An Equivalent Property of a Hilbert-Type Integral Inequality and Its Applications

B. Yang, D. Andrica, O. Bagdasar, and M. Th. Rassias

Abstract Making use of complex analytic techniques as well as methods involving weight functions, we study a few equivalent conditions of a Hilbert-type integral inequality with nonhomogeneous kernel and parameters. As applications we deduce a few equivalent conditions of a Hilbert-type integral inequality with homogeneous kernel, and we also consider operator expressions.

Key words Hilbert-type integral inequality; weight function; equivalent form; operator; norm

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1 Introduction

In 1925, Hardy [1] presented the following result, which is currently known in the literature as the classic Hardy-Hilbert integral inequality. This states that for the positive real numbers p,q with $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, and functions $f(x),g(y)\geq 0$, with

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$$0 < \int_0^\infty f^p(x)dx < \infty \text{ and } 0 < \int_0^\infty g^q(y)dy < \infty,$$

we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \quad (1)$$

where the constant factor

$$\frac{\pi}{\sin(\pi/p)}$$

is the best possible.

For $p = q = 2$, (1) recovers the well known Hilbert integral inequality. Both (1), as well as Hilbert's integral inequality play an important role in analysis and its applications (cf. [2], [3]).

In 1934, Hardy et al. established the following extension of (1):
If $k_1(x,y)$ is a nonnegative homogeneous function of degree -1, and one defines

$$k_p = \int_0^\infty k_1(u,1)u^{-\frac{1}{p}}du \in \mathbb{R}_+ := (0,\infty),$$

then we have the following Hardy-Hilbert-type integral inequality:

$$\int_0^\infty \int_0^\infty k_1(x,y)f(x)g(y)dxdy < k_p \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \quad (2)$$

where the constant factor k_p is the best possible (cf. [2], Theorem 319).

Additionally, the following Hilbert-type integral inequality with nonhomogeneous kernel is proved:
If $h(u) > 0, \phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in \mathbb{R}_+$, then

$$\int_0^\infty \int_0^\infty h(xy)f(x)g(y)dxdy < \phi\left(\frac{1}{p}\right) \left(\int_0^\infty x^{p-2}f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, (3)$$

where the constant factor $\phi\left(\frac{1}{p}\right)$ is the best possible (cf. [2], Theorem 350).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang established an extension of Hilbert's integral inequality, namely the following (cf. [4], [5]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) \left(\int_0^\infty x^{1-\lambda}f^2(x)dx \int_0^\infty y^{1-\lambda}g^2(y)dy\right)^{\frac{1}{2}}, \quad (4)$$

where the constant factor $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ is the best possible ($B(u,v)$ is the beta function).

In 2004, by introducing two pairs of conjugate exponents (p,q) and (r,s), Yang [6] proved the following extension of (1):
If $\lambda > 0$, $p,r > 1$, $\frac{1}{p}+\frac{1}{q} = \frac{1}{r}+\frac{1}{s} = 1$, and $f(x), g(y) \geq 0$, satisfy

$$0<\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx<\infty \ and \ 0<\int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy<\infty,$$

then we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy$$
$$<\frac{\pi}{\lambda\sin(\pi/r)}\left[\int_0^\infty x^{p\left(1-\frac{\lambda}{r}\right)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q\left(1-\frac{\lambda}{s}\right)-1}g^q(y)dy\right]^{\frac{1}{q}}, \quad (5)$$

where the constant factor

$$\frac{\pi}{\lambda\sin(\pi/r)}$$

is the best possible. For $\lambda=1, r=q, s=p$, (5) reduces to (1).

In 2005, the paper [7] also provided an extension of (1) and (4) with the kernel $\frac{1}{(x+y)^\lambda}$ and two pairs of conjugate exponents. Krnić et al. [8]-[16] proved some extensions and particular cases of (1), (2) and (3) with parameters. In 2009, Yang established an extension of (2) and (5), namely the following (cf. [17], [19]):
If $\lambda_1+\lambda_2=\lambda\in\mathbb{R}$, $k_\lambda(x,y)$ is a nonnegative homogeneous function of degree $-\lambda$, satisfying

$$k_\lambda(ux,uy)=u^{-\lambda}k_\lambda(x,y)\ (u,x,y>0),$$

and

$$k(\lambda_1)=\int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du\in\mathbb{R}_+,$$

then we have

$$\int_0^\infty\int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy$$
$$<k(\lambda_1)\left[\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\lambda_2)-1}g^q(y)dy\right]^{\frac{1}{q}}, \quad (6)$$

where the constant factor $k(\lambda_1)$ is the best possible.
For $\lambda=1$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$, (6) reduces to (2), while for $\lambda>0$, $\lambda_1=\frac{\lambda}{r}$, $\lambda_2=\frac{\lambda}{s}$, $k_\lambda(x,y)=\frac{1}{x^\lambda+y^\lambda}$, (6) reduces to (5).

