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NEW PROOFS OF LINEAR RECURRENCE IDENTITIES FOR TERMS OF THE HORADAM SEQUENCE

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Abstract. We state, and prove using matrices, two (related) linear recurrence identities for terms of the so called Horadam sequence; each result expresses the general term of the sequence as a linear combination of terms with particular initial values. First offered by A.F. Horadam himself in the 1960s, our approach to their formulation is quite different and, we believe, new.

1 Introduction

Denote by $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a, b; p, q)\}_0^{\infty}$, in standard format, the four-parameter Horadam sequence arising from the second order linear recursion

$$w_{n+2} = pw_{n+1} - qw_n, \qquad n \geq 0, \tag{1.1}$$

for which $w_0 = a$ and $w_1 = b$ are initial values. Relations between terms of the sequence (and full/partial specialisations thereof) are many and varied in the literature that exists on it (see the surveys [7, 6][1]), and here we formulate related linear recurrence identities that retain almost complete symbolic generality (and combine to offer a third one). Of note is the fact that while the methodology introduces naturally two arbitrary parameters *en route* to the results, their final forms are independent of them. We also point out that our approach is quite different from that of Horadam in whose work the identities appeared together in 1965 [2, (2.14), p. 164].

2 Two Identities and Proofs

2.1 Identities

We will establish simultaneously, by proof, the following two (non-independent) identities:

Identity I. For $n \geq 2$,

$$w_n(a, b; p, q) = bw_{n-1}(1, p; p, q) - qaw_{n-2}(1, p; p, q).$$

Identity II. For $n \geq 1$,

$$w_n(a, b; p, q) = aw_n(1, p; p, q) - (pa - b)w_{n-1}(1, p; p, q).$$

2.2 Proofs

Before we proceed, we introduce a family of polynomials $\alpha_0(x), \alpha_1(x), \alpha_2(x), \ldots$, where, for $n \geq 0$,

$$\alpha_n(x) = \alpha_n(A(x), B(x), C(x)) = (1, 0) \begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.1}$$

[1]Published in 2013 and 2017, these works together attempt to cover (as comprehensively as possible) those relevant publications from the time of the 1960s, when Alwyn F. Horadam—after whom the sequence is known—released two seminal papers [2, 3] that brought the sequence to the attention of the academic community in a way not seen before; Horadam passed away in 2016, with subsequent tributes available as [5, 11] and, more recently, [4].

the first few of which are

$$\begin{aligned}
\alpha_0(x) &= 1,\\
\alpha_1(x) &= -B(x),\\
\alpha_2(x) &= B^2(x) - A(x)C(x),\\
\alpha_3(x) &= 2A(x)B(x)C(x) - B^3(x),\\
\alpha_4(x) &= B^4(x) - 3A(x)B^2(x)C(x) + A^2(x)C^2(x),\\
\alpha_5(x) &= 4A(x)B^3(x)C(x) - 3A^2(x)B(x)C^2(x) - B^5(x), \qquad (2.2)
\end{aligned}$$

etc., with general closed form

$$\alpha_n(x) = \frac{1}{2^{n+1}}\frac{[-B(x)+\rho(x)]^{n+1} - [-B(x)-\rho(x)]^{n+1}}{\rho(x)}, \qquad n \geq 0, \qquad (2.3)$$

$\rho(x) = \rho(A(x), B(x), C(x)) = \sqrt{B^2(x) - 4A(x)C(x)}$ being a 'discriminant' function. These have, in previous work, been associated with integer sequences whose individual governing (ordinary) generating function $z(x)$, say, satisfies a quadratic equation $0 = A(x)z^2 + B(x)z + C(x)$, where $A(x), B(x), C(x) \in \mathbb{Z}[x]$; instances seen in [1] are families of Catalan, (Large) Schröder and Motzkin polynomials (characterised, resp., as $\alpha_n(x,-1,1), \alpha_n(x, x-1, 1)$ and $\alpha_n(x^2, x-1, 1)$ by their namesake sequences) for whom 'auto-identities' were developed (that is, identities generated algorithmically by computer), and any reader seeking further context for these polynomials is referred to articles [8, 9, 10] by the authors. We shall make reference to the established results [1, Lemma 1.1, p. 10]

$$0 = A(x)C(x)\alpha_n(x) + B(x)\alpha_{n+1}(x) + \alpha_{n+2}(x), \qquad n \geq 0, \qquad (2.4)$$

and [1, (L5), p. 11]

$$\begin{pmatrix} \alpha_n(x) \\ -C(x)\alpha_{n-1}(x) \end{pmatrix} = \begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n \geq 1, \qquad (2.5)$$

accordingly.

