Closed Form Evaluations of Some Series Involving Catalan Numbers

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Abstract

Closed form evaluations of some infinite series comprising sums of exponentiated multiples of Catalan numbers are yielded by a known expansion of the trigonometric function $\sin(2\alpha)$, and then re-formulated independently as verification.

1 Introduction

In very recent work [1] some previously well documented series expansions of the sine function are examined in relation to issues of convergence. Writing

$$\sin(2p\alpha) = R_p(\alpha) \tag{1}$$

then, for integer $p \geq 1$, it is known that $R_p(\alpha)$ takes the form of an infinite series in odd powers of $\sin(\alpha)$, with p instances of $R_p(\alpha)$ of interest to the mathematics community due to the historic roots to their formulation and the appearance of the celebrated Catalan numbers within each one as $p = 1, 2, 3, 4, \ldots$

It has long been known that the principal interval of convergence for $R_p(\alpha)$ is $|\alpha| < \frac{\pi}{2}$, which is extended to $|\alpha| \leq \frac{\pi}{2}$ in [1] by appropriate analysis. This short note considers the $p = 1$ case of (1) from which infinite series evaluations are delivered in closed form by some particular values of α that lend themselves to the requisite algebraic manipulation. These are then re-

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formulated easily, and so verified for both interest and completeness, using the Catalan sequence generating function directly.

2 Results

We denote by c_n the $(n+1)$th term of the Catalan sequence $\{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{1, 1, 2, 5, 14, \ldots\}$, with closed form $c_n = \frac{1}{n+1}\binom{2n}{n}$ $(n \geq 0)$ and (ordinary) generating function $G(x) = \frac{1}{2x}(1 - \sqrt{1-4x}) = \sum_{n\geq 0} c_n x^n$.

2.1 Particular Series

For $p = 1$ (1) reads (see [1], or [2, Result I, p.41] where it is derived)

$$\sin(2\alpha) \;=\; R_1(\alpha) \;=\; 2\left\{\sin(\alpha) - \sum_{n=1}^{\infty}\left[\frac{c_{n-1}}{2^{2n-1}}\right]\sin^{2n+1}(\alpha)\right\} \tag{2}$$

which, when evaluated at $\alpha = \frac{\pi}{2}$, yields

$$\sum_{n\geq 1}(1/4)^n c_{n-1} = \frac{1}{2} \tag{3}$$

after a little re-arrangement; this recovers a result due originally to Euler (in the equivalent form $\sum_{n\geq 0} c_{n+1}/4^n = 4$) which was the basis of a paper [3] where its background was discussed and three different proofs offered. We leave it as a straightforward reader exercise to confirm that further values of $\alpha = \frac{\pi}{3}, \frac{\pi}{4}$ and $\frac{\pi}{6}$, similarly applied to (2), produce the following:

$$\begin{aligned}
\sum_{n\geq 1}(3/16)^n c_{n-1} &= \frac{1}{4}, \\
\sum_{n\geq 1}(1/8)^n c_{n-1} &= \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right), \\
\sum_{n\geq 1}(1/16)^n c_{n-1} &= \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right).
\end{aligned} \tag{4}$$

2.2 Alternative Formulation

Consider the sum

$$S(\beta) \;=\; \sum_{n=1}^{\infty}\beta^n c_{n-1} \;=\; \beta\sum_{n=0}^{\infty}c_n\beta^n \;=\; \beta G(\beta). \tag{5}$$

In other words,

$$S(\beta) = \frac{1}{2}(1 - \sqrt{1-4\beta}), \tag{6}$$

of which we seek four cases, the series evaluations of (3),(4) now immediate as $S(\frac{1}{4})$, $S(\frac{3}{16})$, $S(\frac{1}{8})$ and $S(\frac{1}{16})$, resp.

We finish with a couple of remarks.

Remark 1 The hypergeometric form of the sum $S(\beta)$ is $S(\beta) = \beta\, {}_2F_1(\frac{1}{2}, 1; 2\,|4\beta)$. Setting $\beta = \frac{1}{4}$ gives (3) readily through evaluation of the ${}_2F_1(\frac{1}{2}, 1; 2|1)$ series by Gauss' Theorem (see, for example, the Appendix of [3]).

