A New Formulation of a Result by McLaughlin for an Arbitrary Dimension 2 Matrix Power

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Abstract

We obtain an existing 2004 result of J. McLaughlin which gives explicit entries for a general dimension 2 matrix raised to an arbitrary power. Our formulation employs so called Catalan polynomials related to the crucial parameter of McLaughlin's statement, and is a new one running along a different line of argument.

1 Introduction

1.1 Background

Let $\mathbf{M}$ be a general 2×2 matrix

$$\mathbf{M} = \mathbf{M}(A, B, C, D) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1}$$

for which it is assumed that each of A, B, C, D is non-zero and, further, that the matrix $\mathbf{M}$ has both a non-zero trace $T = T(A, D) = A + D$ and non-zero determinant $M = M(A, B, C, D) = |\mathbf{M}| = AD - BC$. With $\alpha_1 = A$, $\beta_1 = B$, $\gamma_1 = C$, $\delta_1 = D$, suppose the matrix has nth power ($n \geq 1$)

$$\mathbf{M}^n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}, \tag{2}$$

where $\alpha_n = \alpha_n(A, B, C, D), \ldots, \delta_n = \delta_n(A, B, C, D)$. A theorem, published recently by the author [1], states that the ratio β_n/γ_n $(= B/C)$ is a quantity independent of matrix power n. It is noted therein, where various proofs are presented, that this invariance property is delivered trivially by a result of McLaughlin [2, Theorem 1, p.3] who, on defining a parameter

$$\begin{aligned} y_n &= y_n(A, B, C, D) = y_n(T(A, D), M(A, B, C, D)) \\ &= \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i} T^{n-2i}(-M)^i, \end{aligned} \tag{3}$$

proved inductively that, for $n \geq 1$,

$$\mathbf{M}^n(A, B, C, D) = \begin{pmatrix} y_n - Dy_{n-1} & By_{n-1} \\ Cy_{n-1} & y_n - Ay_{n-1} \end{pmatrix}; \tag{4}$$

the anti-diagonals ratio of $\mathbf{M}^n$ is immediate as $By_{n-1}/Cy_{n-1} = B/C$, and (4) was used to formulate a number of combinatorial identities by him.

1.2 This Paper

In this paper we give an independent proof of (4) via a different route based on the notion of the so called Catalan polynomial closely related to McLaughlin's parameter, ending with some remarks in order to place the work in context. Its broader origins lie in a condition found for anti-diagonals product invariance across powers of 2×2 matrix sets [3] in which $A = A(x)$, $B = B(x)$ and $C = C(x)$ are drawn from $\mathbf{Z}[x]$ and characterise a particular class of polynomial families (being those functional coefficients of a quadratic equation satisfied by the ordinary generating function of a suite of associated integer sequences; see Remark 1 later). *As an isolated result of linear algebra the invariance of the anti-diagonals ratio of an arbitrary 2-square matrix with respect to its power would seem little known, which is surprising.*

The general $(n + 1)$th Catalan polynomial $P_n(x)$ is, for $n \geq 0$, defined as

$$P_n(x) = \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i} (-x)^i, \tag{5}$$

with the connecting relation

$$y_n(A, B, C, D) = y_n(T(A, D), M(A, B, C, D)) = T^n P_n(M/T^2) \tag{6}$$

a simple one to which we will obviously make appeal. The first few polynomials are $P_0(x) = P_1(x) = 1$, $P_2(x) = 1 - x$, $P_3(x) = 1 - 2x$, $P_4(x) =$

$1 - 3x + x^2$, $P_5(x) = 1 - 4x + 3x^2$, $P_6(x) = 1 - 5x + 6x^2 - x^3$, $P_7(x) = 1 - 6x + 10x^2 - 4x^3$, and so on, and the crucial result enabling our formulation of (4) is that, for any x, y [4, (21), p.142],

$$\begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}^n = \begin{pmatrix} P_n(-xy) & xP_{n-1}(-xy) \\ yP_{n-1}(-xy) & xyP_{n-2}(-xy) \end{pmatrix}; \tag{7}$$

evidently (7) gives directly the said invariance result (the statement of which is unaffected) in the particular $D = 0$ case for $\mathbf{M}$.

