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Traffic Assignment: On the Interplay between Optimisation and Equilibrium Problems

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ABSTRACT
Motorists often have to choose routes helping them to realise faster journey times. Route choices between an origin and a destination might involve direct main roads, shorter routes through narrow side streets, or longer but (potentially) faster journeys using motorways or ring-roads. In the absence of effective traffic control measures, an approximate equilibrium travel time may result between the routes available, which is generally expected to be far from optimal. In this paper we investigate discrete and continuous optimisation and equilibrium-type problems, for a simplified traffic assignment problem on a simple network with parallel links and fixed demand. We explore the interplay between solutions of certain optimisation and equilibrium problems which can be solved by dynamic programming. The results are supported by numerical simulations, in which the price of anarchy is calculated to highlight the demand levels where there is a change in road choice and usage.

KEYWORDS
Traffic assignment problem; equilibrium state; discrete dynamic programming; multi-objective optimisation.

1. Introduction

Whenever journey time is the main criterion for choosing their route, drivers tend to act selfishly while trying to minimise their own journey times. As a consequence, route switching by the travellers to the perceived fastest route, will ensue a steady state where all (used) routes have an approximately equal travel time. The resultant total travel time at this *equilibrium* is generally greater than the *optimum* one, achieved in the presence of a perfect traffic control system. Such models may lead to the decidedly counter-intuitive result that additions to road capacity may result in increased, rather than the expected slower journey times.

The classical Traffic Assignment Problem (TAP) was first formulated by Dafermos and Sparrow [1]. Since then, numerous mathematical programs have been designed for solving variations of the fixed demand problem. In the book of Patriksson [2] and in the references therein, various minimisation problems with separable goal function and simple constraints for the TAP problem have been treated, including the particular case of networks involving parallel routes between an origin and a destination.

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In [3, Section 7.3] we have investigated a TAP model with fixed demand and parallel links, using route travel times inspired from Youn *et al.* [4]. For this model we have first formulated a discrete optimisation problem with separable objective function, and a discrete equilibrium problem under the assumption that *all routes are being used and individual travel costs are all equal*, represented via a system of equations. Alternative formulations and solution methodologies based on dynamic programming, tabu search heuristics and numerical techniques were further investigated in [5].

This paper explores the interplay between the above mentioned optimisation and equilibrium problems. Section 2 presents some cost functions for TAP models, discrete optimisation and equilibrium problems, and a dynamic programming approach used to solve the discrete optimisation problem. In Section 3 we formulate some continuous counterparts of the discrete optimisation and equilibrium problems, highlighting the links between these two types of problems. Specifically, we first show that the solution of the continuous optimisation problem corresponds to the solution of an equilibrium-type problem (Theorem 3.1). Then we demonstrate that the solution of the continuous equilibrium problem is in fact the solution of an abstract optimisation problem with separable variables, whose travel cost functions match the pattern of the original optimisation problem (Theorem 3.2). We also formulate an optimisation counterpart for the discrete equilibrium problem, which can be solved by dynamic programming. In Section 4 we compare the solutions of the above mentioned problems through numerical simulations, with a focus on the price of anarchy.

2. Preliminaries

2.1. General cost of "origin - destination" traffic flow

Car travel time is dependent on the number of cars accessing a route, as well as speed limit, length and capacity of the road. If there are $m \geq 2$ routes between the origin and destination points, the travel time t_i for a car accessing route i ($i = 1, \dots, m$) is a monotonic increasing polynomial function of the traffic flow $x_i \in \mathbb{N} = \{0, 1, 2, \dots\}$, as measured in "units of vehicles" per "unit of time" accessing route i, namely:

$$t_i = f_i(x_i) = a_i + b_i \left(\frac{x_i}{c_i}\right)^{p_i}, \quad \text{where } p_i \geq 1,\ a_i \geq 0 \text{ and } b_i, c_i > 0, \tag{1}$$

as in Youn *et al.* [4], also called the *BPR Formula* (Bureau of Public Roads) [6].

Note that as $x_i \to \infty$ one has $f_i(x_i) \to \infty$. Also, if $x_i = 0$ then $f_i(x_i) = a_i$, while for $x_i = c_i$ we have $f_i(x_i) = a_i + b_i$. The term x_i/c_i is the (traffic) flow to road capacity ratio, and traffic flows may often be well above the road design capacity ($x \gg c$).