Additionally, the following extension of (3) was proved:

$$\int_0^\infty\int_0^\infty h(xy)f(x)g(y)dxdy$$
$$<\phi(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \quad (7)$$

where the constant factor $\phi(\sigma)$ is the best possible (cf. [18]).
For $\sigma=\frac{1}{p}$, (7) reduces to (3). Some equivalent inequalities of (6) and (7) were constructed by [19]. In 2013, Yang [18] also studied the equivalence of (6) and (7) by adding a condition $h(u)=k_\lambda(u,1)$. In 2017, Hong [20] studied an equivalent

condition for (6) involving certain parameters, and some further related results were given in [21]-[25].

In the present paper, making use of complex analytic techniques as well as methods involving weight functions, we study a few equivalent conditions of a Hilbert-type integral inequality with the nonhomogeneous kernel

$$\frac{1}{\prod_{k=1}^{s}[(xy)^{\lambda}+c_k]}\ (c_k>0)$$

and a best possible constant factor. In the form of applications we deduce a few equivalent conditions of a Hilbert-type integral inequality with homogeneous kernel. We also consider operator expressions.

2 Some lemmas

Lemma 1. *(cf. [26]) If $\mathbb{C}$ is the set of complex numbers and $\mathbb{C}_\infty=\mathbb{C}\cup\{\infty\}$,*

$$z_k\in\mathbb{C}\backslash\{z\,|\,Re(z)\geq 0,\ \ Im(z)=0\}\ \ (k=1,2,\ldots,n)$$

are different points, the function $f(z)$ is analytic in $\mathbb{C}_\infty$ except for z_i $(i=1,2,\ldots,n)$, and $z=\infty$ is a zero point of $f(z)$ whose order is not less than 1, then for $\alpha\in\mathbb{R}$, we have

$$\int_0^\infty f(x)x^{\alpha-1}dx=\frac{2\pi i}{1-e^{2\pi\alpha i}}\sum_{k=1}^{n}Re(s)[f(z)z^{\alpha-1},z_k], \tag{8}$$

where $0<Im(\ln z)=\arg z<2\pi$. In particular, if z_k $(k=1,\ldots,n)$ are all poles of order 1, setting

$$\varphi_k(z)=(z-z_k)f(z)\ (\varphi_k(z_k)\neq 0),$$

then

$$\int_0^\infty f(x)x^{\alpha-1}dx=\frac{\pi}{\sin\pi\alpha}\sum_{k=1}^{n}(-z_k)^{\alpha-1}\varphi_k(z_k). \tag{9}$$

Example 1. For $s\in\mathbb{N}=\{1,2,\ldots\}$ and $0<c_1\leq\cdots\leq c_s$, $0<\sigma<s\lambda,\varepsilon>0$, we set

$$h(u):=\frac{1}{\prod_{k=1}^{s}(u^{\lambda}+c_k)},\quad (u>0),$$

and

$$\widetilde{c}_k=c_k+(k-1)\varepsilon\ (k=1,\ldots,s).$$

By (9), for $z_k=-\widetilde{c}_k$, we derive that

$$
\begin{aligned}
\widetilde{k}_s(\sigma) &= \int_0^\infty \frac{1}{\prod_{k=1}^s (t^\lambda + \widetilde{c}_k)} t^{\sigma-1} dt \\
&= \frac{1}{\lambda} \int_0^\infty \frac{1}{\prod_{k=1}^s (u + \widetilde{c}_k)} u^{\frac{\sigma}{\lambda}-1} du \\
&= \frac{\pi}{\lambda \sin \frac{\pi\sigma}{\lambda}} \sum_{k=1}^s \frac{\widetilde{c}_k^{\frac{\sigma}{\lambda}-1}}{\prod_{j=1(j\neq k)}^s (\widetilde{c}_j - \widetilde{c}_k)}.
\end{aligned}
$$

