -P_{13}(x;4) \\ P_{12}(x;4) \end{bmatrix} &= [\mathbf{P}_4(x)]^{16} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & -x \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}^{16} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 1-12x+28x^2-4x^3 & \cdots \\ -1+11x-21x^2+x^3 & \cdots \\ 1-10x+15x^2 & \cdots \\ -1+9x-10x^2 & \cdots \\ 1-8x+6x^2 & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{(I.6)}$$

giving $P_{16}(x;4) = 1-12x+28x^2-4x^3$, $P_{15}(x;4) = 1-11x+21x^2-x^3$, $P_{14}(x;4) = 1-10x+15x^2$, $P_{13}(x;4) = 1-9x+10x^2$, $P_{12}(x;4) = 1-8x+6x^2$.

The characteristic equation $0 = \lambda^{m+1} - \lambda^m + x$ asociated with $P_n(x;m)$ is given immediately by the linear recurrence (I.1) (or (I.3) more indirectly by consideration of $|\mathbf{P}_m(x) - \lambda \mathbf{I}_{m+1}|$, see Appendix), and, with x unassigned, proves problematic to solve for roots $\lambda_1^{(m)}, \ldots, \lambda_{m+1}^{(m)}$ for values of m beyond $m = 1$. Whilst $\lambda_{1,2}^{(1)} = (1 \pm \sqrt{1-4x})/2$ trivially (from which the closed form for $P_n(x) = P_n(x;1)$ is easily constructed [1, Section 5.1.1, p.17]), when $m = 2$ the simplest root, for example, within the solution set $\lambda_{1,2,3}^{(2)}$ of zeros to the cubic $\lambda^3 - \lambda^2 + x$ has the awkward form $\{[8-108x+12\sqrt{3x(27x-4)}]^{\frac{2}{3}} + 4 + 2[8-108x+12\sqrt{3x(27x-4)}]^{\frac{1}{3}}\}/6[8-108x+12\sqrt{3x(27x-4)}]^{\frac{1}{3}}$ (with the other two roots more complicated); for $m \geq 3$ the characteristic equation is, for all practical purposes, largely intractable symbolically and so no general closed form for $P_n(x;m)$ is available through this route (see also the Acknowledgement). We can, however, say something further regarding the roots of the characteristic equation:

<u>Lemma</u> *Suppose $x \neq 0, \frac{m^m}{(m+1)^{m+1}}$. Then, for $m \geq 1$, $f_x(\lambda) = \lambda^{m+1} - \lambda^m + x$ has distinct roots.*

<u>Proof</u> We argue by contradiction. The Lemma holds trivially for $m = 1$. Suppose, for $m \geq 2$, $f_x(\lambda)$ has a multiple root α, say. Then we can write

$$f_x(\lambda) = (\lambda - \alpha)^2 \prod_{i=1}^{m-1} (\lambda - \alpha_i), \qquad \text{(L1)}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ are other roots of $f_x(\lambda)$ (one or more of which may co-incide with α). Differentiating,

$$\begin{aligned} f'_x(\lambda) &= \frac{df_x(\lambda)}{d\lambda} \\ &= (\lambda - \alpha)\left[(\lambda-\alpha)\frac{d}{d\lambda}\left\{\prod_{i=1}^{m-1}(\lambda-\alpha_i)\right\} + 2\prod_{i=1}^{m-1}(\lambda-\alpha_i)\right], \qquad \text{(L2)} \end{aligned}$$

so that $\lambda = \alpha$ is a root of both $f_x(\lambda)$ and $f'_x(\lambda)$. Now the actual roots of $f'_x(\lambda)$ are simply the solutions $\lambda_a = 0$ (repeated), $\lambda_b = \frac{m}{m+1}$ of the equation $0 = f'_x(\lambda) = (m+1)\lambda^m - m\lambda^{m-1}$, so in order to achieve the required contradiction it remains but to show that $f_x(\lambda_a)$ and $f_x(\lambda_b)$ are each non-zero. This is trivial: $f_x(\lambda_a) = f_x(0) = x \neq 0$ by assumption, whilst $f_x(\lambda_b) = f_x(\frac{m}{m+1}) = (\frac{m}{m+1})^{m+1} - (\frac{m}{m+1})^m + x = \cdots = x - \frac{m^m}{(m+1)^{m+1}} \neq 0$, again by assumption.□ (See the Appendix for the cases $x = 0, \frac{m^m}{(m+1)^{m+1}}$).

