

**ALWYN FRANCIS HORADAM, 1923–2016:  
A PERSONAL TRIBUTE TO THE  
MAN AND HIS SEQUENCE**

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ABSTRACT. Having received news of the passing of Alwyn Horadam this last July, I was determined that I should write something in his honour in which my own contact with him is described and combined with some introductory details on what I feel is his major endowment to the community of mathematicians—the so called and pre-eminent Horadam sequence whose specialisations thereof are great in number.

1. MY OWN REFLECTIONS

1.1. **Background and Context.** Anthony G. Shannon, in the May 1987 issue of *The Fibonacci Quarterly*, wrote a lovely essay about Alwyn Francis Horadam ('A. F. Horadam—*Ad Multos Annos*', pp. 100–105) on the occasion of his retirement, capturing the essence of his life and work up to that time from the viewpoint of a former student, close friend and professional colleague; clearly, Alwyn had achieved a great deal even then, and—aside from his wider family role which was important to him—he was a highly rated and much admired academic in the field of discrete mathematics. It was with sadness that I heard of (Associate) Professor Horadam's death on July 22nd, 2016, and I leave it to people such as Tony—with whom his relationship ran much longer and deeper than with myself—to offer elsewhere their own recollections and thoughts on the man, his career, and

his intellectual output as a mathematician which extended well into the current millenium [5]. Over the last few years I have come to know each of them in different ways: since my first publication on Horadam sequences Professor Shannon has been a valuable source of advice and help regarding my work, from which time I also corresponded with Alwyn by letter here from Derby in England; indeed it was Tony who introduced Alwyn to my research—something for which I will always be grateful as, although cut short by his subsequent illness and passing, it opened up a channel of discourse between us to which I would not otherwise have had access.

In our pressurised ‘rush-to-publish’ scientific environment of higher education the gifted expositor is a rare talent whose efforts are, by and large, still all too often given but a meagre and sometimes begrudged nod of approval. Tony’s 1987 article was a well crafted composition, celebrating the accomplishments of a “real university man” as it was put—evidently he was not only a very good mathematician but a person of integrity with a well defined sense of his own place within academia, holding in his time a range of different posts where his skills were manifest and appreciated by all with whom he came into contact. It falls to me now to write a testimony of differing nature, done with a desire that such a monograph leaves—in its own small way—strong and favourable impressions of the character of Alwyn and his main mathematical gift to us as formed by one who came into his circle of contact late on in what had been a busy and fulfilling life; through it I would like to think his memory will remain an enduring one.

**1.2. My Contact with Alwyn.** I co-authored a survey article on Horadam sequences [3] in 2013, and it was dedicated to Alwyn in acknowledgement of their longevity within discrete mathematics research; it was my idea, and seemed an appropriate thing to do. I had

then only been looking at them for a relatively short while—having just started work on periodicity properties—but I was immediately struck by the impact of his two well-known and seminal papers of 1965 [1,2], evidenced by both the quantity and breadth of research motivated by them as a consequence (in fact a solo paper of mine, titled ‘Horadam Sequences: A Survey Update and Extension’, will be appearing in this *Bulletin*). Tony had already shown and read to him a reasonably complete draft of the document on a visit to the family home in Armidale (New South Wales, Australia) in the spring of 2012, and Alwyn—from a message passed on by Tony—was delighted with the acknowledgement we gave him. His comments actually appeared in the published version, and his pride was evident in a very first direct communication to me almost a year later on 20th March, 2013: “I am writing to thank you . . . for your interest in my number theoretic work and for furthering research investigations in this area. From what you say, my results may encourage new discoveries and directions in discrete mathematics. This would be wonderful.” He ended with more thanks “. . . for publicising and extending [his findings]. I am flattered and a little humbled by your appreciation of my research contributions and especially by your generous Dedication . . .”, before signing off “Yours in appreciation and friendship . . .” I know, too, from an e-mail I received from his daughter Kathy—shortly after publication of the article and the celebration of Alwyn’s 90th birthday with her father and two sisters—that she was much touched by our gesture.