Denoting the cost of transporting x_i vehicles along route i ($i = 1, \dots, m$) by

$$g_i(x_i) = x_i f_i(x_i), \tag{2}$$

the total travel time for n vehicles distributed on m routes is given by formula:

$$T(x) = \sum_{i=1}^{m} g_i(x_i) = \sum_{i=1}^{m} x_i f_i(x_i), \tag{3}$$

where $x = (x_1, \dots, x_m) \in \mathbb{N}^m$ and $x_1 + \cdots + x_m = n$.

The consideration of various road types having different associated travel times is well justified by the real world. For example, travellers to a city centre may have to choose between main highway to city centre (high capacity, direct and popular), side routes (also called rat runs, which offer short distance but are easily congested), or ring-roads and motorways (often faster, but longer and indirect).

The travel time functions given in (1) for these routes are shown in Table 1, where the parameters a_i, b_i and p_i considered in [5], match the classical *BPR Formula* [6]. The movement of x_i cars along route i costs $g_i(x_i) = x_i f_i(x_i)$, $i = 1, 2, 3$, and the total travel time for moving $n = x_1 + x_2 + x_3$ cars along these routes is given by the formula $T(x) = g_1(x_1) + g_2(x_2) + g_3(x_3)$, as seen in (3).

Road Parameters and Road Travel Time per Vehicle					
Road No.	a_i	b_i	c_i	p_i	$f_i(x_i) = a_i + b_i \left(\frac{x_i}{c_i}\right)^{p_i}$
1	1.85	0.2775	4000	2	$f_1(x_1) = 1.85 + 0.2775 \left(\frac{x_1}{4000}\right)^2$
2	1.5	0.225	1500	3	$f_2(x_2) = 1.5 + 0.225 \left(\frac{x_2}{1500}\right)^3$
3	2.15	0.3225	1000	5	$f_3(x_3) = 2.15 + 0.3225 \left(\frac{x_3}{1000}\right)^5$

Table 1. Travel cost functions $f_i(x_i)$ for a model involving 3 roads.

For instance, for $n = 10000$ and $x = (x_1, x_2, x_3) = (6804, 2179, 1017)$, the total travelling cost is $T(x) = 25365.2607$, which will be shown to be optimal in Section 4.

2.2. Optimality vs. Equilibrium

In this paper two main types of traffic management approaches are considered.

First, one may try to find the number of cars to be directed along each route, in order to minimize travel time. The following discrete optimisation problem is obtained:

$$\begin{cases} \text{Minimise } T(x_1, \ldots, x_m) \\ \text{subject to} \\ x_1 + \cdots + x_m = n \\ x_1, \ldots, x_m \in \mathbb{N}. \end{cases} \tag{4}$$

Second, admitting that all drivers are allowed to seek to minimise their own individual travel times, an equilibrium would develop where all travel times are equal and the steady state traffic flow along each route could be obtained as a solution of the following system of equations:

$$\begin{cases} f_1(x_1) = \ldots = f_m(x_m) \\ x_1 + \cdots + x_m = n \\ x_1, \ldots, x_m \in \mathbb{N}. \end{cases} \tag{5}$$

As this system may be inconsistent, we consider the mean and normalised standard deviation of the individual travel times vector $(f_1(x_1), \ldots, f_m(x_m))$, defined by

$$\mu(x) = \frac{1}{m} \sum_{i=1}^{m} f_i(x_i) \text{ and } \sigma(x) = \sqrt{\frac{1}{m-1} \sum_{i=1}^{m} \left|f_i(x_i) - \mu(x)\right|^2}.$$

An alternative to the discrete equilibrium system (5) is the optimisation problem:

$$\begin{cases} \text{Minimise } \sigma(x_1,\dots,x_m) \\ \text{subject to} \\ x_1+\cdots+x_m=n \\ x_1,\dots,x_m\in\mathbb{N}. \end{cases} \tag{6}$$

If the equilibrium system (5) is consistent, then its solutions are optimal solutions of problem (6). Conversely, if $x^0=(x_1^0,\dots,x_m^0)$ is an optimal solution of (6), then x^0 is a solution of (5) if and only if $\sigma(x^0)=0$.

A measure of the difference between optimal and equilibrium solutions is the *price of anarchy* introduced by Koutsoupias and Papadimitriou [7], defined by:

$$P_A=\frac{\text{Total cost at equilibrium}}{\text{Total cost at optimum}}. \tag{7}$$

If $P_A\gg 1$, time travel savings can be gained through effective traffic management.