Setting $\mu = s\lambda - \sigma (> 0)$, we obtain that

$$
\begin{aligned}
0 < \widetilde{k}_s(\sigma) &= \frac{1}{\lambda} \int_0^\infty \frac{1}{\prod_{k=1}^s (u + \widetilde{c}_k)} u^{\frac{\sigma}{\lambda}-1} du \\
&\leq \frac{1}{\lambda} \int_0^\infty \frac{1}{(u + c_1)^s} u^{\frac{\sigma}{\lambda}-1} du \\
&= \frac{1}{\lambda c_1^{\mu/\lambda}} \int_0^\infty \frac{1}{(v+1)^s} v^{\frac{\sigma}{\lambda}-1} dv \\
&= \frac{1}{\lambda c_1^{\mu/\lambda}} B\left(\frac{\sigma}{\lambda}, \frac{\mu}{\lambda}\right) < \infty,
\end{aligned}
$$

and by Levi's theorem (cf. [28]), it follows that

$$
\begin{aligned}
k_s(\sigma) &= \int_0^\infty \frac{t^{\sigma-1}}{\prod_{k=1}^s (t^\lambda + c_k)} dt = \lim_{\varepsilon \to 0^+} \int_0^\infty \frac{t^{\sigma-1}}{\prod_{k=1}^s (t^\lambda + \widetilde{c}_k)} dt \\
&= \lim_{\varepsilon \to 0^+} \widetilde{k}_s(\sigma) = \frac{\pi}{\lambda \sin \frac{\pi\sigma}{\lambda}} \sum_{k=1}^s \frac{c_k^{\frac{\sigma}{\lambda}-1}}{\prod_{j=1(j\neq k)}^s (c_j - c_k)} \in \mathbb{R}_+. \qquad (10)
\end{aligned}
$$

In particular:
(i) for $s = 1$, we obtain

$$
k_1(\sigma) = \frac{1}{\lambda} \int_0^\infty \frac{u^{(\sigma/\lambda)-1}}{u + c_1} du = \frac{\pi}{\lambda c_1^{\mu/\lambda} \sin\left(\frac{\pi\sigma}{\lambda}\right)};
$$

(ii) for $s = 2$, we get that

$$
\begin{aligned}
k_2(\sigma) &= \int_0^\infty \frac{1}{(t^\lambda + c_1)(t^\lambda + c_2)} t^{\sigma-1} dt \\
&= \frac{\pi}{\lambda \sin \frac{\pi\sigma}{\lambda}} \frac{c_1^{\frac{\sigma}{\lambda}-1} - c_2^{\frac{\sigma}{\lambda}-1}}{c_2 - c_1};
\end{aligned}
$$

(iii) for $c_s = \cdots = c_1$ in (10), we have

$$
k(\sigma) := \int_0^\infty \frac{t^{\sigma-1}}{(t^\lambda + c_1)^s} dt = \frac{s}{\lambda c_1^{\mu/\lambda}} B\left(\frac{\sigma}{\lambda}, \frac{\mu}{\lambda}\right).
$$

If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $s\in\mathbb{N}$, $0<c_1\leq\cdots\leq c_s$, $0<\sigma<s\lambda$, $\sigma_1\in\mathbb{R}$, then for $n\in\mathbb{N}$, we define the following two expressions:

$$I_1:=\int_1^\infty\left\{\int_0^1\frac{1}{\prod_{k=1}^s[(xy)^\lambda+c_k]}x^{\sigma+\frac{1}{pn}-1}dx\right\}y^{\sigma_1-\frac{1}{qn}-1}dy, \tag{11}$$

$$I_2:=\int_0^1\left\{\int_1^\infty\frac{1}{\prod_{k=1}^s[(xy)^\lambda+c_k]}x^{\sigma-\frac{1}{pn}-1}dx\right\}y^{\sigma_1+\frac{1}{qn}-1}dy. \tag{12}$$