Properties II: For fixed $m \geq 1$, define a function

$$\ell(x;m) = \lim_{n\to\infty} \left\{ \frac{P_n(x;m)}{P_{n+1}(x;m)} \right\}. \tag{II.1}$$

Assuming $\ell(x;m)$ exists, then (I.1) yields easily (reader exercise) that it must satisfy the governing equation

$$0 = 1 - \ell(x;m) + x\ell^{m+1}(x;m), \tag{II.2}$$

whence we can write down its series form

$$\ell(x;m) = \sum_{i=0}^{\infty} \frac{1}{(m+1)i+1} \binom{(m+1)i+1}{i} x^i; \tag{II.3}$$

any equation in $\ell(x;m)$ of the form $\ell = \alpha + x\phi(\ell)$ (for arbitrary constant α and infinitely differentiable function ϕ), of which (II.2) is a special case, lends itself to a series form of solution using the well known technique of Lagrange Inversion (mentioned briefly, for instance, in a paper by Larcombe and Wilson [4, pp.104-105] on the Catalan sequence o.g.f.). Hypergeometric forms are available for $\ell(x;m)$ (with closed forms for $m = 1, 2$), the first few of which are

$$\begin{aligned}
\ell(x;1) &= {}_2F_1\left(\begin{matrix}\frac{1}{2}, 1\\ 2\end{matrix}\,\middle|\, 4x\right) = C(x),\\
\ell(x;2) &= {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ \frac{3}{2}\end{matrix}\,\middle|\, \frac{27x}{4}\right) = \frac{2}{\sqrt{3x}} \sin\left[\frac{1}{3}\arcsin\left(\frac{3\sqrt{3x}}{2}\right)\right],\\
\ell(x;3) &= {}_3F_2\left(\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\\ \frac{2}{3}, \frac{4}{3}\end{matrix}\,\middle|\, \frac{256x}{27}\right)
\end{aligned} \tag{II.4}$$

etc., where, for $m \geq 2$,

$$\ell(x;m) = {}_mF_{m-1}\left(\begin{matrix}\frac{1}{m+1}, \frac{2}{m+1}, \ldots, \frac{m}{m+1}\\ \frac{2}{m}, \frac{3}{m}, \ldots, \frac{m-1}{m}, \frac{m+1}{m}\end{matrix}\,\middle|\, \frac{(m+1)^{m+1}x}{m^m}\right). \tag{II.5}$$

Equations (8),(II.5) are, of course, formulated empirically based on computations (see the Acknowledgement).

Remark 2 Assuming that any solution of (II.2) exists $\in \mathbf{R}[[x]]$ (we have identified a particular solution $\ell(x;m) \in \mathbf{Z}[[x]]$ as given in (II.3)), this solution must be unique, for suppose two Taylor series $f(x), g(x) \in \mathbf{R}[[x]]$ are solutions, $f(x) \neq g(x)$. We can write $0 = 1 - f(x) + xf^{m+1}(x) = 1 - g(x) + xg^{m+1}(x)$, and in turn $0 = f(x) - g(x) - x[f^{m+1}(x) - g^{m+1}(x)] = [f(x) - g(x)][1 - x\Omega(x)]$, where $\Omega(x) = f^m(x) + f^{m-1}(x)g(x) + f^{m-2}(x)g^2(x) + \cdots + f^2(x)g^{m-2}(x) + f(x)g^{m-1}(x) + g^m(x) \in \mathbf{R}[[x]]$. Now since $f(x) - g(x) \neq 0$ then $1 - x\Omega(x) = 0$, and at $x = 0$ this gives a contradiction; hence, $f(x) = g(x)$ and the two solutions co-incide.

