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Research Article

Complementary Inequalities Involving the Stolarsky Mean

Ovidiu Bagdasar

Department of Mathematical Sciences, The University of Nottingham, University Park, Nottingham NG7 2RD, UK

Correspondence should be addressed to Ovidiu Bagdasar, ovidiubagdasar@yahoo.com

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Let n be a positive integer and p, q, a, and b real numbers satisfying $p > q > 0$ and $0 < a < b$. It is proved that for the real numbers $a_1, \ldots, a_n \in [a,b]$, the maximum of the function $f_{p,q}(a_1, \ldots, a_n) = (a_1^p + \cdots + a_n^p)/n - ((a_1^q + \cdots + a_n^q)/n)^{p/q}$ is attained if and only if $k(n)$ of the numbers $a_1, \ldots, a_n$ are equal to a and the other $n - k(n)$ are equal to b, while $k(n)$ is one of the values $[(b^q - D_{p,q}^q(a,b))/(b^q - a^q) \cdot n]$, $[(b^q - D_{p,q}^q(a,b))/(b^q - a^q) \cdot n] + 1$, where $[\cdot]$ denotes the integer part and $D_{p,q}(a,b)$ represents the Stolarsky mean of a and b, of powers p and q. Some asymptotic results concerning $k(n)$ are also discussed.

1. Introduction

Let us begin with some definitions. Given the positive real numbers a and b and the real numbers p and q, the difference mean or Stolarsky mean $D_{p,q}(a,b)$ of a and b is defined by (see, e.g., [1] or [2])

$$D_{p,q}(a,b) := \begin{cases} \left(\dfrac{q(a^p - b^p)}{p(a^q - b^q)}\right)^{1/(p-q)}, & \text{if } pq(p-q)(b-a) \neq 0, \\ \left(\dfrac{a^p - b^p}{p(\ln a - \ln b)}\right)^{1/p}, & \text{if } p(a-b) \neq 0,\ q = 0, \\ \left(\dfrac{q(\ln a - \ln b)}{(a^q - b^q)}\right)^{-1/q}, & \text{if } q(a-b) \neq 0,\ p = 0, \\ \exp\left(-\dfrac{1}{p} + \dfrac{a^p \ln a - b^p \ln b}{a^p - b^p}\right), & \text{if } q(a-b) \neq 0,\ p = q, \\ (ab)^{1/2}, & \text{if } a - b \neq 0,\ p = q = 0, \\ a, & \text{if } a - b = 0. \end{cases} \tag{1.1}$$

The power mean of power $p \in \mathbb{R}$ corresponding to the real numbers $a_1, \ldots, a_n$ is defined by

$$M_p(a_1, \ldots, a_n) = \begin{cases} \left(\dfrac{a_1^p + \cdots + a_n^p}{n} \right)^{1/p}, & \text{if } p \neq 0, \\ (a_1 \cdots a_n)^{1/n}, & \text{if } p = 0. \end{cases} \tag{1.2}$$

The relation between the Stolarsky mean and the power mean can be written as

$$D_{2p,p}(a,b) = M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2} \right)^{1/p}, & \text{if } p \neq 0, \\ (ab)^{1/2}, & \text{if } p = 0. \end{cases} \tag{1.3}$$

It is well known that for fixed $a_1, \ldots, a_{\mathrm{n}}$ and $r \geq s$, we have the inequality

$$0 \leq M_r(a_1, \ldots, a_n) - M_s(a_1, \ldots, a_n), \tag{1.4}$$

with equality for $r = s$ (independent of $a_1, \ldots, a_n$), or for $a_1 = \cdots = a_n$ (see [3–5] or [6]).

Shisha and Mond [7] obtained a complementary result which examines the upper bounds of (1.4) for weighted versions of the power means. Also, we have a considerable amount of work regarding the complementary means done by many authors, including Diaz and Metcalf [8], Beck [9], and Páles [10].