My correspondence lasted but a year or so, during which time I came to grow fond of Alwyn and always looked forward to receiving his letters, written from a care home in Armidale where he had been living since the middle of 2012. They gave updates on his health, the odd reference to an article, information about his children and late

wife Mollie (Eleanor Mollie Horadam, 1921–2002)—who was also a mathematician—and so on. For my part I wrote about my work and family life, and sent him any reading material I thought might appeal (he especially requested this). The overriding feature of his missives was one of genuine personal rapport (a short second letter, written on 5th July, 2013, finished “All the best for the future in health, happiness and achievement ...”), and one could not help but feel the goodwill they contained. A later letter—on hearing of my internal promotion—began “Firstly, and most importantly, my sincere congratulations on your appointment as foundation Professor of Mathematics at Derby, a conspicuous and well-merited honour. You must be deservedly feeling on top of the world with the ball at your feet (to mix metaphors). I sincerely hope that you will obtain pleasure and satisfaction in overcoming the challenges ahead, though no doubt these will not be achieved without occasional frustration.” Wise words, certainly.

A most unfortunate reaction to prescribed antibiotics brought about the end of Alwyn’s mathematical affairs around 2007 (Kathy described it to me as a “catastrophic surgical episode”), after which other aspects of his health started to cause him difficulties. By the time I became aware of his work he was, therefore, a man who knew his productive days as a mathematician were over, but also one who still maintained a connection with mathematics and clearly took pleasure in my work; he was always keen to hear of my research news, and generous in his support. There was also more than a touch of wistfulness about him, though, brought on no doubt by his daily struggles at times and quite understandable. Writing just before the Christmas of 2013, for instance, he was slightly apologetic for his “nostalgic indulgence for the remembrance of things past”, adding “Lassitude and the declining desire to put pen to paper deter me from much

correspondence these days, but when the spirit moves me the vestige of the old stimulus to communicate comes upon me, to my great pleasure, as must be obvious. However, my poor diminished brain seems light years from the creative sources which used to activate it mathematically.” That said, there was humour, too—which was great to see—the same letter containing the words “Please excuse the unorthodox and unexpected numbering of the pages 1 and 2 of this letter. Life is full of surprises, especially for me! For possible interest, I append a few Horadamia miscellanies: Do they constitute some weird kind of sequence?”

On returning from a trip to see him in the autumn of 2014, Tony brought news of a decline in Alwyn’s health which was to foreshadow his eventual departure from us—his letters to me before then, though few in number, I will always treasure as they have the unmistakable mark of a gentleman running through them.

**1.3. Some Further Thoughts.** One recurring theme of my experiences as a professional mathematician has been an appreciation of those people who create things which I consider to be noteworthy or captivating, or both. Far too many people in society, it would appear, exist in a rather passive mode (for a variety of legitimate reasons) that makes individuality and outstanding attainment difficult. Mathematicians are fortunate, however, in that they have the opportunity—with sufficient talent and possibly luck—to leave some kind of legacy from their professional work, and I view the Horadam sequence as Alwyn’s lasting bequest which should be accorded due recognition because of its prominence across a variety of journals over a sustained period of time. If, therefore, this article should ever help to bring about even a mini renaissance of interest in it then that is to the good for we all, as academics, carry an obligation to ensure both the preservation and continuance of our subject through the

natural evolutionary stages of inheritance, possession, enhancement and transmission with each passing generation.

The leisurely and reflective timescales to which universities were at one time geared have gone forever, and the madness of today's 'outputs' driven mindset permeating the university sector has simply taken us to the other end of the spectrum somehow. The current state of affairs—while now seemingly accepted and embraced by a majority as the norm—is, though, by no means an ideal one as it singularly fails to grasp the intellectual demands made by some academic disciplines and the time required to produce deep and thoughtful results within them (mathematics being, of course, a case in point); “less is more, sometimes”, as the adage goes. A minority of people still have the capability to produce articles on a regular basis that are well written, informative, mathematically sound and sufficiently distinct, however difficult that might seem—Alwyn was without doubt one of these, and being in possession of 1st class honours degrees in both english and mathematics he crafted papers with grammatical correctness accorded as much care as mathematical content, a policy we would all do well to observe as we produce papers whose purpose is to be read and not simply to exist.