Properties III: Instances of the generalised Catalan polynomial appear to have considerable connections with other existing sequences. By way of illustration, we list the first few terms of each of the sequences $\{P_n(x;1)\}_0^\infty$, $\{P_n(x;2)\}_0^\infty$ for all integer values of $x \in [-10, 10]$, along with those named O.E.I.S. sequences described thereby:

Value $m = 1$:

$$
\begin{aligned}
\{P_n(-10;1)\}_0^\infty &= \{1,1,11,21,131,341,1651,5061,21571,72181,287891,\ldots\} \\
&= \text{Generalised Fibonacci Sequence A015446}, \\
\{P_n(-9;1)\}_0^\infty &= \{1,1,10,19,109,280,1261,3781,15130,49159,185329,\ldots\} \\
&= \text{Generalised Fibonacci Sequence A015445}, \\
\{P_n(-8;1)\}_0^\infty &= \{1,1,9,17,89,225,937,2737,10233,32129,113993,\ldots\} \\
&= \text{Generalised Fibonacci Sequence A015443}, \\
\{P_n(-7;1)\}_0^\infty &= \{1,1,8,15,71,176,673,1905,6616,19951,66263,\ldots\} \\
&= \text{Generalised Fibonacci Sequence A015442}, \\
\{P_n(-6;1)\}_0^\infty &= \{1,1,7,13,55,133,463,1261,4039,11605,35839,\ldots\} \\
&= \text{Generalised Fibonacci Sequence A015441}, \\
\{P_n(-5;1)\}_0^\infty &= \{1,1,6,11,41,96,301,781,2286,6191,17621,\ldots\} \\
&= \text{Generalised Fibonacci Sequence A015440}, \\
\{P_n(-4;1)\}_0^\infty &= \{1,1,5,9,29,65,181,441,1165,2929,7589,\ldots\} \\
&= \text{Sequence A006131}, \\
\{P_n(-3;1)\}_0^\infty &= \{1,1,4,7,19,40,97,217,508,1159,2683,\ldots\} \\
&= \text{Sequence A006130}, \\
\{P_n(-2;1)\}_0^\infty &= \{1,1,3,5,11,21,43,85,171,341,683,\ldots\} \\
&= \text{Jacobsthal Sequence A001045}, \\
\{P_n(-1;1)\}_0^\infty &= \{1,1,2,3,5,8,13,21,34,55,89,\ldots\} \\
&= \text{Fibonacci Sequence A000045}, \\
\{P_n(0;1)\}_0^\infty &= \{1,1,1,1,1,1,1,1,1,1,1,\ldots\} \\
&= \text{``All 1s'' Sequence A000012},
\end{aligned}
$$

$$
\begin{aligned}
\{P_n(1;1)\}_0^\infty &= \{1,1,0,-1,-1,0,1,1,0,-1,-1,\ldots\} \\
&= \text{Inverse of 6th Cyclotomic Polynomial Sequence A010892}, \\
\{P_n(2;1)\}_0^\infty &= \{1,1,-1,-3,-1,5,7,-3,-17,-11,23,\ldots\} \\
&= \text{Lucas/Lehmer Sequence A107920}, \\
\{P_n(3;1)\}_0^\infty &= \{1,1,-2,-5,1,16,13,-35,-74,31,253,\ldots\} \\
&= \text{Sequence A106852}, \\
\{P_n(4;1)\}_0^\infty &= \{1,1,-3,-7,5,33,13,-119,-171,305,989,\ldots\} \\
&= \text{Sequence A106853}, \\
\{P_n(5;1)\}_0^\infty &= \{1,1,-4,-9,11,56,1,-279,-284,1111,2531,\ldots\} \\
&= \text{Sequence A106854}, \\
\{P_n(6;1)\}_0^\infty &= \{1,1,-5,-11,19,85,-29,-539,-365,2869,5059,\ldots\}, \\
\{P_n(7;1)\}_0^\infty &= \{1,1,-6,-13,29,120,-83,-923,-342,6119,8513,\ldots\}, \\
\{P_n(8;1)\}_0^\infty &= \{1,1,-7,-15,41,161,-167,-1455,-119,11521,12473,\ldots\}, \\
\{P_n(9;1)\}_0^\infty &= \{1,1,-8,-17,55,208,-287,-2159,424,19855,16039,\ldots\}, \\
\{P_n(10;1)\}_0^\infty &= \{1,1,-9,-19,71,261,-449,-3059,1431,32021,17711,\ldots\}, \\
&\vdots
\end{aligned} \tag{III.1}
$$