2 The Proof

We begin by defining matrices

$$\mathbf{Q}(C, D) = \begin{pmatrix} 1 & -D/C \\ 0 & 1 \end{pmatrix} \tag{P1}$$

and

$$\mathbf{R}(A, B, C, D) = \begin{pmatrix} 1 & -M/CT \\ C/T & 0 \end{pmatrix}, \tag{P2}$$

so that, conveniently,

$$T\mathbf{Q}(C, D)\mathbf{R}(A, B, C, D)\mathbf{Q}^{-1}(C, D) = \mathbf{M}(A, B, C, D), \tag{P3}$$

and in turn

$$\mathbf{M}^n(A, B, C, D) = T^n\mathbf{Q}(C, D)\mathbf{R}^n(A, B, C, D)\mathbf{Q}^{-1}(C, D). \tag{P4}$$

Equation (7), with $x = -M/CT$, $y = C/T$ (so that $-xy = M/T^2 = r(T, M)$, say), allows us to write down the nth power of $\mathbf{R}$ as

$$\mathbf{R}^n(A, B, C, D) = \begin{pmatrix} P_n(r) & -\frac{M}{CT}P_{n-1}(r) \\ \frac{C}{T}P_{n-1}(r) & -rP_{n-2}(r) \end{pmatrix}, \tag{P5}$$

whence (P4) reads

$$\mathbf{M}^n(A, B, C, D) = T^n \begin{pmatrix} M'_{11}(A, B, C, D; n) & M'_{12}(A, B, C, D; n) \\ M'_{21}(A, B, C, D; n) & M'_{22}(A, B, C, D; n) \end{pmatrix}, \tag{P6}$$

say, and where, after a little algebra, the precise entries of $\mathbf{M}^n$ in (P6) are obtained through (P1),(P5) and matched with those of (4) as now shown. Firstly, we find that

$$T^n M'_{11}(A, B, C, D; n) \quad = \quad T^n \left(P_n(r) - \frac{D}{T}P_{n-1}(r) \right)$$

$$
\begin{aligned}
&= T^n P_n(r) - DT^{n-1}P_{n-1}(r) \\
&= y_n - Dy_{n-1}, \\
T^n M'_{21}(A,B,C,D;n) &= T^n \cdot \frac{C}{T} P_{n-1}(r) \\
&= CT^{n-1}P_{n-1}(r) \\
&= Cy_{n-1}, \qquad \text{(P7)}
\end{aligned}
$$

having used (6). To deal with the remaining entries, however (in order to arrive at (4) as required), we need a further sub-result. For $n \geq 2$ (given $P_0(x) = P_1(x) = 1$), the Catalan polynomials satisfy the linear order 2 recurrence [5, (69), p.17]

$$
P_n(x) = P_{n-1}(x) - xP_{n-2}(x), \qquad \text{(P8)}
$$

from which, incidentally, the closed form

$$
P_n(x) = \frac{1}{2^{n+1}} \frac{(1+\sqrt{1-4x}\,)^{n+1} - (1-\sqrt{1-4x}\,)^{n+1}}{\sqrt{1-4x}} \qquad \text{(P9)}
$$

is readily established[1] and by (6) infers

$$
y_n = \frac{1}{2}\left(\frac{T}{2}\right)^n \frac{(1+\sqrt{1-4M/T^2}\,)^{n+1} - (1-\sqrt{1-4M/T^2}\,)^{n+1}}{\sqrt{1-4M/T^2}}, \qquad \text{(P10)}
$$

which is absent from McLaughlin's work [2]. Setting $x = r = M/T^2$ in (P8) and multiplying throughout by T^n, we have the corresponding recursion

$$
y_n = Ty_{n-1} - My_{n-2} \qquad \text{(P11)}
$$

for McLaughlin's parameter, which we deploy as seen below (note that in [2] (P11) is necessarily derived so as to actually complete the inductive proof of (4), and holds for $n \geq 2$ with $y_0 = 1$, $y_1 = T$). Continuing,

$$
\begin{aligned}
T^n M'_{22}(A,B,C,D;n) &= T^n \left(\frac{D}{T}P_{n-1}(r) - rP_{n-2}(r)\right) \\
&= DT^{n-1}P_{n-1}(r) - MT^{n-2}P_{n-2}(r)
\end{aligned}
$$