Returning to our problem, by defining the function

$$\begin{aligned} F_s(a_1, \ldots, a_n) &= \frac{a_1 + \cdots + a_n}{n} - \left(\frac{a_1^s + \cdots + a_n^s}{n} \right)^{1/s} \\ &= M_1(a_1, \ldots, a_n) - M_s(a_1, \ldots, a_n), \end{aligned} \tag{1.5}$$

we obtain

$$f_{p,q}(a_1, \ldots, a_n) = \frac{a_1^p + \ldots + a_n^p}{n} - \left(\frac{a_1^q + \cdots + a_n^q}{n} \right)^{p/q} = F_{q/p}\left(a_1^p, \ldots, a_n^p \right). \tag{1.6}$$

Using the inequalities between power means (1.4), $F_s \geq 0$ if and only if $1 \geq s$, therefore $f_{p,q} \geq 0$ if and only if $1 \geq q/p$. This condition is more general than $q > p > 0$, but there are details in the subsequent proofs which would not be satisfied in the other cases.

As the minimum of $f_{p,q}$ over $[0, \infty)^{\mathrm{n}}$ is 0 (possible only for $a_1 = \cdots = a_n$), it is natural to question what the maximum of $f_{p,q}$ is, and, eventually, to find the configuration where this is attained. Since $\sup_{a_1, \ldots, a_n \in [0,\infty)} f_{p,q}(a_1, \ldots, a_n) = \infty$, the problem of finding the maximum of $f_{p,q}$ only makes sense when all the variables $a_1, \ldots, a_n$ of $f_{p,q}$ are restricted to the compact interval $[a, b] \subseteq [0, \infty)$.

The first theorem in the next section, deals with finding the maximum and the corresponding optimal configuration. The result enables one to obtain elegant proofs for some related inequalities. In the end of the present work we obtain some asymptotic limits relative to the configuration where the maximum of $f_{p,q}$ is attained.

2. Results

Theorem 2.1. *Given the positive integer n, the real numbers $p > q > 0$ and $0 < a < b$. Consider the function $f_{p,q} : [a,b]^n \to \mathbb{R}$, defined by* (1.4)*. Then the following assertions are true.*

(1) *The function $f_{p,q}$ attains its maximum at a point $(a_1, \ldots, a_n)$ if and only if $k(n)$ of the variables are equal to a, while the other $n - k(n)$ are equal to b, where $k(n)$ can be*

$$k(n) \in \left\{ \left[\frac{b^q - D^q_{p,q}(a,b)}{b^q - a^q} \cdot n \right], \left[\frac{b^q - D^q_{p,q}(a,b)}{b^q - a^q} \cdot n \right] + 1 \right\}. \tag{2.1}$$

(2) *If n, p, and q are held fixed while $b \to a$, it can be proven that*

$$\frac{1}{2} - \frac{1}{n} \le \lim_{b \searrow a} \frac{k(n)}{n} \le \frac{1}{2} + \frac{1}{n}, \tag{2.2}$$

provided the limit exists.

As an application of Theorem 2.1, the following problem (see [3, pages 70–72]) is solved.

Corollary 2.2. *Given the positive integer n, determine the smallest value of α such that the inequality*

$$\frac{a_1^2 + \cdots + a_n^2}{n} - \left(\frac{a_1 + \cdots + a_n}{n} \right)^2 \le \alpha \max_{1 \le i \le j \le n} \frac{\left(a_i - a_j\right)^2}{n^2} \tag{2.3}$$

holds true for all positive real numbers $a_1, \ldots, a_n$.

Theorem 2.3. *Given the positive integer n, the smallest value of α such that* (2.3) *holds true for all positive real numbers $a_1, \ldots, a_n$ is*

$$\alpha = \left[\frac{n}{2} \right] \left[\frac{n+1}{2} \right]. \tag{2.4}$$

In the following theorem we examine the behavior of $k(n)$ when the numbers p, q in Theorem 2.1, are terms of a sequence with certain properties.