Setting $u=xy$ in (11) and (12), by Fubini's theorem (cf. [28]), we obtain

$$\begin{aligned}
I_1 &= \int_1^\infty\left[\int_0^y\frac{1}{\prod_{k=1}^s(u^\lambda+c_k)}\left(\frac{u}{y}\right)^{\sigma+\frac{1}{pn}-1}\frac{1}{y}du\right]y^{\sigma_1-\frac{1}{qn}-1}dy\\
&= \int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}\left[\int_0^y\frac{1}{\prod_{k=1}^s(u^\lambda+c_k)}u^{\sigma+\frac{1}{pn}-1}du\right]dy\\
&= \int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}dy\int_0^1\frac{1}{\prod_{k=1}^s(u^\lambda+c_k)}u^{\sigma+\frac{1}{pn}-1}du\\
&\quad+\int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}\int_1^y\frac{1}{\prod_{k=1}^s(u^\lambda+c_k)}u^{\sigma+\frac{1}{pn}-1}dudy\\
&= \int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}dy\int_0^1\frac{1}{\prod_{k=1}^s(u^\lambda+c_k)}u^{\sigma+\frac{1}{pn}-1}du\\
&\quad+\int_1^\infty\left[\int_u^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}dy\right]\frac{1}{\prod_{k=1}^s(u^\lambda+c_k)}u^{\sigma+\frac{1}{pn}-1}du,
\end{aligned} \tag{13}$$

$$\begin{aligned}
I_2 &= \int_0^1\left\{\int_y^\infty\frac{1}{\prod_{k=1}^s(u^\lambda+c_k)}\left(\frac{u}{y}\right)^{\sigma-\frac{1}{pn}-1}\frac{1}{y}du\right\}y^{\sigma_1+\frac{1}{qn}-1}dy\\
&= \int_0^1 y^{(\sigma_1-\sigma)+\frac{1}{n}-1}\left[\int_y^\infty\frac{1}{\prod_{k=1}^s(u^\lambda+c_k)}u^{\sigma-\frac{1}{pn}-1}du\right]dy\\
&= \int_0^1 y^{(\sigma_1-\sigma)+\frac{1}{n}-1}dy\int_y^1\frac{1}{\prod_{k=1}^s(u^\lambda+c_k)}u^{\sigma-\frac{1}{pn}-1}du\\
&\quad+\int_0^1 y^{(\sigma_1-\sigma)+\frac{1}{n}-1}\int_1^\infty\frac{1}{\prod_{k=1}^s(u^\lambda+c_k)}u^{\sigma-\frac{1}{pn}-1}dudy\\
&= \int_0^1\left[\int_0^u y^{(\sigma_1-\sigma)+\frac{1}{n}-1}dy\right]\frac{1}{\prod_{k=1}^s(u^\lambda+c_k)}u^{\sigma-\frac{1}{pn}-1}du\\
&\quad+\int_0^1 y^{(\sigma_1-\sigma)+\frac{1}{n}-1}dy\int_1^\infty\frac{1}{\prod_{k=1}^s(u^\lambda+c_k)}u^{\sigma-\frac{1}{pn}-1}du.
\end{aligned} \tag{14}$$

In what follows we suppose that $p>1, \frac{1}{p}+\frac{1}{q}=1, s\in\mathbb{N}, 0<c_1\leq\cdots\leq c_s$, $\sigma,\mu>0$, $\sigma+\mu=s\lambda$, $\sigma_1\in\mathbb{R}$.

Lemma 2. *If there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in $(0,\infty)$, the following inequality*

$$I := \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dxdy$$
$$\leq M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}} \tag{15}$$

holds true, then we have $\sigma_1 = \sigma$. In this case, it follows that $M \geq k_s(\sigma)$.

Proof. If $\sigma_1 < \sigma$, then for $n > \frac{1}{\sigma-\sigma_1}$ $(n \in \mathbb{N})$, we set two functions

$$f_n(x) := \begin{cases} 0,\ 0 < x < 1 \\ x^{\sigma - \frac{1}{pn} - 1},\ x \geq 1 \end{cases},\quad g_n(y) := \begin{cases} y^{\sigma_1 + \frac{1}{qn} - 1},\ 0 < y \leq 1 \\ 0,\ y > 1 \end{cases}.$$

Hence, we obtain that

$$J_2 := \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y)dy\right]^{\frac{1}{q}}$$
$$= \left(\int_1^\infty x^{-\frac{1}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_0^1 y^{\frac{1}{n}-1}dy\right)^{\frac{1}{q}} = n.$$