Remark 2 The use of $\beta = \frac{1}{4}$ in (6) is vaild since it can be shown, without too much difficulty, that the interval of convergence of the power series representation of the function $\sqrt{1-4\beta}$ is the closed one $\beta \in [-\frac{1}{4}, \frac{1}{4}]$. For this reason (*i.e.*, being on the edge of the interval, and consistent with the corresponding value $\alpha = \frac{\pi}{2}$ in (2) on the edge of the convergence interval $|\alpha| \leq \frac{\pi}{2}$ for $R_1(\alpha)$) the convergence rate of the series of (3) is vastly slower compared to those in (4) whose summed values are attained more quickly in line with the order of their listing. By way of example, the series of (3),(4) are found to be within 10^{-3} of their closed form sums using the respective decreasing number of terms $79578, 9, 4, 2$; for a tighter tolerance of 10^{-4} these become $O(10^6), 15, 7, 3$.

3 Summary

We have presented different evaluation methods for four infinite series of certain type which involve Catalan numbers and are believed to be new ones. It is hoped to report on others, of a similar kind, in future.

References

[1] Larcombe, P.J., O'Neill, S.T. and Fennessey, E.J. On certain series expansions of the sine function: Catalan numbers and convergence, *Fib. Quart.*, to appear.

[2] Larcombe, P.J. (2000). On Catalan numbers and expanding the sine function, *Bull. I.C.A.*, **28**, pp.39-47.

[3] Larcombe, P.J. and Fennessey, E.J. (1999). Using an old combinatorial problem to teach analysis: Euler's borderline convergence, *Cong. Num.*, **138**, pp.211-220.

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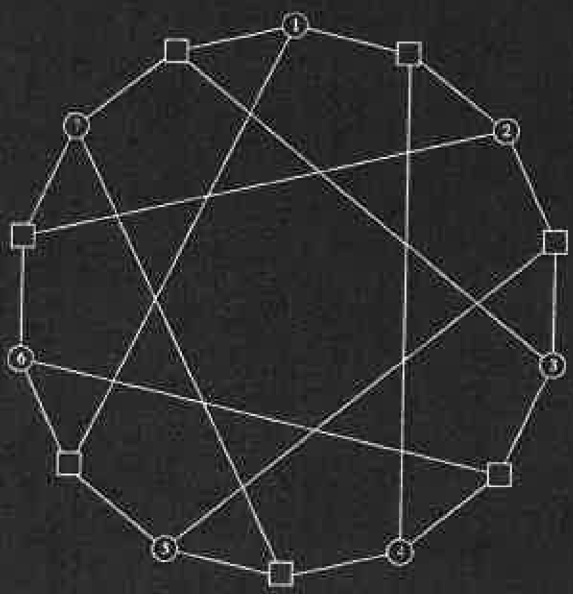

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Commemorating 200 Years Since the Birth of Eugène Charles Catalan

Guest Editor
Peter J. Larcombe

Dedication

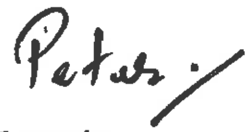
This Special May 2014 Bulletin Issue is Dedicated to the Memory of
David R. French ('Frenchy')
1943—2014

The Catalan sequence has an almost unparalleled ubiquity in discrete mathematics, arising as, or in, the solution of a wide variety of apparently disparate and unconnected counting problems. Throughout the major part of the 19th century the accepted version of its discovery linked the initial identification of the sequence to Leonhard Euler, who in 1751 wrote of its elements as providing solutions to the so called *triangulated decompositions of polygons*—a problem which is today well known and through which the Catalan sequence was to eventually bear the name of Catalan himself, seemingly after a flurry of activity (by Catalan and some contemporaries) during the 1830s and 1840s. This false attribution (and others) continued until 1988 when a Chinese historian, J. Luo, detailed a new context as evidence of an even earlier awareness of the Catalan sequence by the scholar Antu Ming (who during the first half of the 1700s examined, via geometrical considerations, a certain type of infinite series containing Catalan numbers).