2.3. Methodology

Methods for solving the discrete optimisation and equilibrium problems (4) and (6), and some of their continuous counterparts by exhaustive search, tabu search, dynamic programming, and steepest descent, were given in [3, Section 7.3] and [5]. For n vehicles and m routes, the complexity of exhaustive search was $O(n^{m-1})$.

Since the optimisation problem (4) has a finite feasible set and a cost function with separable variables, it can be solved by Bellman's algorithm [8].

As explained in detail in [3, Section 7.3], one must define recursively the functions $G_1,\dots,G_m:[0,n]\cap\mathbb{N}\to\mathbb{R}$ for all $c\in[0,n]\cap\mathbb{N}$ by Bellman's functional equations:

$$\begin{cases} G_1(c) &= g_1(c); \\ G_k(c) &= \min_{s\in[0,c]\cap\mathbb{N}}\left[g_k(s)+G_{k-1}(c-s)\right], \quad k=2,3,\dots,m. \end{cases} \tag{8}$$

The optimal value of problem (4) is given by the explicit formula $\min T(x)=G_m(n)$, where $x=(x_1,\dots,x_m)\in\mathbb{N}^m$ and $x_1+\cdots+x_m=n$.

An optimal solution $x^0=(x_1^0,\dots,x_m^0)$ of (4) is obtained by going backwards.

Let $c:=n$ and choose $x_m^0\in\underset{s\in[0,c]\cap\mathbb{N}}{\operatorname{argmin}}\left[g_m(s)+G_{m-1}(c-s)\right]$,

Let $c:=n-x_m^0$ and choose $x_{m-1}^0\in\underset{s\in[0,c]\cap\mathbb{N}}{\operatorname{argmin}}\left[g_{m-1}(s)+G_{m-2}(c-s)\right]$,

...

Let $c:=n-x_m^0-\ldots-x_3^0$ and choose $x_2^0\in\underset{s\in[0,c]\cap\mathbb{N}}{\operatorname{argmin}}\left[g_2(s)+G_1(c-s)\right]$,

Let $x_1^0:=n-x_m^0-\ldots-x_3^0-x_2^0$.

In Section 4 we use this method to solve the discrete optimisation problem (4), and a counterpart of the equilibrium system (5), formulated as a discrete optimisation problem, whose objective function has separable variables.

3. On the interplay between Optimisation and Equilibrium problems

Here we consider some continuous counterparts of the optimisation and equilibrium problems (4) and (5). We show that the solution of the optimisation problem is the solution of an equilibrium problem, while the continuous equilibrium problem can be interpreted as the solution of a certain optimisation problem, related to (4).

3.1. Continuous counterparts of the optimisation problem (4)

We consider the following relaxation of the optimisation problem (4)

$$\begin{cases} \text{Minimise } T(x_1,\dots,x_m) \\ \text{subject to} \\ x_1+\cdots+x_m=n \\ x_1,\dots,x_m \geq 0, \end{cases} \tag{9}$$

and its counterpart

$$\begin{cases} \text{Minimise } T(x_1,\dots,x_m) \\ \text{subject to} \\ x_1+\cdots+x_m=n \\ x_1,\dots,x_m > 0. \end{cases} \tag{10}$$

Denoting the feasible domain of problem (9) by

$$S=\left\{x=(x_1,\dots,x_m)\in\mathbb{R}^m_+ : x_1+\cdots+x_m=n\right\}, \tag{11}$$

it is easily seen that the feasible domain of problem (4) is $S\cap\mathbb{N}^m$. Moreover, the feasible domain of problem (10) is the relative interior of S, denoted by $\operatorname{ri} S$, as usual in Convex Analysis [9]. Taking into account that $S\cap\mathbb{N}^m\subseteq S=\operatorname{cl}(\operatorname{ri} S)$ and T is continuous, we infer that the optimal values of problems (4), (9) and (10) satisfy

$$\min_{x\in S\cap\mathbb{N}^m} T(x) \geq \min_{x\in S} T(x) = \inf_{x\in\operatorname{ri} S} T(x).$$

Notice that problems (4) and (9) always have optimal solutions, while problem (10) possesses minimizing sequences, but not necessarily minimal solutions.

Theorem 3.1. *The optimisation problem* (10) *is equivalent to the following equilibrium-type system*

$$\begin{cases} g_1'(x_1)=\ldots=g_m'(x_m) \\ x_1+\cdots+x_m=n \\ x_1,\dots,x_m>0. \end{cases} \tag{12}$$

Proof. Consider a point $x^0=(x_1^0,\dots,x_m^0)\in\operatorname{int}\mathbb{R}^m_+$.