By (14) and (15), we have

$$\int_0^1 \left[\int_0^u y^{(\sigma_1-\sigma)+\frac{1}{n}-1}dy\right] \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{pn} - 1} du$$
$$\leq I_2 = \int_0^\infty \int_0^\infty \frac{f_n(x)g_n(y)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dxdy \leq MJ_2 = Mn. \tag{16}$$

Since $(\sigma_1 - \sigma) + \frac{1}{n} < 0$, it follows that for any $u \in (0,1)$,

$$\int_0^u y^{(\sigma_1-\sigma)+\frac{1}{n}-1}dy = \infty.$$

By (16), in view of

$$\frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{pn} - 1} > 0, \quad u \in (0,1),$$

we deduce that $\infty \leq Mn < \infty$, which is a contradiction.

If $\sigma_1 > \sigma$, then for $n > \frac{1}{\sigma_1 - \sigma}$ $(n \in \mathbb{N})$, we set

$$\widetilde{f}_n(x) := \begin{cases} x^{\sigma + \frac{1}{pn} - 1},\ 0 < x \leq 1 \\ 0,\ x > 1 \end{cases},\quad \widetilde{g}_n(y) := \begin{cases} 0,\ 0 < y < 1 \\ y^{\sigma_1 - \frac{1}{qn} - 1},\ y \geq 1 \end{cases}.$$

Hence, we derive that

$$\begin{aligned}\widetilde{J}_2 \ : \ &= \left[\int_0^\infty x^{p(1-\sigma)-1}\widetilde{f}_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1}\widetilde{g}_n^q(y)dy\right]^{\frac{1}{q}} \\ &= \left(\int_0^1 x^{\frac{1}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\frac{1}{n}-1}dy\right)^{\frac{1}{q}} = n.\end{aligned}$$

By (13) and (15), we have

$$\begin{aligned}&\int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}dy \int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1}du \\ &\leq I_1 = \int_0^\infty \int_0^\infty \frac{\widetilde{f}_n(x)\widetilde{g}_n(y)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dxdy \leq M\widetilde{J}_2 = Mn. \qquad (17)\end{aligned}$$

Since $(\sigma_1 - \sigma) - \frac{1}{n} > 0$, it follows that

$$\int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}dy = \infty.$$

By (17), in view of

$$\int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1}du > 0,$$

we have $\infty \leq Mn < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, we reduce (13) and then apply (17) as follows:

$$\begin{aligned}\frac{1}{n}I_1 &= \frac{1}{n}\left[\int_1^\infty y^{-\frac{1}{n}-1}dy \int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1}du \right. \\ &\quad \left. + \int_1^\infty \left(\int_u^\infty y^{-\frac{1}{n}-1}dy\right) \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1}du\right] \\ &= \int_0^1 \frac{u^{\sigma+\frac{1}{pn}-1}}{\prod_{k=1}^s (u^\lambda + c_k)}du + \int_1^\infty \frac{u^{\sigma-\frac{1}{qn}-1}}{\prod_{k=1}^s (u^\lambda + c_k)}du \\ &\leq \frac{1}{n}M\widetilde{J}_2 = M. \qquad (18)\end{aligned}$$

Since the sequence

$$\left\{\frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1}\right\}_{n=1}^\infty \quad \left(resp.\ \left\{\frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma-\frac{1}{qn}-1}\right\}_{n=1}^\infty\right)$$

is nonnegative and increasing in $(0,1)$ (resp. $(1,\infty)$), by Levi's theorem (cf. [28]), we deduce that

$$k_s(\sigma) = \int_0^1 \lim_{n\to\infty} \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1} du + \int_1^\infty \lim_{n\to\infty} \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma-\frac{1}{qn}-1} du$$
$$= \lim_{n\to\infty} \left[\int_0^1 \frac{u^{\sigma+\frac{1}{pn}-1} du}{\prod_{k=1}^s (u^\lambda + c_k)} + \int_1^\infty \frac{u^{\sigma-\frac{1}{qn}-1} du}{\prod_{k=1}^s (u^\lambda + c_k)} \right] \le M < \infty. \quad (19)$$

This completes the proof of the lemma. □

3 Main results

Theorem 1. *The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \ge 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following inequality:

$$J := \left\{ \int_0^\infty y^{p\sigma_1 - 1} \left[\int_0^\infty \frac{f(x)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx \right]^p dy \right\}^{\frac{1}{p}}$$
$$< M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}; \quad (20)$$

(ii) there exists a constant M, such that for any $f(x), g(y) \ge 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

we have the following Hilbert-type integral inequality with nonhomogeneous kernel:

$$I = \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx dy$$
$$< M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}; \quad (21)$$

(iii) $\sigma_1 = \sigma$.