Alwyn lived and worked in an era much different than mine and, I imagine, many readers of this piece, when the world of academia moved at a slower pace and scholars published more when they felt they had something worthwhile to say rather than—as seems to be the case nowadays—largely in response to internal/external institutional pressures or as a perceived requisite for job promotion. This made for a more relaxed atmosphere in which to work, and I sense that Alwyn was very much a part of that time in so far as he was able to apportion his energies between research, teaching and administrative duties with equal diligence and enjoyment from what I have

learnt of him. It also seems to me that the vicissitudes of life had not diminished a gentleness in his character, and it was a commendable thing to see in his letters. With this in mind, then through this offering I simply want to pay my own respects to him and, in a tutorial style of narrative, to bring the uninitiated to the mathematical construct that is a Horadam sequence by looking at basic closed forms (and some very fundamental properties thereof) by way of an easy introduction. I do so in the hope that it will entice readers into the world of the Horadam sequence, some of whom might go on to add to the existing body of knowledge on it and so extend its life beyond that of a kind and unassuming founder—Alwyn F. Horadam; I feel he would approve of this.

## 2. THE HORADAM SEQUENCE:

### A SHORT BACKGROUND AND INTRODUCTION TO CLOSED FORMS

**2.1. Background.** Consider the second-order linear recurrence

$$(2.1) \quad w_n = pw_{n-1} - qw_{n-2}, \quad n \geq 2,$$

parameterised by variables  $p, q$  and subject to arbitrary initial values  $w_0 = a, w_1 = b$ . The resulting recurrence sequence is known as a Horadam sequence, and written  $\{w_n(a, b; p, q)\}_{n=0}^{\infty} = \{w_n(a, b; p, q)\}_0^{\infty}$  (a notation fixed by Alwyn himself and commonly employed to this day), with associated characteristic equation

$$(2.2) \quad 0 = \lambda^2 - p\lambda + q$$

from whose roots closed forms of the general  $(n + 1)$ th term  $w_n$  are given. We tend to use the term Horadam sequence to mean an order two recurrence sequence for which any of  $a, b, p$  or  $q$  is a free variable.

Properties of sequence terms, and inter-relations across different sequences, have been embedded within a rich seam of results mined by mathematicians for a long time. Having been first announced

through the two aforementioned 1965 publications [1,2] as a generalisation of the (already familiar) Fibonacci and Lucas sequences, the Horadam sequence holds its own in terms of importance due to the potential it has offered to researchers for analysis over many years. Given so called (fundamental) generalised Fibonacci and (primordial) generalised Lucas sequences noted by E. Lucas as long ago as the 1890s, it was perhaps inevitable that someone would regard them as merely overtures to a broader line of theory and conceive of a fully general order two recurrence sequence. That distinction goes to Alwyn Horadam, whose early presentations came at an opportune moment within discrete mathematics and sowed seeds of enquiry that quickly flourished in the work of contemporaries and scores of others to follow. One of the reasons for this is, of course, the fact that closed forms for sequence terms of (2.1) are readily available, in fully symbolic form, for both degenerate and non-degenerate root cases of the quadratic characteristic equation (2.2). This permits certain types of theoretical interrogation rendered well nigh impossible for sequences delivered by defining recursions of degree three or more, keeping levels of manual algebra manageable; such closed forms are the foundation of many a paper as they provide an essential platform on which to build results.

As I have stated, to mark its standing within the field of linear recurrences I published jointly, in 2013, a survey article [3] as an appraisal of some of the work related to the Horadam sequence, serving not simply as a reference chronicle but hopefully also as a tribute to its ongoing presence since being introduced into mainstream literature over half a century ago. I guess in the back of my mind was an awareness that at the time no overview had been given, or stewardship taken, of the topic, and I wanted to produce a document that I felt would be of benefit come the day of *redde rationem* when work



had all but ceased and some sort of account was required so as to place the efforts of Alwyn—and those he inspired such as myself—in historical context. Given that the vast majority of us who devote a good part of our lives to the service of mathematics in the pursuit of knowledge and understanding neither expect nor receive little in the way of relative fame or reward, this was an opportunity to flag up Alwyn as the originator of, and major contributor to, a serious research area that has been exploited consistently over many years. This is not to say that the Horadam sequence has eclipsed or usurped in some way the Fibonacci/Lucas sequences—whose venerated existence continues to enrapture both professional and amateur mathematicians—rather that the Horadam sequence, being more general, itself offers by default a very wide scope for research and is easily specialised through the defining parameters  $a, b, p, q$ . It is quite clear that the generalised recursion (2.1) has influenced the work of numerous people, both directly and indirectly, and as such the sequence deserves an esteemed position within the discrete mathematics community. The cumulative weight of theoretical results on it *per se*, and the plethora of links between derivative sequences, are both impressive and significant, and while visibility of an entity/idea often eventually militates against new findings even now fresh insights continue to manifest themselves—it is not the place here, however, to detail these.