Assume that x^0 is an optimal solution of problem (10). Let us attach to (10) the Lagrangian function $L:(\operatorname{int}\mathbb{R}^m_+)\times\mathbb{R}\to\mathbb{R}$ defined by

$$L(x,\lambda)=T(x)+\lambda H(x), \tag{13}$$

with the objective function $T(x)$ given by (3), and the constraint function

$$H(x) := x_1 + \cdots + x_m - n, \tag{14}$$

for all $x = (x_1, \ldots, x_m) \in \operatorname{int} \mathbb{R}^m_+$. By the necessary optimality condition, there exists $\lambda_0 \in \mathbb{R}$ such that (x^0, λ_0) is a stationary point of L, that is

$$\begin{cases} \frac{\partial L}{\partial x_i}(x^0, \lambda_0) = g_i'(x_i^0) + \lambda_0 = 0, \quad i = 1, \ldots, m \\ \frac{\partial L}{\partial \lambda}(x^0, \lambda_0) = H(x^0) = x_1^0 + \cdots + x_m^0 - n = 0. \end{cases} \tag{15}$$

In particular, this shows that x^0 is a solution of the system (12).

Conversely, assuming that x^0 is a solution of (12), the number below is well-defined

$$\lambda_0 := -g_1'(x_1^0) = \cdots = -g_m'(x_m^0).$$

Let $L_0 : \operatorname{int} \mathbb{R}^m_+ \to \mathbb{R}$ be defined for all $x \in \operatorname{int} \mathbb{R}^m_+$ by

$$L_0(x) = L(x, \lambda_0).$$

By means of (1) and (2), we deduce that

$$d^2 L_0(x^0)(h) = \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^2 L_0}{\partial h_i \partial h_j}(x^0) h_i h_j = \sum_{i=1}^m g_i''(x_i^0) h_i^2 = \sum_{i=1}^m \frac{b_i(1+p_i)p_i}{c_i^{p_i}} x_i^{p_i - 1} h_i^2 > 0,$$

for all $h = (h_1, \ldots, h_m) \in \mathbb{R}^m \setminus \{0_m\}$ such that $\mathrm{d}H(h) = h_1 + \cdots + h_m = 0$. Hence, by the sufficient optimality condition, x^0 is an optimal solution of (10). $\square$

Comparing (12) to (5), it is natural to consider the equilibrium-type system

$$\begin{cases} g_1'(x_1) = \ldots = g_m'(x_m) \\ x_1 + \cdots + x_m = n \\ x_1, \ldots, x_m \in \mathbb{N}. \end{cases} \tag{16}$$

For any $x = (x_1, \ldots, x_m) \in \mathbb{N}^m$ such that $x_1 + \cdots + x_m = n$ we consider the mean and normalised standard deviation of the vector $(g_1'(x_1), \ldots, g_m'(x_m))$, defined by

$$\overline{\mu}(x) = \frac{1}{m} \sum_{i=1}^m g_i'(x_i) \text{ and } \overline{\sigma}(x) = \sqrt{\frac{1}{m-1} \sum_{i=1}^m \left| g_i'(x_i) - \overline{\mu}(x) \right|^2}.$$

As an alternative to the equilibrium system (16) we may consider the following optimisation problem, which always has a solution (having finite state space):

$$\begin{cases} \text{Minimise } \overline{\sigma}(x_1, \ldots, x_m) \\ \text{subject to} \\ x_1 + \cdots + x_m = n \\ x_1, \ldots, x_m \in \mathbb{N}. \end{cases} \tag{17}$$

This problem can be solved using a heuristic algorithm or numerically, as in [3].

3.2. Continuous counterparts of the equilibrium system (5)

As mentioned in Section 2.2, the equilibrium system (5) does not have integer solutions in general. Also, in contrast to (4), the optimisation problem (6) cannot be solved by Bellman's algorithm, because its objective function σ does not have separable variables. Therefore, one may consider the relaxed version of (5) over positive reals:

$$\begin{cases} f_1(x_1) = \ldots = f_m(x_m) \\ x_1 + \cdots + x_m = n \\ x_1, \ldots, x_m > 0. \end{cases} \tag{18}$$