[1] We remark that the 2008 article [5] is the first one in which these polynomials made a named appearance in the literature, having been discovered in a study of processes underlying the sequential production of what has been termed iterated generating functions—they were found to give rise to a simple mechanism (an order 1 recursive scheme) to iteratively deliver polynomials which act as ordinary generating functions for an increasing term count of a pre-specified (and finite) Catalan number sequence; equations (P8),(P9), and other properties of the Catalan polynomials, are given in Section 5 therein (see also Remark 2(a) here).

$$
\begin{aligned}
&= Dy_{n-1} - My_{n-2} \\
&= Dy_{n-1} + (y_n - Ty_{n-1}) \\
&= Dy_{n-1} + y_n - (A+D)y_{n-1} \\
&= y_n - Ay_{n-1}, \qquad \text{(P12)}
\end{aligned}
$$

and, finally,

$$
\begin{aligned}
&T^n M'_{12}(A,B,C,D;n) \\
&= T^n \left(\frac{D}{C} P_n(r) - \frac{1}{CT}(M+D^2)P_{n-1}(r) + \frac{D}{C} r P_{n-2}(r) \right) \\
&= \frac{1}{C}[DT^n P_n(r) - (M+D^2)T^{n-1}P_{n-1}(r) + DMT^{n-2}P_{n-2}(r)] \\
&= \frac{1}{C}[Dy_n - (M+D^2)y_{n-1} + DMy_{n-2}] \\
&= \frac{1}{C}[Dy_n - (M+D^2)y_{n-1} + D(Ty_{n-1} - y_n)] \\
&= \frac{1}{C}[DT - (M+D^2)]y_{n-1} \\
&= \frac{1}{C}[D(A+D) - (AD - BC + D^2)]y_{n-1} \\
&= \frac{1}{C}[BC]y_{n-1} \\
&= By_{n-1}. \qquad \text{(P13)}
\end{aligned}
$$

This ends the proof.□

We conclude the paper with some remarks which place it in a better context for the reader.

3 Some Additional Remarks

Remark 1 Being in a position to do so, then for completeness and interest we recover an unusual result from an earlier paper. Let $A(x), B(x), C(x) \in \mathbf{Z}[x]$, and suppose the (ordinary) generating function $T(x)$ of a sequence of integers satisfies a general quadratic governing equation

$$0 = A(x)T^2(x) + B(x)T(x) + C(x). \qquad (8)$$

The functional coefficients $A(x), B(x), C(x)$ can be considered to give rise to a family of associated polynomials $\alpha_0(x), \alpha_1(x), \alpha_2(x), \ldots$, defined as

$$
\begin{aligned}
\alpha_n(x) &= \alpha_n(A(x), B(x), C(x)) \\
&= (1,0) \begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n \geq 0, \qquad (9)
\end{aligned}
$$

as alluded to earlier. Now, setting $D = 0$ in (4) then

$$\begin{aligned}(1,0)\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= (1,0)\mathbf{M}^n(A,B,C,0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= y_n(A,B,C,0) \\ &= A^n P_n(-BC/A^2), \end{aligned} \tag{10}$$

by (6). Thus, from comparison with (9), we have

$$\alpha_n(A(x), B(x), C(x)) = [-B(x)]^n \, P_n\left(\frac{A(x)C(x)}{B^2(x)}\right), \tag{11}$$

a curious relation first given in [6, Theorem 3, p.21].

It is felt that Remarks 2(a),(b) which follow are important ones, as they clarify the relationship between the Catalan polynomials and what are understood to be Fibonacci polynomials, addressing also an ambiguity within the definition of the latter which from the literature would appear to be unresolved still (the author would like to thank a referee for motivating this addition to the paper). It would be fair to say that the connection between Catalan polynomials and a matrix power (7) is one that has proven itself to be surprisingly rich in its application, of which that here is but one particular instance.