<u>Value $m = 2$</u> :

$$
\begin{aligned}
\{P_n(-10;2)\}_0^\infty &= \{1,1,1,11,21,31,141,351,661,2071,5581,\ldots\}, \\
\{P_n(-9;2)\}_0^\infty &= \{1,1,1,10,19,28,118,289,541,1603,4204,\ldots\}, \\
\{P_n(-8;2)\}_0^\infty &= \{1,1,1,9,17,25,97,233,433,1209,3073,\ldots\}, \\
\{P_n(-7;2)\}_0^\infty &= \{1,1,1,8,15,22,78,183,337,883,2164,\ldots\}, \\
\{P_n(-6;2)\}_0^\infty &= \{1,1,1,7,13,19,61,139,253,619,1453,\ldots\}, \\
\{P_n(-5;2)\}_0^\infty &= \{1,1,1,6,11,16,46,101,181,411,916,\ldots\}, \\
\{P_n(-4;2)\}_0^\infty &= \{1,1,1,5,9,13,33,69,121,253,529,\ldots\} \\
&= \text{Sequence A089977}, \\
\{P_n(-3;2)\}_0^\infty &= \{1,1,1,4,7,10,22,43,73,139,268,\ldots\} \\
&= \text{“Number of Pairs of Rabbits\ldots” Sequence A084386}, \\
\{P_n(-2;2)\}_0^\infty &= \{1,1,1,3,5,7,13,23,37,63,109,\ldots\} \\
&= \text{Sequence A077949}, \\
\{P_n(-1;2)\}_0^\infty &= \{1,1,1,2,3,4,6,9,13,19,28,\ldots\} \\
&= \text{Sequence A000930}, \\
\{P_n(0;2)\}_0^\infty &= \{1,1,1,1,1,1,1,1,1,1,1,\ldots\} \\
&= \text{“All 1s” Sequence A000012}, \\
\{P_n(1;2)\}_0^\infty &= \{1,1,1,0,-1,-2,-2,-1,1,3,4,\ldots\}
\end{aligned}
$$

$$
\begin{aligned}
&= \text{Sequence A050935}, \\
\{P_n(2;2)\}_0^\infty &= \{1,1,1,-1,-3,-5,-3,3,13,19,13,\ldots\} \\
&= \text{Sequence A077950}, \\
\{P_n(3;2)\}_0^\infty &= \{1,1,1,-2,-5,-8,-2,13,37,43,4,\ldots\}, \\
\{P_n(4;2)\}_0^\infty &= \{1,1,1,-3,-7,-11,1,29,73,69,-47,\ldots\}, \\
\{P_n(5;2)\}_0^\infty &= \{1,1,1,-4,-9,-14,6,51,121,91,-164,\ldots\}, \\
\{P_n(6;2)\}_0^\infty &= \{1,1,1,-5,-11,-17,13,79,181,103,-371,\ldots\}, \\
\{P_n(7;2)\}_0^\infty &= \{1,1,1,-6,-13,-20,22,113,253,99,-692,\ldots\}, \\
\{P_n(8;2)\}_0^\infty &= \{1,1,1,-7,-15,-23,33,153,337,73,-1151,\ldots\}, \\
\{P_n(9;2)\}_0^\infty &= \{1,1,1,-8,-17,-26,46,199,433,19,-1772,\ldots\}, \\
\{P_n(10;2)\}_0^\infty &= \{1,1,1,-9,-19,-29,61,251,541,-69,-2579,\ldots\}, \\
&\vdots
\end{aligned} \tag{III.2}
$$

Not surprisingly, overlaps with other existing sequences occur for further m instances of $P_n(x;m)$, but these are not listed here for reason of brevity.

4 Interpretations

4.1 Polygon Partitioning

Equations (II.2),(II.3) are known, having a basic connection with the historical problem of polygon decomposition (it is so called triangularisation that is the special case through which the Catalan numbers are commonly interpreted combinatorially). They have in this context been discussed in detail in [4, (A1),(A2),(A6), pp.104-105] and, accordingly, we may write as follows: $[x^i]\{\ell(x;m)\} = \frac{1}{(m+1)i+1}\binom{(m+1)i+1}{i} = \frac{1}{mi+1}\binom{(m+1)i}{i}$ (*i.e.*, the coefficient of the term x^i in $\ell(x;m)$) being an instance of what is sometimes referred to in the literature as a 'generalised' or 'higher' Catalan number - -*is the number of ways to internally partition, or subdivide, an* $(mi+2)$*-gon into* i $(m+2)$*-gons by* $i-1$ *diagonals* [4, (A7), p.105]; $m=1$ describes the famous decomposition of an $(i+2)$-gon into i triangles by $i-1$ diagonals, enumerated by the Catalan number c_i (and in this case $\ell(x;1) = \sum_{i=0}^{\infty}\frac{1}{i+1}\binom{2i}{i}x^i = C(x)$, as we have already noted). As an aside, the reader is also directed to [5, (22), p.198] (see the Further Remarks section thereof), where the expression $\frac{1}{(m+1)i+1}\binom{(m+1)i+1}{i}$ can be linked to what is known as a Raney sequence and corresponds to a particular "Fuss-Catalan" number, the terminology having been used elsewhere (for related background material on this point see [4], and also its forerunner article [6] in which an attempt was made to set out the full history of the Catalan sequence).