If Condition (iii) is satisfied, then $M \ge k_s(\sigma)$ and the constant factor $M = k_s(\sigma)$ in (20) and (21) is the best possible.

Proof. $(i) \Rightarrow (ii)$. By Hölder's inequality (cf. [29]), we have

$$
\begin{aligned}
I &= \int_0^\infty \left\{ y^{\sigma_1 - \frac{1}{p}} \int_0^\infty \frac{f(x)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx \right\} \left(y^{\frac{1}{p} - \sigma_1} g(y) \right) dy \\
&\leq J \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \qquad (22)
\end{aligned}
$$

Then by (20), we derive (21).

$(ii) \Rightarrow (iii)$. By Lemma 1, we have $\sigma_1 = \sigma$.

$(iii) \Rightarrow (i)$. Setting $u = xy$ for $y > 0$, we obtain the following weight function

$$
\begin{aligned}
\omega(\sigma, y) \ :&= y^\sigma \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} x^{\sigma-1} dx \\
&= \int_0^\infty \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma-1} du = k_s(\sigma). \qquad (23)
\end{aligned}
$$

By Hölder's weighed inequality and (23), we have

$$
\begin{aligned}
&\left\{ \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} f(x) dx \right\}^p \\
&= \left\{ \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \left[\frac{y^{(\sigma-1)/p}}{x^{(\sigma-1)/q}} f(x) \right] \left[\frac{x^{(\sigma-1)/q}}{y^{(\sigma-1)/p}} \right] dx \right\}^p \\
&\leq \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \\
&\quad \times \left\{ \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} dx \right\}^{p/q} \\
&= \left[\frac{\omega(\sigma, y)}{y^{q(\sigma-1)+1}} \right]^{p-1} \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \\
&= \frac{(k_s(\sigma))^{p-1}}{y^{p\sigma-1}} \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \qquad (24)
\end{aligned}
$$

If (24) assumes the form of equality for some $y \in (0, \infty)$, then (cf. [29]) there exist constants A and B, such that they are not both zero, and

$$
A \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) = B \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} \text{ a.e. in } \mathbb{R}_+.
$$

We suppose that $A \neq 0$ (otherwise $B = A = 0$). Then it follows that

$$
x^{p(1-\sigma)-1} f^p(x) = y^{q(1-\sigma)} \frac{B}{Ax} \text{ a.e. in } \mathbb{R}_+,
$$

which contradicts the fact that

$$
0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty.
$$

Hence, (24) assumes the form of strict inequality.

For $\sigma_1 = \sigma$, by Fubini's theorem, we have

$$\begin{aligned}
J &< (k_s(\sigma))^{\frac{1}{q}} \left\{ \int_0^\infty \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx dy \right\}^{\frac{1}{p}} \\
&= (k_s(\sigma))^{\frac{1}{q}} \left\{ \int_0^\infty \left[\int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)(p-1)}} dy \right] f^p(x) dx \right\}^{\frac{1}{p}} \\
&= (k_s(\sigma))^{\frac{1}{q}} \left[\int_0^\infty \omega(\sigma, x) x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \\
&= k_s(\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}.
\end{aligned}$$

Setting $M \geq k_s(\sigma)$, then (20) follows.

Therefore, the conditions (i), (ii) and (iii) are equivalent.

When Condition (iii) is satisfied, if there exists a constant $M < k_s(\sigma)$, such that (21) is valid, then by Lemma 3, we have $M \geq k_s(\sigma)$. By this contradiction it follows that the constant factor $M = k_s(\sigma)$ in (21) is the best possible. The constant factor $M = k_s(\sigma)$ in (20) is still the best possible. Otherwise, by (22) (for $\sigma_1 = \sigma$), we would conclude that the constant factor $M = k_s(\sigma)$ in (21) is not the best possible. □

Setting $y = \frac{1}{Y},\ G(Y) = Y^{s\lambda-2} g\left(\frac{1}{Y}\right), \mu_1 = s\lambda - \sigma_1$ in Theorem 4, then replacing Y (respectively $G(Y)$) by y (respectively $g(y)$), we deduce the following result.