Proof. Let

$$\mathbf{H}(p,q) = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}, \qquad (I.1)$$

from which the recursion (1.1) readily delivers the matrix power relation

$$\begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} = \mathbf{H}^{n-1}(p,q)\begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \qquad (I.2)$$

that holds for $n \geq 1$. If we now define, with γ, δ arbitrary, a matrix

$$\mathbf{A}(w_0, w_1, \gamma, \delta) = \begin{pmatrix} w_1 & \gamma \\ w_0 & \delta \end{pmatrix}, \qquad (I.3)$$

then we can write

$$\begin{pmatrix} w_1 \\ w_0 \end{pmatrix} = \mathbf{A}(w_0, w_1, \gamma, \delta)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (I.4)$$

and so (I.2) becomes

$$\begin{aligned}
\begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} &= \mathbf{H}^{n-1}(p,q)\mathbf{A}(w_0, w_1, \gamma, \delta)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\\
&= \mathbf{A}(w_0, w_1, \gamma, \delta)\mathbf{A}^{-1}(w_0, w_1, \gamma, \delta)\mathbf{H}^{n-1}(p,q)\mathbf{A}(w_0, w_1, \gamma, \delta)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\\
&= \mathbf{A}(w_0, w_1, \gamma, \delta)\mathbf{C}^{n-1}(p, q, w_0, w_1, \gamma, \delta)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (I.5)
\end{aligned}$$

where (denoting $|\mathbf{A}|$ as $A = A(w_0, w_1, \gamma, \delta) = \delta w_1 - \gamma w_0$, and assumed non-zero)

$$\begin{aligned} \mathbf{C}(p,q,w_0,w_1,\gamma,\delta) &= \mathbf{A}^{-1}(w_0,w_1\gamma,\delta)\mathbf{H}(p,q)\mathbf{A}(w_0,w_1,\gamma,\delta) \\ &= \frac{1}{A}\begin{pmatrix} \delta & -\gamma \\ -w_0 & w_1 \end{pmatrix}\begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}\begin{pmatrix} w_1 & \gamma \\ w_0 & \delta \end{pmatrix} \\ &= \frac{1}{A}\begin{pmatrix} \delta(pw_1 - qw_0) - \gamma w_1 & \delta(p\gamma - q\delta) - \gamma^2 \\ (w_1)^2 - w_0(pw_1 - qw_0) & \gamma w_1 - w_0(p\gamma - q\delta) \end{pmatrix}, \end{aligned} \tag{I.6}$$

after a little algebra. We will require the bottom right-hand entry of $\mathbf{C}(p,q,w_0,w_1,\gamma,\delta)$ to be zero for our purpose, whence

$$\delta = \frac{\gamma}{qw_0}(pw_0 - w_1), \tag{I.7}$$

and A can be expressed as

$$\begin{aligned} A(w_0,w_1,p,q,\gamma) &= \delta w_1 - \gamma w_0 \\ &= \frac{\gamma}{qw_0}(pw_0 - w_1)w_1 - \gamma w_0 \\ &= \frac{\gamma}{qw_0}[pw_0w_1 - q(w_0)^2 - (w_1)^2]. \end{aligned} \tag{I.8}$$

After some further manipulation the remaining entries of $\mathbf{C}(p,q,w_0,w_1,\gamma,\delta)$ (I.6) may be rewritten with δ similarly absent (reader exercise), whereupon, with reference to (I.8), it is found that

$$\mathbf{C}(p,q,w_0,w_1,\gamma) = \frac{1}{A}\begin{pmatrix} pA & \gamma A/w_0 \\ -qw_0A/\gamma & 0 \end{pmatrix} = \begin{pmatrix} p & \gamma/w_0 \\ -qw_0/\gamma & 0 \end{pmatrix}, \tag{I.9}$$

and so (I.5) reads

$$\begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} w_1 & \gamma \\ w_0 & \delta \end{pmatrix}\begin{pmatrix} p & \gamma/w_0 \\ -qw_0/\gamma & 0 \end{pmatrix}^{n-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{I.10}$$