The inimitable Gian-Carlo Rota once wrote an opening passage to a 1970 paper on combinatorial theory (published in the Proceedings of the International Congress of Mathematicians held in Nice, France, in September of that year), opining that “Combinatorial analysis, or combinatorial theory, as it has come to be called, is currently enjoying an outburst of activity. This can be partly attributed to the abundance of new and highly relevant problems brought to the fore

by advances in discrete applied mathematics, and partly to the fact that only lately has the field ceased to be the private preserve of mathematical acrobats, and attempts have been made to develop coherent theories, thereby bringing it closer to the mainstream of mathematics.” Going on to outline specifically the theory of combinatorial geometries—still in its infancy and rooted historically in the then unsolved four-colour conjecture—he was unquestionably correct in that it was indeed an exciting time for those working in discrete mathematics which, having already separated itself from classic continuum-based analytical mathematics, was finding its feet as a distinct field of study possessing its own general/idiosyncratic problems, and attracting new personalities accordingly. I would venture to say that Alwyn’s 1965 papers rode this wave of enthusiasm and opportunity which—along with one or two other concurrent studies—gave instant impetus and immediacy to work on linear recurrences, setting the scene for many years of research as an initial period of energetic endeavour was followed by one of consolidation and then maturity. There is no doubt, too, that the timing of these publications fitted well with the prevailing mathematical *zeitgeist*, and this was reflected in the nature of work carried out over the subsequent decade or so—as Tony pointed out to me, similar ideas (although much limited in scope and quantity) had surfaced for a few years, but until the mid 1960s there was no real audience ready to receive the concepts/theory and move the topic forward in the way we have seen since then.

**2.2. Closed Forms and Basic Properties.** I want to finish by taking a brief look at some basic properties of the Horadam sequence—for the casual reader I feel this is instructive, as it necessarily involves treatment of the aforementioned characteristic root cases of (2.2). Note that although construction of sequence term closed forms

is very much a standard and well known algebraic operation, a most unusual variant is due to Niven and Zuckerman [4] which is set out in an appendix here as it would appear to be quite novel; seemingly lost in the wealth of material published since its low key appearance in 1960, the reader is encouraged to work it through as I have. I alluded to their technique in a 2014 article, before which I sent it to both Tony and Alwyn. Neither had ever seen it (which is perhaps not too surprising since it predates the two 1965 papers by a few years, and books were slow to arrive at Antipodean libraries in those days) and this, to my mind, makes its inclusion here even more worthwhile. Alwyn described it subsequently as “neat” in one of his letters to me, commenting on his oversight of the method: “Perhaps I was too busy gathering the sea-shells at my feet while the great ocean of knowledge rolled along beside me!”; this is a feeling with which most research mathematicians are all too familiar at times.

**Non-Degenerate Case:** For  $p^2 \neq 4q$  ( $p, q \neq 0$ ), there are two distinct characteristic roots  $\alpha(p, q) = (p + \sqrt{p^2 - 4q})/2$ ,  $\beta(p, q) = (p - \sqrt{p^2 - 4q})/2$ , with  $\alpha + \beta = p$ ,  $\alpha\beta = q$  and, for  $n \geq 0$ , a so called Binet closed form

$$(2.3) \quad \begin{aligned} w_n(a, b; p, q) &= w_n(\alpha(p, q), \beta(p, q), a, b) \\ &= \frac{(b - a\beta)\alpha^n - (b - a\alpha)\beta^n}{\alpha - \beta}. \end{aligned}$$

Geometric type sequences arise through particular instances thus:

Case A:  $p = 0$  With  $q \neq 0$  implied, the characteristic roots are merely  $\alpha(q) = \sqrt{qi}$ ,  $\beta(q) = -\sqrt{qi}$  (as solutions of  $\lambda^2 + q = 0$ ), and (2.3) reads

$$(2.4) \quad \begin{aligned} w_n(a, b; 0, q) &= w_n(\alpha(q), \beta(q), a, b) \\ &= \frac{(\sqrt{qi})^n [(b + a\sqrt{qi}) - (-1)^n (b - a\sqrt{qi})]}{2\sqrt{qi}}, \end{aligned}$$

yielding sequence terms as follows. Let  $m = 0, 1, 2, 3, \dots$ . For  $n$  (even)  $= 2m \geq 0$  (2.4) reduces to  $w_{2m} = a(-q)^m$ , whilst for  $n$  (odd)  $= 2m + 1 \geq 1$  it contracts to  $w_{2m+1} = b(-q)^m$ , delivering a sequence

$$(2.5) \quad \{w_n(a, b; 0, q)\}_0^\infty = \{a, b, -aq, -bq, aq^2, bq^2, -aq^3, -bq^3, aq^4, bq^4, \dots\}$$

comprising two geometric subsequences; this is, of course, also available directly from the simplified form of (2.1)  $w_n = -qw_{n-2}$  ( $n \geq 2$ ).

Case B:  $q = 0$  With  $p \neq 0$  implied, the characteristic roots are  $\alpha(p) = p, \beta(p) = 0$  (as solutions of  $\lambda^2 - p\lambda = 0$ ), and

$$(2.6) \quad w_n(a, b; p, 0) = w_n(\alpha(p), \beta(p), a, b) = p^{n-1}b$$

by (2.3), which holds for  $n \geq 1$ . Given  $w_0 = a$ , the resulting sequence is

$$(2.7) \quad \{w_n(a, b; p, 0)\}_0^\infty = \{a, b, bp, bp^2, bp^3, bp^4, \dots\}$$

and consistent with (2.1) ( $w_n = pw_{n-1}$ ,  $n \geq 2$ ; note that, as an alternative, if this first order recurrence is regarded as having just one initial value  $w_0 = a$ , the geometric sequence  $\{a, ap, ap^2, ap^3, ap^4, \dots\}$  is generated).

Geometric type sequences were alluded to by Alwyn in one of his 1965 papers [1, p. 166] as a passing remark.

**Degenerate Case:** For  $p^2 = 4q$  the characteristic roots co-incide as simply  $\alpha(p) = \beta(p) = p/2$  and, for  $n \geq 0$ ,

$$(2.8) \quad w_n(a, b; p, p^2/4) = w_n(\alpha(p), a, b) = bn\alpha^{n-1} - a(n-1)\alpha^n.$$

We cannot set  $p$  or  $q$  to zero separately in this instance as  $p = 0$  iff  $q = 0$  trivially (in which case  $\alpha = \beta = 0$  and (2.8) becomes valid

only for  $n \geq 2$ , with  $\{w_n(a, b; 0, 0)\}_0^\infty = \{a, b, 0, 0, 0, 0, \dots\}$ ; equally, (2.1) collapses to  $w_n = 0$  ( $n \geq 2$ )).

#### APPENDIX

Here I set out Niven and Zuckerman's novel formulation of [4, pp. 90–92] for degenerate and non-degenerate characteristic root Horadam sequence term closed forms; noting that  $w_0, w_1$  correspond to (resp.)  $a, b$  of (2.3) and (2.8), their notation has been modified accordingly to suit my purpose.

First I establish (2.3), for which the (distinct, non-zero) characteristic roots of (2.2) are  $\alpha(p, q), \beta(p, q) = \frac{1}{2}(p \pm \sqrt{p^2 - 4q})$  as stated earlier. Using (2.1) the difference  $w_{n+1} - \lambda w_n$  may be written in a mathematically judicious way (adding and subtracting certain like terms) as

$$\begin{aligned} w_{n+1} - \lambda w_n &= (pw_n - qw_{n-1}) - \lambda w_n \\ (A.1) \qquad &= (p - \lambda)(w_n - \lambda w_{n-1}) - (\lambda^2 - p\lambda + q)w_{n-1}, \end{aligned}$$