4.2 Dyck Paths

A so called m-Dyck path of order i is a 2D lattice path from the origin $(0,0)$ to the point (mi, i) which never goes below the main diagonal $\{(ms, s) : 0 \leq s \leq i\}$ using combinations of steps $(0,1)$ (north) and $(1,0)$ (east). From the popular reference work of Hilton and Pedersen [7] we have the immediate observation that $\frac{1}{mi+1}\binom{(m+1)i}{i}$ *is the number of (order i) m-Dyck paths.*

5 Summary

Based on previous work involving the authors, a generalised type of polynomial has been presented. Containing the Catalan polynomials as a particular class,[1] this generalised polynomial has some interesting mathematical properties of its own, allied to a couple of associated combinatorial interpretations. It remains to be seen whether or not any non-linear identities can be formulated for $P_n(x;m)$ such as those isolated ones observed in [1, Section 5.1.5, pp.21-22], or results developed in [2,3] and elsewhere, for the Catalan polynomials.

We finish the paper with one or two remarks.

Final Remarks In a 1973 publication Riordan [12] introduced a set of polynomials in re-visiting the classic bracketing problem of Catalan (counting so called "clutches of nests"). These are termed "Catalan polynomials" in the text by Koshy ([13, p.321]; Touchard's well known reccurrence formula for the Catalan numbers is also recovered) but, for the record, we emphasise that they are not the same polynomials as ours. Finally, for completeness, the interested reader is directed to some additional results for Catalan polynomials [14,15]; these have been given by the authors as part of those for a wider class of polynomial families of which the Catalan polynomials whose generalised form is considered here — are but one instance (others are, for example, classes of polynomials termed (Large) Schröder and Motzkin polynomials (so named after their namesake sequences from which they are derived through their governing o.g.f.), neither of which have a generalised version at this moment in time).

[1] For completeness we note that prior to the 2008 and 2009 publications [1-3] the Catalan polynomials arose originally in work by W. Lang [8] (see O.E.I.S. Sequence No. A115139). We also remark here that Catalan polynomials have appeared in (1.70),(1.71) of Gould's 1972 listing [9, p.9], though with no context which relates them to the Catalan sequence; a couple of further results loosely relevant to the generalised Catalan polynomials are (1.120),(1.121) on p.15. Note that properties of the Catalan polynomials are explored further in other papers [10,11] appearing in this *Bulletin* Special Issue.

6 Acknowledgement

The authors would like to thank Prof. Dr. Wolfram Koepf for undertaking computations (using his customised software package "hsum9.mpl") which led to the empiric formulations (8),(II.5). For arbitrary m the summand in the binomial sum form of $P_n(x;m)$ (6) is not a hypergeometric term so that a general closed form does not exist. For individual integer $m \geq 1$ this is not the case, but there are still no closed forms in these instances other than for $m = 1$ where the software returns that established for $P_n(x;1) = P_n(x)$ [1, (70), p.17].

Appendix

Here we derive the characteristic equation associated with $P_n(x;m)$ mentioned in Properties I. Consider, with $\mathbf{P}_m(x)$ defined in (I.3), its characteristic polynomial

$$\begin{aligned} |\mathbf{P}_m(x) - \lambda \mathbf{I}_{m+1}| &= (1-\lambda)|A(\lambda)| - 0 + 0 - 0 + \cdots \\ &\quad \cdots + (-1)^{m+2} \cdot (-1)^{m+1} x |B(\lambda)| \qquad \text{(A1)} \end{aligned}$$