Now, setting $A(x) = \gamma/w_0$, $B(x) = -p$ and $C(x) = qw_0/\gamma$, then in terms of elements $\alpha_n(\gamma/w_0, -p, qw_0/\gamma)$ (2.5) offers the above as

$$\begin{aligned} \begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} &= \begin{pmatrix} w_1 & \gamma \\ w_0 & \delta \end{pmatrix}\begin{pmatrix} \alpha_{n-1} \\ -(qw_0/\gamma)\alpha_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} w_1\alpha_{n-1} - qw_0\alpha_{n-2} \\ w_0\alpha_{n-1} - (qw_0\delta/\gamma)\alpha_{n-2} \end{pmatrix}; \end{aligned} \tag{I.11}$$

in other words,

$$w_n = w_1\alpha_{n-1} - qw_0\alpha_{n-2}, \qquad n \geq 2, \tag{I.12}$$

and (having removed δ by (I.7))

$$w_n = w_0\alpha_n - (pw_0 - w_1)\alpha_{n-1}, \qquad n \geq 1. \tag{I.13}$$

Furthermore, equating w_n in (I.12),(I.13) delivers immediately

$$0 = \alpha_n - p\alpha_{n-1} + q\alpha_{n-2}, \tag{I.14}$$

which recovers the special case of (2.4) for $\alpha_n(\gamma/w_0, -p, qw_0/\gamma)$ and, moreover, is the same linear recurrence (1.1) as for Horadam terms. It follows, therefore, that $\alpha_n(\gamma/w_0, -p, qw_0/\gamma)$ is a *particular Horadam element*, and with $\alpha_0(A(x), B(x), C(x)) = 1$ and $\alpha_1(A(x), B(x), C(x)) = -B(x)$ corresponding to $\alpha_0(\gamma/w_0, -p, qw_0/\gamma) = 1$ and $\alpha_1(\gamma/w_0, -p, qw_0/\gamma) = p$, we infer

$$\alpha_n(\gamma/w_0, -p, qw_0/\gamma) = w_n(1, p; p, q), \qquad n \geq 0; \tag{I.15}$$

equations (I.12),(I.13) now read as Identities I and II, completing their proofs. □

2.3 Example and a Further Identity

Noting that

$$\{w_n(a,b;p,q)\}_0^\infty = \{a, b, pb-qa, p^2b-pqa-qb, p^3b-p^2qa-2pqb+q^2a, \ldots\} \qquad (2.6)$$

and

$$\{w_n(1,p;p,q)\}_0^\infty = \{1, p, p^2-q, p^3-2pq, p^4-3p^2q+q^2, \ldots\}, \qquad (2.7)$$

Identity I is readily verified for $n=4$, say, for which the r.h.s. is $bw_3(1,p;p,q)-qaw_2(1,p;p,q) = b(p^3-2pq)-qa(p^2-q) = p^3b-p^2qa-2pqb+q^2a = w_4(a,b;p,q) =$ l.h.s. Alternatively, the r.h.s. of Identity II is $aw_4(1,p;p,q)-(pa-b)w_3(1,p;p,q) = a(p^4-3p^2q+q^2)-(pa-b)(p^3-2pq) = \cdots = p^3b-p^2qa-2pqb+q^2a = w_4(a,b;p,q) =$ l.h.s.

Finally, although non-independent of each other (for they are clearly, and trivially, connected by the Horadam recursion (1.1)), Identities I and II may be combined to give a further result:

Identity (Additional). For $n \geq 2$,

$$2w_n(a,b;p,q) = aw_n(1,p;p,q) + (2b-pa)w_{n-1}(1,p;p,q) - qaw_{n-2}(1,p;p,q).$$

We see this holds for $n=2$, for instance, whose r.h.s. $= aw_2(1,p;p,q)+(2b-pa)w_1(1,p;p,q)$ $-qaw_0(1,p;p,q) = a(p^2-q)+(2b-pa)p-qa(1) = 2(pb-qa) = 2w_2(a,b;p,q) =$ l.h.s.