Theorem 2.4. *Consider the sequences $\{p_n\}_{n \in \mathbb{N}}$ and $\{q_n\}_{n \in \mathbb{N}}$ satisfying $p_n > q_n, n \in \mathbb{N}$, with $\lim_{n \to \infty} q_n = \infty$ and $\lim_{n \to \infty}(p_n / q_n) = 1$. For each $n \in \mathbb{N}$ define $k(n)$ as in* (2.1)*, for the powers p_n and q_n. Then the $k(n)$ verifies*

$$\lim_{n \to \infty} \frac{k(n)}{n} = \frac{e-1}{e}. \tag{2.5}$$

3. Proofs

Proof of Theorem 2.1. (1) We first prove that the point $(a_1, \ldots, a_n)$ where the maximum of $f_{p,q}$ is attained lies on the boundary of the hypercube $[a,b]^n$ and moreover, it is a vertex. This result is the subject of Lemma 3.1. We then find the configuration where the maximum is realized. □

Lemma 3.1. *The function $f_{p,q}$ attains its maximum at the point $(a_1, \ldots, a_n)$ if and only if $a_i \in \{a, b\}$ for all $i \in 1, \ldots, n$.*

Proof of Lemma 3.1. Since $f_{p,q}$ is continuous on the compact interval $[a,b]^n$, there is a point $(\overline{a}_1, \ldots, \overline{a}_n) \in [a,b]^n$ where $f_{p,q}$ attains its maximum. If $(\overline{a}_1, \ldots, \overline{a}_n)$ is an interior point of $[a,b]^n$, then $(\partial f_{p,q}/\partial a_i)(\overline{a}_1, \ldots, \overline{a}_n) = 0$ for all $i = 1, \ldots, n$, therefore

$$p \cdot \frac{\overline{a}_i^{\,p-1}}{n} - \frac{p}{q} \cdot \frac{q\overline{a}_i^{\,q-1}}{n} \left(\frac{\overline{a}_1^{\,q} + \cdots + \overline{a}_n^{\,q}}{n} \right)^{p/q-1} = 0, \tag{3.1}$$

which implies

$$\overline{a}_i = \left(\frac{\overline{a}_1^{\,q} + \cdots + \overline{a}_n^{\,q}}{n} \right)^{1/q} \tag{3.2}$$

for all $i = 1, \ldots, n$. However, if $\overline{a}_1 = \cdots = \overline{a}_n$, then $f_{p,q}(\overline{a}_1, \ldots, \overline{a}_n) = 0$ which clearly is not the maximum of $f_{p,q}$. Consequently, $(\overline{a}_1, \ldots, \overline{a}_n)$ lies on the boundary of $[a,b]^n$. Due to symmetry and since $f_{p,q}(a, \ldots, a) = f_{p,q}(b, \ldots, b) = 0$, there exist $k \in \{1, \ldots, n-1\}$ and $l \in \{k+1, \ldots, n\}$ such that

$$\overline{a}_1 = \cdots = \overline{a}_k = a, \qquad \overline{a}_{k+1} = \cdots = \overline{a}_l = b. \tag{3.3}$$

If $l < n$ then $\overline{a}_{l+1}, \ldots, \overline{a}_n \in (a,b)$. For this case, consider the function $g_l : (a,b)^{n-l} \to \mathbb{R}$, defined by

$$g_l(a_{l+1}, \ldots, a_n) = f_{p,q}\left(\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{l-k}, a_{l+1}, \ldots, a_n \right). \tag{3.4}$$