(only two signed minors (*i.e.*, cofactors) of $\mathbf{P}_m(x) - \lambda \mathbf{I}_{m+1}$, expanding the determinant along the top row, are non-zero and contribute), where

$$A(\lambda) = \begin{bmatrix} -\lambda & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -\lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & -\lambda \end{bmatrix},$$

$$B(\lambda) = \begin{bmatrix} -1 & -\lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & -\lambda & \cdots & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}, \qquad \text{(A2)}$$

are respective lower and upper triangular $m \times m$ matrices (albeit sparse ones). The determinant of an upper or a lower triangular matrix is merely the product of its diagonal terms, whence we proceed readily from (A1) as follows:

$$\begin{aligned} |\mathbf{P}_m(x) - \lambda \mathbf{I}_{m+1}| &= (1-\lambda)|A(\lambda)| - x|B(\lambda)| \\ &= (1-\lambda)(-\lambda)^m - x(-1)^m \end{aligned}$$

$$
\begin{aligned}
&= (-1)^m[\lambda^m(1-\lambda) - x] \\
&= (-1)^{m+1}(\lambda^{m+1} - \lambda^m + x), \qquad \text{(A3)}
\end{aligned}
$$

with characteristic equation $0 = \lambda^{m+1} - \lambda^m + x$.

The Lemma says nothing about the zeros of the characteristic equation (*i.e.*, roots of $f_x(\lambda) = \lambda^{m+1} - \lambda^m + x$) when x takes the two specific values $x = 0, \frac{m^m}{(m+1)^{m+1}}$. We can, however, deal with these and we do so here in brief for completeness:

Case $x = x(m) = m^m/(m+1)^{m+1}$

Experimental computations suggest that for any $m \geq 1$ then $f_{x(m)}(\lambda)$ possesses a sole *repeated* root[2] $\lambda = m/(m+1)$, together with either (a) $m-1$ complex pairs of roots if $m = 1, 3, 5, 7, \ldots$, or (b) $m-2$ complex pairs of roots, plus an additional (negative) distinct real root, if $m = 2, 4, 6, 8, \ldots$. Thus, we can seek a closed form for the (degree $m-1$) polynomial

$$
f^*_{x(m)}(\lambda) = \frac{f_{x(m)}(\lambda)}{\left(\lambda - \frac{m}{m+1}\right)^2} \qquad \text{(A4)}
$$

which delivers the remaining $m-1$ roots of $f_{x(m)}(\lambda)$ according to (a) or (b), and, with some work, we find a general closed form

$$
f^*_{x(m)}(\lambda) = \sum_{i=0}^{m-1} \frac{i+1}{m}\left(\frac{m}{m+1}\right)^{m-1-i}\lambda^i, \qquad \text{(A5)}
$$

verification of which we leave as a reader exercise.

Case $x = 0$

This is trivial, for the characteristic equation reads $0 = f_0(\lambda) = \lambda^m(\lambda - 1)$, with m repeated roots of zero and a single distinct root of unity.

References

[1] Clapperton, J.A., Larcombe, P.J. and Fennessey, E.J. (2008). On iterated generating functions for integer sequences, and Catalan polynomials, *Util. Math.*, **77**, pp.3-33.

[2] Clapperton, J.A., Larcombe, P.J., Fennessey, E.J. and Levrie, P. (2008). A class of auto-identities for Catalan polynomials, and Padé approximation, *Cong. Num.*, **189**, pp.77-95.

[2]Of course it is immediate (see the proof of the Lemma), that $f_{x(m)}(\frac{m}{m+1}) = 0$, but it is not obvious that $\lambda = m/(m+1)$ is a *double* root.

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[3] Clapperton, J.A., Larcombe, P.J. and Fennessey, E.J. (2009). Some new identities for Catalan polynomials, *Util. Math.*, **80**, pp.3-10.

[4] Larcombe, P.J. and Wilson, P.D.C. (2001). On the generating function of the Catalan sequence: a historical perspective, *Cong. Num.*, **149**, pp.97-108.

[5] Larcombe, P.J. and French, D.R. (2003). The Catalan number k-fold self-convolution identity: the original formulation, *J. Comb. Math. Comb. Comp.*, **46**, pp.191-204.

[6] Larcombe, P.J. and Wilson, P.D.C. (1998). On the trail of the Catalan sequence, *Math. Today*, **34**, pp.114-117.

[7] Hilton, P. and Pedersen, J. (1991). Catalan numbers, their generalization, and their uses, *The Math. Intell.*, **13**(2), pp.64-75.

[8] Lang, W. (2000). On polynomials related to powers of the generating function of Catalan's numbers, *Fib. Quart.*, **38**, pp.408-419.