3 Summary

This paper presents two (non-independent) linear recurrence identities (and a third one therefrom) involving terms of the long established Horadam sequence. It is clear that (where T denotes transposition) the representation of $(w_1, w_0)^T$ in (I.4) (through the introduction of the matrix $\mathbf{A}(w_0, w_1, \gamma, \delta)$ (I.3)) is critical to the formulation which is both elegant and subtle. Horadam gives no details as such in [2], but alludes to the fact that Identity I or II follows directly from the closed forms of the terms $w_n(a,b;p,q)$ and $w_n(1,p;p,q)$ that are listed therein—for completeness, we refer the reader to the Appendix where we give a flavour of his line of thinking.

Appendix

Here we illustrate how Identity I or II can be generated directly, choosing to deal with a different characteristic root case for each.

The characteristic polynomial $\lambda^2 - p\lambda + q$ associated with (1.1) leads to well known closed (Binet) forms of the Horadam sequence general term. In the non-degenerate characteristic roots case (with distinct roots $\alpha(p,q) = (p+\sqrt{p^2-4q})/2$, $\beta(p,q) = (p-\sqrt{p^2-4q})/2$ ($p^2 \neq 4q$)),

$$w_n(a,b;p,q) = w_n(\alpha(p,q), \beta(p,q), a, b) = \frac{(b-a\beta)\alpha^n - (b-a\alpha)\beta^n}{\alpha-\beta}, \qquad n \geq 0, \qquad (A.1)$$

while in the degenerate characteristic roots case (with non-distinct roots $\alpha(p) = \beta(p) = p/2$ ($p^2 = 4q$)),

$$w_n(a,b;p,p^2/4) = w_n(\alpha(p), a, b) = bn\alpha^{n-1} - a(n-1)\alpha^n, \qquad n \geq 0, \qquad (A.2)$$

noting that the relations

$$\alpha + \beta = p, \qquad \alpha\beta = q, \qquad (A.3)$$

cover both root types; it follows immediately that, using (A.3), equations (A.1) and (A.2) give

$$w_n(1,p;p,q) = \frac{(p-\beta)\alpha^n - (p-\alpha)\beta^n}{\alpha-\beta} = \frac{(\alpha)\alpha^n - (\beta)\beta^n}{\alpha-\beta} = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}, \qquad (A.4)$$

and

$$w_n(1,p;p,p^2/4) = pn\alpha^{n-1} - (n-1)\alpha^n = (2\alpha)n\alpha^{n-1} - (n-1)\alpha^n = (n+1)\alpha^n. \qquad (A.5)$$

The identities are easily derived as follows, again appealing to (A.3) as necessary.

Identity I (Non-Degenerate Roots Case). Consider

$$\begin{aligned} bw_{n-1}(1,p;p,q) - qaw_{n-2}(1,p;p,q) \\ &= \frac{1}{\alpha-\beta}[b(\alpha^n-\beta^n) - qa(\alpha^{n-1}-\beta^{n-1})] \\ &= \frac{1}{\alpha-\beta}[b(\alpha^n-\beta^n) - (\alpha\beta)a(\alpha^{n-1}-\beta^{n-1})] \\ &= \frac{1}{\alpha-\beta}[b(\alpha^n-\beta^n) - a(\beta\alpha^n-\alpha\beta^n)] \\ &= \frac{1}{\alpha-\beta}[(b-a\beta)\alpha^n - (b-a\alpha)\beta^n] \\ &= w_n(a,b;p,q). \end{aligned} \tag{A.6}$$

Identity II (Degenerate Roots Case). Consider

$$\begin{aligned} aw_n(1,p;p,p^2/4) - (pa-b)w_{n-1}(1,p;p,p^2/4) \\ &= a(n+1)\alpha^n - (pa-b)n\alpha^{n-1} \\ &= a(n+1)\alpha^n - ([2\alpha]a-b)n\alpha^{n-1} \\ &= a[(n+1)-2n]\alpha^n + bn\alpha^{n-1} \\ &= bn\alpha^{n-1} - a(n-1)\alpha^n \\ &= w_n(a,b;p,p^2/4). \end{aligned} \tag{A.7}$$

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