Corollary 1. *The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following integral inequality:

$$\begin{aligned}
&\left\{ \int_0^\infty y^{p\mu_1 - 1} \left[\int_0^\infty \frac{f(x)}{\prod_{k=1}^s (x^\lambda + c_k y^\lambda)} dx \right]^p dy \right\}^{\frac{1}{p}} \\
&< M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}; \qquad (25)
\end{aligned}$$

(ii) There exists a constant M, such that for any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1-\mu_1)-1} g^q(y) dy < \infty,$$

we have the following Hilbert-type integral inequality with homogeneous kernel:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\prod_{k=1}^s (x^\lambda + c_k y^\lambda)} dxdy$$
$$< M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\mu_1)-1} g^q(y)dy\right]^{\frac{1}{q}}; \tag{26}$$

(iii) $\mu_1 = \mu$.

If Condition (iii) holds, then we have $M \geq k_s(\sigma)$*, and the constant factor* $M = k_s(\sigma)$ *in (25) and (26) is the best possible.*

Remark 1. On the other hand, setting $y = \frac{1}{Y}$, $G(Y) = Y^{s\lambda-2} g(\frac{1}{Y}), \sigma_1 = s\lambda - \mu_1$, in Corollary 5, then replacing Y (resp. $G(Y)$) by y (resp. $g(y)$), we deduce Theorem 4. Hence, Theorem 4 and Corollary 5 are equivalent.

4 Operator expressions

We set the following functions:
$\varphi(x) := x^{p(1-\sigma)-1}, \psi(y) := y^{q(1-\sigma)-1}, \phi(y) := y^{q(1-\mu)-1}$, wherefrom,

$$\psi^{1-p}(y) = y^{p\sigma-1}, \phi^{1-p}(y) = y^{p\mu-1} \ (x, y \in \mathbb{R}_+).$$

Define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbb{R}_+) = \left\{ f : \|f\|_{p,\varphi} := \left(\int_0^\infty \varphi(x)|f(x)|^p dx\right)^{\frac{1}{p}} < \infty \right\},$$
$$L_{q,\psi}(\mathbb{R}_+) = \left\{ g : \|g\|_{q,\psi} := \left(\int_0^\infty \psi(y)|g(y)|^q dy\right)^{\frac{1}{q}} < \infty \right\},$$
$$L_{q,\phi}(\mathbb{R}_+) = \left\{ g : \|g\|_{q,\phi} := \left(\int_0^\infty \phi(y)|g(y)|^q dy\right)^{\frac{1}{q}} < \infty \right\},$$
$$L_{p,\psi^{1-p}}(\mathbb{R}_+) = \left\{ h : \|h\|_{p,\psi^{1-p}} = \left(\int_0^\infty \psi^{1-p}(y)|h(y)|^p dy\right)^{\frac{1}{p}} < \infty \right\},$$
$$L_{q,\phi^{1-p}}(\mathbb{R}_+) = \left\{ h : \|h\|_{p,\phi^{1-p}} = \left(\int_0^\infty \phi^{1-p}(y)|h(y)|^p dy\right)^{\frac{1}{p}} < \infty \right\}.$$

(a) In view of Theorem 4 (setting $\sigma_1 = \sigma$), for $f \in L_{p,\varphi}(\mathbb{R}_+)$, setting

$$h_1(y) := \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} f(x)dx \ (y \in \mathbb{R}_+),$$

by (20), we have

$$\|h_1\|_{p,\psi^{1-p}} = \left(\int_0^\infty \psi^{1-p}(y)h_1^p(y)dy\right)^{\frac{1}{p}} < M\|f\|_{p,\varphi} < \infty. \tag{27}$$

Definition 1. Define a Hilbert-type integral operator with nonhomogeneous kernel $T^{(1)} : L_{p,\varphi}(\mathbb{R}_+) \to L_{p,\psi^{1-p}}(\mathbb{R}_+)$ as follows: For any $f \in L_{p,\varphi}(\mathbb{R}_+)$, there exists a unique representation $T^{(1)}f = h_1 \in L_{p,\psi^{1-p}}(\mathbb{R}_+)$, satisfying $T^{(1)}f(y) = h_1(y)$, for any $y \in \mathbb{R}_+$.