[9] Gould, H.W. (1972). Combinatorial identities, Rev. Ed., University of West Virginia, U.S.A.

[10] Jarvis, A.F., Larcombe, P.J. and Fennessey, E.J. (2014). Some factorisation and divisibility properties of Catalan polynomials, *Bull. I.C.A.*, to appear (this issue).

[11] Larcombe, P.J. and Fennessey, E.J. (2014). On cyclicity and density of some Catalan polynomial sequences, *Bull. I.C.A.*, to appear (this issue).

[12] Riordan, J. (1973). A note on Catalan parentheses, *Amer. Math. Month.*, **80**, pp.904-906.

[13] Koshy, T. (2009). Catalan numbers with applications, Oxford University Press, New York, U.S.A.

[14] Clapperton, J.A., Larcombe, P.J. and Fennessey, E.J. (2010). New theory and results from an algebraic application of Householder root finding schemes, *Util. Math.*, **83**, pp.3-36.

[15] Clapperton, J.A., Larcombe, P.J. and Fennessey, E.J. (2011). Two new identities for polynomial families, *Bull. I.C.A.*, **62**, pp.25-32.

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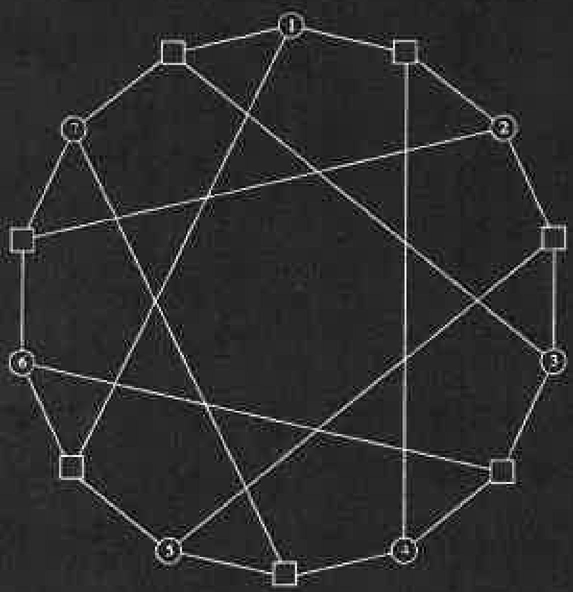

Bulletin of the ICA, Volume 71, May, 2014
Table of Contents

Commemorating 200 Years Since the Birth of Eugène Charles Catalan

Guest Editor
Peter J. Larcombe

Dedication

This Special May 2014 Bulletin Issue is Dedicated to the Memory of
David R. French ('Frenchy')
1943—2014

The Catalan sequence has an almost unparalleled ubiquity in discrete mathematics, arising as, or in, the solution of a wide variety of apparently disparate and unconnected counting problems. Throughout the major part of the 19th century the accepted version of its discovery linked the initial identification of the sequence to Leonhard Euler, who in 1751 wrote of its elements as providing solutions to the so called ***triangulated decompositions of polygons***—a problem which is today well known and through which the Catalan sequence was to eventually bear the name of Catalan himself, seemingly after a flurry of activity (by Catalan and some contemporaries) during the 1830s and 1840s. This false attribution (and others) continued until 1988 when a Chinese historian, J. Luo, detailed a new context as evidence of an even earlier awareness of the Catalan sequence by the scholar Antu Ming (who during the first half of the 1700s examined, via geometrical considerations, a certain type of infinite series containing Catalan numbers).

From such beginnings well over 250 years ago, the Catalan sequence has continued to make regular appearances in the literature—sometimes in surprising ways—whilst the Catalan numbers have interesting mathematical properties in their own right which link with other integer sequences. My own personal interest in the Catalan sequence took off when it arose in an enumeration problem on which I was working with an undergraduate final year student in the mid 1990s (strangely, it took many years for this work to be disseminated), and—after the assimilation and translation of the relevant material—I wrote, and co-wrote, a series of short pieces on the origins of the Catalan sequence in an attempt to clarify that part of its history. Since then both Catalan and the Catalan numbers have at times figured in my work, most recently through the so called Catalan polynomials which I discovered with a Ph.D. student (James Clapperton) and great friend Dr. Eric Fennessey (in our study of iterated generating functions) and which form the basis of my joint contributions to this Special Issue. I am, of course, not alone in my Catalan-related pursuits. Professor Richard P. Stanley, for instance, has aptly termed an extreme enthusiasm for all matters Catalan as "Catalania" ("Catalan mania"), a 'condition' whose 'sufferers' will undoubtedly recognise! Richard himself keeps a wonderful Catalan Addendum to Volume 2 of his well known book *Enumerative Combinatorics* active as an up-to-date resource for researchers in which he details new interpretations and problems, and Professor Thomas Koshy has been moved to write a stand alone undergraduate text *Catalan Numbers with Applications* for a less specialised readership (see overleaf for more details on these books). Each, in its own particular way, serves the mathematical community well, along with the numerous articles which have, over the years, formed a substantial body of work on the Catalan sequence and secured its place at the forefront of the world of integer sequences.