From such beginnings well over 250 years ago, the Catalan sequence has continued to make regular appearances in the literature—sometimes in surprising ways—whilst the Catalan numbers have interesting mathematical properties in their own right which link with other integer sequences. My own personal interest in the Catalan sequence took off when it arose in an enumeration problem on which I was working with an undergraduate final year student in the mid 1990s (strangely, it took many years for this work to be disseminated), and—after the assimilation and translation of the relevant material—I wrote, and co-wrote, a series of short pieces on the origins of the Catalan sequence in an attempt to clarify that part of its history. Since then both Catalan and the Catalan numbers have at times figured in my work, most recently through the so called Catalan polynomials which I discovered with a Ph.D. student (James Clapperton) and great friend Dr. Eric Fennessey (in our study of iterated generating functions) and which form the basis of my joint contributions to this Special Issue. I am, of course, not alone in my Catalan-related pursuits. Professor Richard P. Stanley, for instance, has aptly termed an extreme enthusiasm for all matters Catalan as "Catalania" ("Catalan mania"), a 'condition' whose 'sufferers' will undoubtedly recognise! Richard himself keeps a wonderful Catalan Addendum to Volume 2 of his well known book *Enumerative Combinatorics* active as an up-to-date resource for researchers in which he details new interpretations and problems, and Professor Thomas Koshy has been moved to write a stand alone undergraduate text *Catalan Numbers with Applications* for a less specialised readership (see overleaf for more details on these books). Each, in its own particular way, serves the mathematical community well, along with the numerous articles which have, over the years, formed a substantial body of work on the Catalan sequence and secured its place at the forefront of the world of integer sequences.

One wonders what Catalan—who as well as being politically active was quite eclectic in his mathematical endeavours—would have made of the way the sequence has captivated academics eager to understand its fundamental nature and application; certainly, it is testimony to the importance of the Catalan numbers that so many people, at all academic levels, continue to develop and often retain an interest in them, and there is no sign of this ending. It is, therefore, a great pleasure to write this Foreword in my capacity as Guest Editor, as the I.C.A. formally celebrates both the significant and longstanding impact of the Catalan sequence within discrete mathematics. The invited contributions on offer here are as varied as they are interesting, forming a timely and fitting tribute to Catalan and the Catalan sequence.

Enjoy !



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Major Contributions to the Literature on Catalan Numbers by Stanley and Koshy

R.P. Stanley (1999). "Enumerative Combinatorics", Volume 2 (Cambridge Studies in Advanced Mathematics No. 62), Cambridge University Press, Cambridge, U.K.
Some useful background information on the Catalan numbers (with references) appears in the *Notes* section at the end of Chapter 6 of this advanced level book, with the subsequent Exercises 6.19 offering a number of combinatorial illustrations. Stanley continues to update the original presentation in the textbook with an "EC2 Supplement" (available from his M.I.T. homepage) which contains errata, updates a nd new material. In addition, a "Catalan Addendum" offers new problems related to Catalan numbers, with solutions, reflecting his deep and enduring interest in them and a determination to see them disseminated; Catalan interpretations in the Addendum currently stand at over 200 in number, the collation of which is a commendable achievement on the part of Stanley.

T. Koshy (2009). "Catalan Numbers with Applications", Oxford University Press, New York, U.S.A.
Koshy's text is aimed at a broad readership (of mathematical amateurs, high school students/teachers, and both undergraduate and postgraduate level students), in which he pulls together and catalogues many different aspects of the Catalan sequence and its numerous contexts. The book—as the author rightly states—is the first to collect and present an orderly treatise on the various occurrences, applications and properties of the Catalan numbers, and Koshy draws on a multitude of reference material to create a very useful resource.

<u>Some Other Works of Note on Catalan</u>

In 1996 the Société Belge des Professeurs de Mathématique d'Expression Française (Mons, Belgium) published "Eugène Catalan: Géomètre sans Patrie, Républicain sans République", a 200+ page book by F. Jongmans on the life and work of Catalan. [Prior to this, and as a precursor, the author had contributed a chapter (Chapter 3, pp.23-41) with the same title in a publication "Regards Sur 175 Ans de Science à l'Université de Liège 1817-1992" (Ed. A.-C. Bernès) which was produced in 1992 under the auspices of the University's Centre d'Histoire des Sciences et des Techniques to mark this period of general scientific activity at the university.]

Other works of note are the articles "Eugène Catalan and the Rise of Russian Science" (*Acad. Roy. Belg. Bull. Class. Sci.*, **2** (1991), pp.59-90) by P.L. Butzer and F. Jongmans, "Les Relations Épistolaires Entre Eugène Catalan et Ernesto Cesàro" (*ibid.*, **10** (1999), pp.223-271) by Butzer *et al.*, and "Quelques Pièces Choisies dans la Correspondance d'Eugène Catalan" (*Bull. Soc. Roy. Sci. Liège*, **50** (1981), pp.287-309) by Jongmans. All bar the final reference are predated by about a century by P. Mansion's "Notice sur les Travaux Mathématiques de Eugène-Charles Catalan" which appeared in *Ann. l'Acad. Roy. Sci. Lett. Beaux-Arts Belg.* in 1896 (**62**, pp.115-172).