Remark 1. The system (18) can be rewritten in a manner similar to (12). Indeed, denoting the antiderivatives of $f_1, \ldots, f_m$, by $\widetilde{g}_1, \ldots, \widetilde{g}_m$, we have

$$\widetilde{g}_i(x_i) = \int_0^{x_i} f_i(t)\,\mathrm{d}t = a_i x_i + \frac{b_i}{(p_i+1)c_i^{p_i}} x_i^{p_i+1}, \quad x_i > 0, \quad i = 1, \ldots, m, \tag{19}$$

and therefore the system (18) is equivalent to:

$$\begin{cases} \widetilde{g}_1'(x_1) = \ldots = \widetilde{g}_m'(x_m) \\ x_1 + \cdots + x_m = n \\ x_1, \ldots, x_m > 0. \end{cases} \tag{20}$$

This allows us to formulate the main result of our paper.

Theorem 3.2. *The equilibrium-type system* (18) *is equivalent to the constrained optimisation problem*

$$\begin{cases} \text{Minimise } \widetilde{T}(x_1, \ldots, x_m) \\ \text{subject to} \\ x_1 + \cdots + x_m = n \\ x_1, \ldots, x_m > 0, \end{cases} \tag{21}$$

whose objective function $\widetilde{T} : \operatorname{int} \mathbb{R}_+^m \to \mathbb{R}$ *is defined for all* $x = (x_1, \ldots, x_m)$ *by*

$$\widetilde{T}(x) = \sum_{i=1}^{m} \widetilde{g}_i(x_i). \tag{22}$$

Proof. Consider a point $x^0 = (x_1^0, \ldots, x_m^0) \in \operatorname{int} \mathbb{R}_+^m$.

Assume that x^0 is an optimal solution of (21). Let $\widetilde{L} : (\operatorname{int} \mathbb{R}_+^m) \times \mathbb{R} \to \mathbb{R}$ be the Lagrangian function associated to the constrained optimisation problem (21), i.e.,

$$\widetilde{L}(x, \lambda) = \widetilde{T}(x) + \lambda H(x), \tag{23}$$

with H given by (14). By the necessary optimality condition, there is $\lambda_0 \in \mathbb{R}$ such that (x^0, λ_0) is a stationary point of $\widetilde{L}$, that is

$$\begin{cases} \frac{\partial \widetilde{L}}{\partial x_i}(x^0, \lambda_0) = \widetilde{g}\,'_i(x^0) + \lambda_0 = f_i(x^0) + \lambda_0 = 0, \quad i = 1, \ldots, m \\ \frac{\partial \widetilde{L}}{\partial \lambda}(x^0, \lambda_0) = H(x^0) = x_1^0 + \cdots + x_m^0 - n = 0. \end{cases} \tag{24}$$

This shows in particular that x^0 is a solution of the system (20), hence of (18) by Remark 1.

Conversely, assume that x^0 is a solution of the equilibrium-type system (18). In view of Remark 1, the number below is well-defined:

$$\lambda_0 := -\widetilde{g}\,'_1(x_1^0) = \cdots = -\widetilde{g}\,'_m(x_m^0).$$

Let $\widetilde{L}_0 : \operatorname{int} \mathbb{R}_+^m \to \mathbb{R}$ be defined for all $x \in \operatorname{int} \mathbb{R}_+^m$ by

$$\widetilde{L}_0(x) = \widetilde{L}(x, \lambda_0).$$

Recalling (19), we deduce that

$$\mathrm{d}^2 \widetilde{L}_0(x^0)(h) = \sum_{i=1}^m \sum_{j=1}^m \frac{\partial^2 \widetilde{L}_0}{\partial h_i \partial h_j}(x^0) h_i h_j = \sum_{i=1}^m \widetilde{g}\,''_i(x_i^0) h_i^2 = \sum_{i=1}^m \frac{b_i p_i}{c_i^{p_i}} x_i^{p_i - 1} h_i^2 > 0,$$

for all $h = (h_1, \ldots, h_m) \in \mathbb{R}^m \setminus \{0_m\}$ such that $\mathrm{d}H(h) = h_1 + \cdots + h_m = 0$. Hence, by the sufficient optimality condition, x^0 is an optimal solution of problem (21).