One wonders what Catalan—who as well as being politically active was quite eclectic in his mathematical endeavours—would have made of the way the sequence has captivated academics eager to understand its fundamental nature and application; certainly, it is testimony to the importance of the Catalan numbers that so many people, at all academic levels, continue to develop and often retain an interest in them, and there is no sign of this ending. It is, therefore, a great pleasure to write this Foreword in my capacity as Guest Editor, as the I.C.A. formally celebrates both the significant and longstanding impact of the Catalan sequence within discrete mathematics. The invited contributions on offer here are as varied as they are interesting, forming a timely and fitting tribute to Catalan and the Catalan sequence.

Enjoy !

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Major Contributions to the Literature on Catalan Numbers by Stanley and Koshy

R.P. Stanley (1999). "Enumerative Combinatorics", Volume 2 (Cambridge Studies in Advanced Mathematics No. 62), Cambridge University Press, Cambridge, U.K.
Some useful background information on the Catalan numbers (with references) appears in the *Notes* section at the end of Chapter 6 of this advanced level book, with the subsequent Exercises 6.19 offering a number of combinatorial illustrations. Stanley continues to update the original presentation in the textbook with an "EC2 Supplement" (available from his M.I.T. homepage) which contains errata, updates a nd new material. In addition, a "Catalan Addendum" offers new problems related to Catalan numbers, with solutions, reflecting his deep and enduring interest in them and a determination to see them disseminated; Catalan interpretations in the Addendum currently stand at over 200 in number, the collation of which is a commendable achievement on the part of Stanley.

T. Koshy (2009). "Catalan Numbers with Applications", Oxford University Press, New York, U.S.A.
Koshy's text is aimed at a broad readership (of mathematical amateurs, high school students/teachers, and both undergraduate and postgraduate level students), in which he pulls together and catalogues many different aspects of the Catalan sequence and its numerous contexts. The book—as the author rightly states—is the first to collect and present an orderly treatise on the various occurrences, applications and properties of the Catalan numbers, and Koshy draws on a multitude of reference material to create a very useful resource.

<u>Some Other Works of Note on Catalan</u>

In 1996 the Société Belge des Professeurs de Mathématique d'Expression Française (Mons, Belgium) published "Eugène Catalan: Géomètre sans Patrie, Républicain sans République", a 200+ page book by F. Jongmans on the life and work of Catalan. [Prior to this, and as a precursor, the author had contributed a chapter (Chapter 3, pp.23-41) with the same title in a publication "Regards Sur 175 Ans de Science à l'Université de Liège 1817-1992" (Ed. A.-C. Bernès) which was produced in 1992 under the auspices of the University's Centre d'Histoire des Sciences et des Techniques to mark this period of general scientific activity at the university.]

Other works of note are the articles "Eugène Catalan and the Rise of Russian Science" (*Acad. Roy. Belg. Bull. Class. Sci.*, **2** (1991), pp.59-90) by P.L. Butzer and F. Jongmans, "Les Relations Épistolaires Entre Eugène Catalan et Ernesto Cesàro" (*ibid.*, **10** (1999), pp.223-271) by Butzer *et al.*, and "Quelques Pièces Choisies dans la Correspondance d'Eugène Catalan" (*Bull. Soc. Roy. Sci. Liège*, **50** (1981), pp.287-309) by Jongmans. All bar the final reference are predated by about a century by P. Mansion's "Notice sur les Travaux Mathématiques de Eugène-Charles Catalan" which appeared in *Ann. l'Acad. Roy. Sci. Lett. Beaux-Arts Belg.* in 1896 (**62**, pp.115-172).