If the point $(\overline{a}_{l+1}, \ldots, \overline{a}_n)$ where the maximum of g_l is attained is interior to $[a,b]^{n-l}$, in virtue of Fermat's theorem, we deduce that

$$\frac{\partial g_l}{\partial a_i}(\overline{a}_{l+1}, \ldots, \overline{a}_n) = 0 \tag{3.5}$$

for all $i = l+1, \ldots, n$. This is equivalent to

$$p \cdot \frac{\overline{a}_i^{\,p-1}}{n} - \frac{p}{q} \cdot \frac{q\overline{a}_i^{\,q-1}}{n} \left(\frac{\overline{a}_1^{\,q} + \cdots + \overline{a}_n^{\,q}}{n} \right)^{p/q-1} = 0, \tag{3.6}$$

hence

$$\overline{a}_i = \left(\frac{\overline{a}_1{}^q + \cdots + \overline{a}_n{}^q}{n}\right)^{1/q} = c. \tag{3.7}$$

A simple computation shows that

$$c^q = \frac{ka^q + (l-k)b^q}{l}, \tag{3.8}$$

and for this configuration we have

$$\begin{aligned} g_l(c,\ldots,c) &= \frac{ka^p + (l-k)b^p + (n-l)c^p}{n} - c^p \\ &= \frac{k(a^p - b^p) + l\left[b^p - (b^q - (k/l)(b^q - a^q))^{p/q}\right]}{n}. \end{aligned} \tag{3.9}$$

Let us define the function $h : [k+1, n] \longrightarrow \mathbb{R}$ as

$$h(x) = x\left[b^p - \left(b^q - \frac{k}{x}(b^q - a^q)\right)^{p/q}\right], \tag{3.10}$$

and prove it is increasing. Indeed, one finds

$$\begin{aligned} h'(x) &= \left[b^p - \left(b^q - \frac{k}{x}(b^q - a^q)\right)^{p/q}\right] - x\cdot\frac{p}{q}\left(b^q - \frac{k}{x}(b^q - a^q)\right)^{p/q-1}\frac{k}{x^2}(b^q - a^q) \\ &= b^p - \left[b^q - \frac{k}{x}(b^q - a^q)\right]^{p/q} - \frac{p}{q}\cdot\frac{k}{x}(b^q - a^q)\left[b^q - \frac{k}{x}(b^q - a^q)\right]^{p/q-1} \\ &= b^p - (b^q - \alpha\eta)^{p/q} - \frac{p}{q}\alpha\eta(b^q - \alpha\eta)^{p/q-1}, \end{aligned} \tag{3.11}$$

where $\alpha = b^q - a^q$, and $\eta = k/x < 1$. Since $a^q < b^q - \alpha\eta = b^q - (k/x)(b^q - a^q) < b^q$, it follows that $h'(x) > 0$ so h is increasing and the upper bound is

$$\begin{aligned} \max g_l = \frac{k(a^p - b^p) + h(l)}{n} &\le \frac{k(a^p - b^p) + h(n)}{n} \\ &= \frac{ka^p + (n-k)b^p}{n} - \left[\frac{ka^q + (n-k)b^q}{n}\right]^{p/q}. \end{aligned} \tag{3.12}$$

This finally proves that k of the numbers $a_1, a_2, \ldots, a_n$ are equal to a, while the other $n-k$ are equal to b as anticipated. This ends the proof of Lemma 3.1. □

The only thing to be done is to find the value of $k \in [0, \dots, n]$ for which the expression

$$\frac{a^p - b^p}{n}k + b^p - \left(\frac{a^q - b^q}{n}k + b^q\right)^{p/q} \tag{3.13}$$

attains its maximum.

To do this, consider the function $g : [0, n] \rightarrow \mathbb{R}$ defined by

$$g(x) = \frac{a^p - b^p}{n}x + b^p - \left(\frac{a^q - b^q}{n}x + b^q\right)^{p/q}, \tag{3.14}$$

and find the points where the maximum of g is attained in the interval $[0, n]$.