Remark 2(a) One version of Fibonacci polynomials $f_n(x)$ ($n \geq 0$) comprises, with $f_0(x) = f_1(x) = 1$, those given by the recursion $f_{n+1}(x) = f_n(x) + xf_{n-1}(x)$ ($n \geq 1$)—see the ordered listing of coefficients as O.E.I.S. Sequence No. A011973. They have been known for many years, appearing within, for example, the evaluation of the exponentiated matrix $\left(\begin{smallmatrix} 1 & x \\ 1 & 0 \end{smallmatrix}\right)$ in a 1919/20 paper by E.E. Jacobsthal [7, pp.44,45]—indeed Koshy, in his 2001 text, refers to them as Jacobsthal polynomials [8, Chapter 39]. They have the property that, for $n \geq 0$, $f_n(x) = P_n(-x)$ (the Catalan polynomial coefficients are found as Sequence No. A115139, and $\{P_n(-1)\}_{n=0}^{\infty} = \{f_n(1)\}_{n=0}^{\infty} = \{1, 1, 2, 3, 5, \ldots\}$ delivers the Fibonacci sequence), which of course means that the identity (7) could be re-cast in terms of these polynomials and McLaughlin's result forced through accordingly.

Remark 2(b) It is worth making the point that the polynomials described above are not the same as what seem to be known in more modern times as Fibonacci polynomials also, but which latter satisfy a slightly different recurrence (*i.e.*, $f_{n+1}(x) = xf_n(x) + f_{n-1}(x)$) and are (after possibly 0 to start) $1, x, x^2 + 1, x^3 + 2x, x^4 + 3x^2 + 1, \ldots$, only reproducing Sequence No. A011973 when coefficients are read in descending power order (ignoring absent powers); these polynomials have their own properties, naturally, and seem to have become properly visible in the literature as named ones from the mid-late 1960s onwards [9-11].[2]

Remark 3 Powers of arbitrary 2×2 matrices have been studied in the past (see, *e.g.*, [13,14]), often using an eigenvalue approach which limits the format of results and so their usefulness as far as we are concerned. An extension of McLaughlin's work was made by Belbachir and Bencherif in which, for $m \geq 2$, an explicit expression for the general term of a degree m linear recurrence—with coefficients drawn from a unitary commutative ring—was found, allowing the generalisation of McLaughlin's result to accommodate powers of a square order m matrix (although the structure of the result [15, Theorem 4, pp.10-11] is not particularly illuminating); this in turn led to some identities involving Fibonacci and Stirling numbers, and the derivation of various other combinatorial relations. McLaughlin and Sury—based on a polynomial identity in k variables—obtained a closed form expression for the entries of the powers of a k-square matrix, using them to derive various combinatorial identities [16]; tractable explicit forms are, however, only available for low values of $k \geq 2$. Our approach does not lend itself to any such development beyond the 2×2 case presented.

4 Summary

Known results involving Catalan polynomials afford the formulation of a theorem by McLaughlin for arbitrary exponentiation of a general 2×2 matrix. The proof presented here—utilising the efficacy of the polynomials—is new and differs significantly in style from McLaughlin's (offering more insight into the result itself, too, we would suggest).

References

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[2] Certainly, these polynomials—seen in papers by Webb and Parberry [9] and Bicknell [10]—are those for which Wikipedia and Wolfram MathWorld currently provide 'hits', and are themselves related to Morgan-Voyce polynomials (first introduced by that author to analyse ladder networks using Fibonacci numbers in 1959); note that Hayes [11, Preface, p.ii] adopts a scaled version of the governing recursion which results in modified polynomials that constitute a small variation within this class of modern day Fibonacci polynomials. Koshy [8, Chapter 37] states that they were first studied by E.C. Catalan (as does Hayes [11]), being introduced as the first generalisation of the Fibonacci numbers to a set of polynomials. Benjamin and Quinn note that their coefficients—including zeros from terms not present—have a combinatorial interpretation in the context of tilings [12, Combinatorial Theorem 12, p.141].

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