Consequently, a point $x^0 = (x_1^0, \ldots, x_m^0) \in \operatorname{int} \mathbb{R}_+^m$ satisfies the equilibrium-type system (18) if and only if it is an optimal solution of problem (21). □

An interesting question arises. Could our new optimisation problem (21) be seen as the minimisation of the total travelling time of some traffic problem? Surprisingly, due to the structure of the cost functions $f_1, \ldots, f_m$ given in (1), we can define new polynomial functions $\widetilde{f}_1, \ldots, \widetilde{f}_m$ (fitting the model proposed by Youn *et al.* [4]) by

$$\widetilde{f}_i(x_i) = a_i + \frac{b_i}{p_i + 1} \left(\frac{x_i}{c_i} \right)^{p_i}, \tag{25}$$

such that the antiderivatives $\widetilde{g}_1, \ldots, \widetilde{g}_m$ of $f_1, \ldots, f_m$ given by (19) satisfy

$$\widetilde{g}_i(x_i) = x_i \widetilde{f}_i(x_i). \tag{26}$$

Hence, our optimisation problem (21) represents a continuous counterpart (over positive reals) of an alternative discrete optimisation problem associated to problem (4):

$$\begin{cases} \text{Minimise } \widetilde{T}(x_1, \ldots, x_m) \\ \text{subject to} \\ x_1 + \cdots + x_m = n \\ x_1, \ldots, x_m \in \mathbb{N}. \end{cases} \tag{27}$$

Similarly to the original optimisation problem (4), problem (27) can also be solved exactly and efficiently by Bellman's algorithm, as shown in the following section.

4. Numerical examples and simulations

In this section we illustrate the interplay between optimisation and equilibrium problems for models involving three and ten links, respectively. In order to compare our results with previous numerical experiments (involving tabu search and interior-point methods), we are using the parameter values considered in [5]. The simulations and diagrams presented in this section have been produced in Matlab®.

Before showing our numerical experiments, recall that the equilibrium system (18) always has a solution if the demand is greater than a certain value.

Proposition 4.1. *Indeed, according to [5, Thm 2.1], if* $f_i : [0, \infty) \to \mathbb{R}$, $i = 1, \dots, m$ *are strictly increasing and unbounded continuous functions, then the system* (18) *has a solution if and only if the demand denoted by* $D = x_1 + \cdots + x_m$ *satisfies*

$$D = \sum_{i=1}^{m} x_i > \sum_{i=1}^{m} f_i^{-1}(M_0) = D^*, \tag{28}$$

where $M_0 = \max_{1 \leq i \leq m}\{f_i(0)\}$. *For the functions* $f_i(x)$ *defined by* (1), *we have*

$$D^* = \sum_{i=1}^{m} c_i \left(\frac{M_0 - a_i}{b_i} \right)^{\frac{1}{p_i}}. \tag{29}$$

4.1. A model with three links

Here we investigate solutions of discrete optimisation and equilibrium problems for a model with three links (i.e., $m = 3$). Discrete solutions obtained by dynamic programming are validated against those obtained by exhaustive search.

We use travel time functions (1) for the 3 road example given in Table 1. In this case, moving x_1 cars along route 1 costs $g_1(x_1) = x_1 f_1(x_1)$. Similarly, $g_2(x_2) = x_2 f_2(x_2)$ and $g_3(x_3) = x_3 f_3(x_3)$. Therefore, the total travel time of $n = x_1 + x_2 + x_3$ cars along these routes is $T(x) = g_1(x_1) + g_2(x_2) + g_3(x_3)$, according to (3).

In Table 2 we compare exact solutions obtained for the optimisation problems (4) and (6) (which represents a counterpart of the equilibrium system (5)), obtained by brute force. The solution which minimises the total travel cost of for $n = 10000$ vehicles is $x_{opt} = (6804, 2179, 1017)$, while the variance of individual travel times is minimal for $x_{equ} = (6427, 2520, 1053)$. Surprisingly, x_{opt} also exhibits a small variance, while the individual travel costs only differ by about 1%.

	Problem (4)				Problem (6)			
Road	x_i	$\frac{x_i}{c_i}$	$\left(\frac{x_i}{c_i}\right)^{p_i}$	$f(x_i)$	x_i	$\frac{x_i}{c_i}$	$\left(\frac{x_i}{c_i}\right)^{p_i}$	$f(x_i)$
1	6804	1.70	2.89	2.6529	6427	1.61	2.58	2.5664
2	2179	1.45	3.07	2.1897	2520	1.68	4.74	2.5669
3	1017	1.02	1.09	2.5009	1053	1.05	1.29	2.5675
$\sigma^2(x)$				0.055				$3.09 \cdot 10^{-7}$
$\mu(x)$				2.447				2.5669
$T(x)$				25365.26				25666.39