The critical points of g are found from the equation

$$g'(x^*) = \frac{a^p - b^p}{n} - \frac{p}{q} \cdot \frac{a^q - b^q}{n}\left(\frac{a^q - b^q}{n}x^* + b^q\right)^{p/q-1} = 0, \tag{3.15}$$

so they satisfy

$$\frac{q(a^p - b^p)}{p(a^q - b^q)} = \left(\frac{a^q - b^q}{n}x^* + b^q\right)^{p/q-1}. \tag{3.16}$$

As seen in the definition of the Stolarsky mean for this case,

$$D_{p,q}^{p-q}(a, b) = \left[\frac{a^q - b^q}{n}x^* + b^q\right]^{(p-q)/q}. \tag{3.17}$$

It is finally found that g has a single critical point

$$x^* = \frac{b^q - D_{p,q}^q(a, b)}{b^q - a^q} \cdot n, \tag{3.18}$$

which (fortunately) is contained in the interior of $[0, n]$.

Taking into account that the second derivative of g is

$$g''(x) = -\frac{p}{q} \cdot \left(\frac{p}{q} - 1\right) \cdot \left(\frac{a^q - b^q}{n}\right)^2 \cdot \left(\frac{a^q - b^q}{n}x + b^q\right)^{p/q-2} < 0, \tag{3.19}$$

the extremal point x^* is a point of maximum for g, and also the function g' is decreasing on the interval $(0, n)$. Because $g'(x^*) = 0$, we obtain $g'(y) > 0$ for $y \in (0, x^*)$, and $g'(y) < 0$ for $y \in (x^*, n)$. Finally, this means that g is increasing on $(0, x^*)$ and decreasing on (x^*, n).

We conclude that

$$\begin{gathered} g(1) < g(2) < \cdots < g([x^*]), \\ g(n) < g(n-1) < \ldots < g([x^*] + 1). \end{gathered} \tag{3.20}$$

The maximum of (3.13) is then attained when k takes one of the values $[x^*]$ and $[x^*]+1$, where

$$x^* = \frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q} \cdot n. \tag{3.21}$$

The value of this k is to be called $k(n)$ from now on.

Remark 3.2. Because in our case

$$pq(p-q)(b-a) \neq 0, \tag{3.22}$$

the Stolarsky mean satisfies the strict inequality $a < D_{p,q}(a,b) < b$, so $0 < x^* < n$.

(2) Using the properties of the integer part $[x^*] \le x^* < [x^*]+1$, we obtain

$$\frac{x^*}{n} - \frac{1}{n} \le \frac{[x^*]}{n} \le \frac{[x^*]+1}{n} \le \frac{x^*}{n} + \frac{1}{n}, \tag{3.23}$$

so

$$\frac{x^*}{n} - \frac{1}{n} \le \frac{k(n)}{n} \le \frac{x^*}{n} + \frac{1}{n}. \tag{3.24}$$

It is then enough to work out the limit

$$\lim_{b \searrow a} \frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q} = \frac{qa^{q-1} - qD_{p,q}^{q-1}(a,a)(\partial D_{p,q}/\partial b)(a,a)}{qa^{q-1}}. \tag{3.25}$$

On the other hand we have

$$1 = \frac{dD_{p,q}(a,a)}{da} = \frac{\partial D_{p,q}}{\partial a}(a,a) + \frac{\partial D_{p,q}}{\partial b}(a,a). \tag{3.26}$$

Due to symmetry the partial derivatives are equal, so the desired limit is

$$\lim_{b \searrow a} \frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q} = \frac{qa^{q-1} - (1/2)qa^{q-1}}{qa^{q-1}} = \frac{1}{2}. \tag{3.27}$$

Taking the limit $b \to a$ in (3.23), we obtain that the limit of $k(n)/n$ as $b \to a$ is confined to the interval $[1/2 - 1/n, 1/2 + 1/n]$.