on which the method hinges. Setting  $\lambda = \alpha$  this reads

$$\begin{aligned} w_{n+1} - \alpha w_n &= (p - \alpha)(w_n - \alpha w_{n-1}) - (\alpha^2 - p\alpha + q)w_{n-1} \\ (A.2) \qquad &= \beta(w_n - \alpha w_{n-1}) \end{aligned}$$

since  $\alpha^2 - p\alpha + q = 0$  and  $\alpha + \beta = p$ . Re-applying once, and then again, gives in turn  $w_{n+1} - \alpha w_n = \beta^2(w_{n-1} - \alpha w_{n-2}) = \beta^3(w_{n-2} - \alpha w_{n-3})$ , and so on, with the exhausted process resulting in

$$(A.3) \qquad w_{n+1} - \alpha w_n = \beta^n(w_1 - \alpha w_0)$$

after the  $(n - 1)$ th re-application. Repeating the whole procedure, on setting  $\lambda = \beta$ , similarly yields

$$(A.4) \qquad w_{n+1} - \beta w_n = \alpha^n(w_1 - \beta w_0),$$

and subtracting (A.3) from (A.4) duly delivers (2.3).

As far as (2.8) is concerned, it is generated directly from (2.3) as the limiting case for which  $\alpha \rightarrow \beta$  (and the distinct roots converge). To this end the numerator and denominator functions of (2.3) are regarded as being primarily those of  $\alpha$  (temporarily suspending dependency on other variables), that is to say,

$$(A.5) \quad g(\alpha) = \alpha^n(w_1 - \beta w_0) - \beta^n(w_1 - \alpha w_0), \quad h(\alpha) = \alpha - \beta.$$

We require  $w_n(\beta) = w_n(\beta, w_0, w_1)$  as the evaluation  $w_n(\alpha)|_{\alpha=\beta} = g(\alpha)/h(\alpha)|_{\alpha=\beta}$ , and since this has indeterminate form 0/0 we find

$$(A.6) \quad w_n(\beta, w_0, w_1) = \lim_{\alpha \rightarrow \beta} \left\{ \frac{dg(\alpha)/d\alpha}{dh(\alpha)/d\alpha} \right\} = w_1 n \beta^{n-1} - w_0(n-1)\beta^n$$

is claimed (correctly) to be the degenerate root closed form using L'Hôpital's Rule, which is (2.8). In the absence of prior knowledge of (2.8) more work would be needed to convince oneself of the authenticity of this solution, which I include here for completeness. Writing  $w_1 n \beta^{n-1} - w_0(n-1)\beta^n = s_n$ , it is necessary to consider

$$(A.7) \quad \begin{aligned} ps_n - qs_{n-1} &= p[w_1 n \beta^{n-1} - w_0(n-1)\beta^n] \\ &\quad - q[w_1(n-1)\beta^{n-2} - w_0(n-2)\beta^{n-1}] \\ &= w_1(n+1)\beta^n - w_0 n \beta^{n+1} \end{aligned}$$

after some simple algebra, having deployed the relations  $p = 2\beta$ ,  $q = \beta^2$  (for this double root case  $\alpha = \beta$ ). Thus, clearly  $ps_n - qs_{n-1} = s_{n+1}$  by inspection, and with the correct sequence initial values  $s_0 = w_0$ ,  $s_1 = w_1$  immediate ( $\beta \neq 0$  remember), then  $w_n = s_n$  for  $n \geq 0$  as required.

For the record, Niven and Zuckerman made no claim that their method could be extended to solve any higher order recurrences, and—having made no progress myself—I leave this problem as an open one.

## REFERENCES

- [1] A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, The Fibonacci Quarterly, **3** (1965), 161–176.
- [2] A. F. Horadam, *Generating functions for powers of a certain generalised sequence of numbers*, Duke Mathematical J., **32** (1965), 437–446.
- [3] P. J. Larcombe, O. D. Bagdasar, and E. J. Fennessey, *Horadam sequences: a survey*, Bulletin of the I.C.A., **67** (2013), 49–72.
- [4] I. Niven and H. S. Zuckerman, *An Introduction to the Theory of Numbers*, Wiley, New York, U.S.A., 1960.
- [5] A. G. Shannon, *Alwyn Horadam: the man and his mathematics*, Notes on Number Theory and Discrete Mathematics, to appear.

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