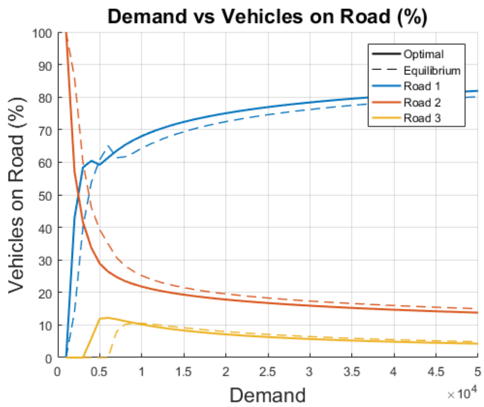
Table 2. Exact solutions for the optimisation and equilibrium problems (4) and (6) for a demand $n = 10000$.

In Table 3 we present solutions for the optimisation problems (4) and (27) (the latter representing a counterpart of the equilibrium system (5)) obtained by dynamic programming. Notice that the traffic levels (x_1, x_2, x_3) match almost perfectly.

	Problem (4)				Problem (27)			
Road	x_i	$\frac{x_i}{c_i}$	$\left(\frac{x_i}{c_i}\right)^{p_i}$	$f(x_i)$	x_i	$\frac{x_i}{c_i}$	$\left(\frac{x_i}{c_i}\right)^{p_i}$	$f(x_i)$
1	6804	1.70	2.89	2.6529	6428	1.61	2.58	2.5666
2	2179	1.45	3.07	2.1897	2520	1.68	4.74	2.5669
3	1017	1.02	1.09	2.5009	1052	1.05	1.29	2.5655
$\sigma^2(x)$				0.055				$5.04 \cdot 10^{-7}$
$\mu(x)$				2.447				2.5663
$T(x)$				25365.26				25665.74

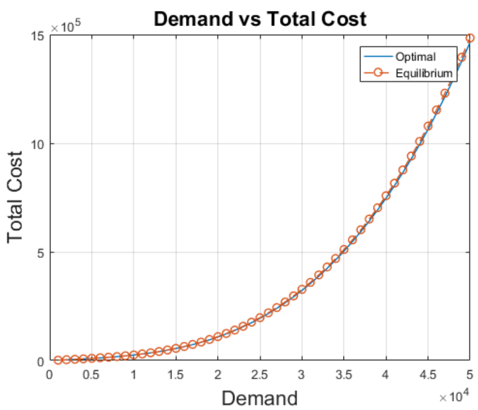
Table 3. Dynamic Programming solutions of the optimisation problems (4) and (27) for a demand $n = 10000$.

In Figure 1 we plot solutions of the optimisation problems (4) and (6), obtained by Tabu Search for demands between 1000 and 50000 given in [5]. Notice that the solutions are very similar. Initially, vehicles prefer link 2, while as demand increases, link 1 starts to attract more traffic. For high demand, road loading (as a percentage of total traffic) is determined by a combination of its capacity c_i and power p_i.

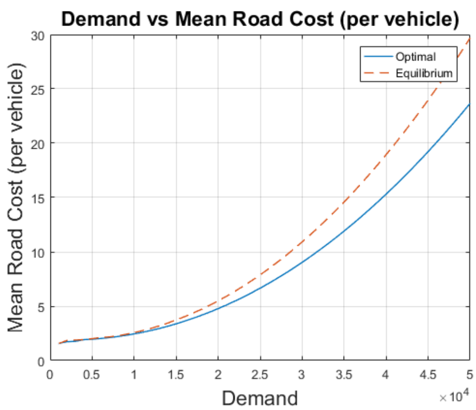


(a) Demand vs % Vehicles on Road

(b) Price of Anarchy



(c) Demand vs % Vehicles on Road



(d) Price of Anarchy

Figure 1. Results for an example with 3 roads (diagrams from [5]).

For low demand, some links may be empty. As the demand increases, new roads are used, which reflects in spikes of the price of anarchy. The two spikes in Figure 1 (b) correspond to the introduction of new roads, as shown by Figure 1 (a). Also, while the total costs between optimal and equilibrium solutions in Figure 1 (c) are similar, the mean cost of a road (per vehicle) may differ significantly (see Figure 1 (d)).