Proof of Theorem 2.3. Considering $p = 2$ and $q = 1$ in Theorem 2.1, we obtain

$$D_{2,1}(a,b) = \frac{1}{2} \cdot \frac{b^2 - a^2}{b - a} = \frac{1}{2}(b + a),$$
$$\frac{k(n)}{n} = \frac{b - (1/2)(b + a)}{b - a} = \frac{1}{2}. \tag{3.28}$$

Out of here, we can immediately obtain the best constant α for which

$$\frac{a_1^2 + \cdots + a_n^2}{n} - \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 \leq \alpha \max_{1 \leq i \leq j \leq n} \left(a_i - a_j\right)^2. \tag{3.29}$$

Following the steps mentioned before, the function gets the maximum only when

$$a_1 = \cdots = a_k = \mathrm{a}, \qquad a_{k+1} = \cdots = a_n = b, \tag{3.30}$$

where $k = [n/2]$, or $k = [(n + 1)/2]$.

This proves that the following inequality holds:

$$\frac{a_1^2 + \cdots + a_n^2}{n} - \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 \leq \frac{(b - a)^2}{n^2}\left(nk - k^2\right), \tag{3.31}$$

so the best constant α will be

$$\alpha = \left[\frac{n}{2}\right]\left[\frac{n + 1}{2}\right]. \tag{3.32}$$

□

Remark 3.3. Although appealing, a result involving arbitrary powers p would depend on which the exact value of $k(n)$ is (out of the two possibilities). At the same time, the power $(b - a)^p$ on the righthand-side can only be obtained for $p = 2$.

Proof of Theorem 2.4. To ease the notations we write $p = p(n) = p_n$ and $q = q(n) = q_n$. The following relation holds:

$$\lim_{n \to \infty} \frac{k(n)}{n} = \lim_{n \to \infty} \frac{b^q - D_{p,q}^q(a,b)}{b^q - a^q}. \tag{3.33}$$

Using the notation $b = at$, the limit can be written as

$$\lim_{n \to \infty} \frac{1 - \left[q(t^p - t^{pq})/p(t^q - t^{pq})\right]^{q/(p-q)}}{1 - 1/t^p}. \tag{3.34}$$

Since the denominator converges to 1, it only remains to examine the limit

$$\lim_{n\to\infty}\left[\frac{q(t^p - t^{pq})}{p(t^q - t^{pq})}\right]^{q/(p-q)}, \tag{3.35}$$

which can be written as

$$\lim_{n\to\infty}\left(\frac{q}{p}\right)^{q/(p-q)}\left[\frac{t^p - t^{pq}}{t^q - t^{pq}}\right]^{q/(p-q)}. \tag{3.36}$$

It can be proven that the two terms of (3.36) converge to finite limits, and analyze each. From the hypothesis $\lim_{n\to\infty}((p_n - q_n)/p_n) = 0$, so the limit of the first term is

$$\lim_{n\to\infty}\left(\frac{q}{p}\right)^{q/(p-q)} = \lim_{n\to\infty}\left(\left(1+\frac{q-p}{p}\right)^{p/(q-p)}\right)^{-q/p} = e^{-1}, \tag{3.37}$$

while second term can be written as

$$\left(1+\frac{t^p - t^q}{t^{pq} - t^p}\right)^{q/(p-q)}. \tag{3.38}$$

Since

$$\lim_{n\to\infty}\frac{t^p - t^q}{t^{pq} - t^p} = 0, \tag{3.39}$$

the same argument as above can be used to obtain

$$\lim_{n\to\infty}\left(1+\frac{t^p - t^q}{t^{pq} - t^p}\right)^{q/(p-q)} = e^L, \tag{3.40}$$

where

$$L = \lim_{n\to\infty}\frac{t^p - t^q}{t^{pq} - t^p}\frac{q}{p-q} = 0. \tag{3.41}$$

In the end we obtain

$$\lim_{n\to\infty}\frac{k(n)}{n} = 1 - e^{-1}. \tag{3.42}$$

□

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