In view of (27), it follows that

$$\|T^{(1)}f\|_{p,\psi^{1-p}} = \|h_1\|_{p,\psi^{1-p}} \le M\|f\|_{p,\varphi},$$

and then the operator $T^{(1)}$ is bounded satisfying

$$\|T^{(1)}\| = \sup_{f(\neq 0)\in L_{p,\varphi}(\mathbb{R}_+)} \frac{\|T^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \le M.$$

If we define the formal inner product of $T^{(1)}f$ and g as follows:

$$(T^{(1)}f,g) := \int_0^\infty \left\{\int_0^\infty \frac{f(x)}{\prod_{k=1}^s[(xy)^\lambda + c_k]}dx\right\} g(y)dy,$$

then we can rewrite Theorem 4 as follows:

Theorem 2. *The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \ge 0$, $f \in L_{p,\varphi}(\mathbb{R}_+)$, $\|f\|_{p,\varphi} > 0$, we have the following inequality:

$$\|T^{(1)}f\|_{p,\psi^{1-p}} < M\|f\|_{p,\varphi}; \tag{28}$$

(ii) there exists a constant M, such that for any $f(x), g(y) \ge 0, f \in L_{p,\varphi}(\mathbb{R}_+)$, $g \in L_{q,\psi}(\mathbb{R}_+)$, $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, we have the following inequality:

$$(T^{(1)}f,g) < M\|f\|_{p,\varphi}\|g\|_{q,\psi}. \tag{29}$$

We still have $\|T^{(1)}\| = k_s(\sigma) \le M$.

(b) In view of Corollary 5 (setting $\mu_1 = \mu$), for $f \in L_{p,\varphi}(\mathbb{R}_+)$, setting

$$h_2(y) := \int_0^\infty \frac{f(x)}{\prod_{k=1}^s(x^\lambda + c_k y^\lambda)}dx$$

defined for every $y \in \mathbb{R}_+$, by (25) we have

$$\|h_2\|_{p,\phi^{1-p}} = \left(\int_0^\infty \phi^{1-p}(y)h_2^p(y)dy\right)^{\frac{1}{p}} < M\|f\|_{p,\varphi} < \infty. \tag{30}$$

Definition 2. Define a Hilbert-type integral operator with the homogeneous kernel $T^{(2)} : L_{p,\varphi}(\mathbb{R}_+) \to L_{p,\phi^{1-p}}(\mathbb{R}_+)$ as follows: For any $f \in L_{p,\varphi}(\mathbb{R})$, there exists a unique representation $T^{(2)}f = h_2 \in L_{p,\phi^{1-p}}(\mathbb{R}_+)$, satisfying $T^{(2)}f(y) = h_2(y)$, for any $y \in \mathbb{R}_+$.

In view of (30), it follows that

$$\|T^{(2)}f\|_{p,\phi^{1-p}} = \|h_2\|_{p,\phi^{1-p}} \leq M\|f\|_{p,\varphi},$$

and then the operator $T^{(2)}$ is bounded satisfying

$$\|T^{(2)}\| = \sup_{f(\neq 0)\in L_{p,\varphi}(\mathbb{R}_+)} \frac{\|T^{(2)}f\|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq M.$$

If we define the formal inner product of $T^{(2)}f$ and g as follows:

$$(T^{(2)}f,g) := \int_0^\infty \left[\int_0^\infty \frac{f(x)}{\prod_{k=1}^s (x^\lambda + c_k y^\lambda)} dx\right] g(y)dy,$$

then we can rewrite Corollary 5 as below:

Corollary 2. *The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \geq 0$, $f \in L_{p,\varphi}(\mathbb{R}_+)$, $\|f\|_{p,\varphi} > 0$, we have the following inequality:

$$\|T^{(2)}f\|_{p,\phi^{1-p}} < M\|f\|_{p,\varphi}; \tag{31}$$

(ii) there exists a constant M, such that for any $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbb{R}_+)$, $g \in L_{q,\phi}(\mathbb{R}_+), \|f\|_{p,\varphi}, \|g\|_{q,\phi} > 0$, we have the following inequality:

$$(T^{(2)}f,g) < M\|f\|_{p,\varphi}\|g\|_{q,\phi}. \tag{32}$$

We still have $\|T^{(2)}\| = k_s(\sigma) \leq M$.

Remark 2. Theorem 8 and Corollary 10 are equivalent.

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