By Proposition 4.1, the solution of the continuous equilibrium problem (18) exists whenever the demand is above a certain value D^*. In our example, by (29) we have $M_0 = \max\{f_1(0), f_2(0), f_3(0)\} = 2.15$ and $D^* = 6295.33$. Hence, the problem (18) has a solution if and only if $x_1 + x_2 + x_3 > D^*$, and the solution satisfies

$$x_1 > f_1^{-1}(2.15) = 4000\sqrt{\frac{2.15 - 1.85}{0.2775}} = 4159.002;$$

$$x_2 > f_2^{-1}(2.15) = 1500\sqrt[3]{\frac{2.15 - 1.5}{0.225}} = 2136.329.$$

4.2. A model with ten links

Here we analyze an extended model with 10 roads (i.e., $m = 10$). The model has the parameter values given in Table 4 and was investigated in [5], where integer and real solutions of the problems described in Sections 2 and 3 have been presented in detail. Here we compare the solutions of the optimisation problems (4) and (27).

Road Parameters and Road Travel Time per Vehicle					
Road No.	a_i	$b_i = 0.15a_i$	c_i	p_i	$f_i(x_i) = a_i + b_i\left(\frac{x_i}{c_i}\right)^{p_i}$
1	1.2	0.18	2000	5	$f_1(x_1) = 1.2 + 0.18\left(\frac{x_1}{2000}\right)^5$
2	1.3	0.195	1500	5	$f_2(x_2) = 1.3 + 0.195\left(\frac{x_2}{1500}\right)^5$
3	0.8	0.12	3500	4	$f_3(x_3) = 0.8 + 0.12\left(\frac{x_3}{3500}\right)^4$
4	0.9	0.135	4000	4	$f_4(x_4) = 0.9 + 0.135\left(\frac{x_4}{4000}\right)^4$
5	1.4	0.21	2000	6	$f_5(x_5) = 1.4 + 0.21\left(\frac{x_5}{2000}\right)^6$
6	1	0.15	3000	6	$f_6(x_6) = 1 + 0.15\left(\frac{x_6}{3000}\right)^6$
7	1.1	0.165	1000	8	$f_7(x_7) = 1.1 + 0.165\left(\frac{x_7}{1000}\right)^8$
8	1.2	0.18	2000	8	$f_8(x_8) = 1.2 + 0.18\left(\frac{x_8}{2000}\right)^8$
9	1.3	0.195	1500	8	$f_9(x_9) = 1.3 + 0.195\left(\frac{x_9}{1500}\right)^8$
10	1.3	0.195	1000	8	$f_{10}(x_{10}) = 1.3 + 0.195\left(\frac{x_{10}}{1000}\right)^8$

Table 4. Travel cost functions $f_i(x_i)$ for a model involving 10 roads.

Solutions of the discrete optimisation problems (4) and (27) (counterpart of the equilibrium problem (5)), for $n = 30000$ vehicles are shown in Table 5. For $i = 1, \ldots, 10$ we show road loading x_i, normalized demand by road capacity $\frac{x_i}{c_i}$, normalized demand to the corresponding power $(x_i/c_i)^{p_i}$ and individual travel cost $f(x_i)$.

The solutions of the two problems produce very similar results in terms of total travel time. As expected, the solution of (4) produces a lower figure than that of (27), but the difference of just about 0.68%. Also, the variance of the individual travel costs $f_i(x_i)$, $i = 1, \ldots, 10$ is less than 10^{-6} for the solution of (27). This is not surprising, as (27) was the counterpart of the equilibrium problem (18). However, the solution of problem (4) also produces small values for the variance of $f_i(x_i)$, $i = 1, \ldots, 10$.

	Problem (4)				Problem (27)			
Road	x_i	$\frac{x_i}{c_i}$	$\left(\frac{x_i}{c_i}\right)^{p_i}$	$f(x_i)$	x_i	$\frac{x_i}{c_i}$	$\left(\frac{x_i}{c_i}\right)^{p_i}$	$f(x_i)$
1	2704	1.35	4.52	2.013	2623	1.31	3.88	1.899
2	1988	1.33	4.09	2.097	1877	1.25	3.07	1.899
3	6029	1.72	8.80	1.857	6088	1.74	9.15	1.899
4	6658	1.66	7.68	1.936	6596	1.65	7.39	1.899
5	2426	1.21	3.19	2.069	2310	1.16	2.37	1.899
6	3902	1.30	4.84	1.726	4043	1.35	5.99	1.899
7	1163	1.16	3.35	1.652	1218	1.22	4.84	1.899
8	2296	1.15	3.02	1.743	2369	1.18	3.88	1.899
9	1700	1.13	2.72	1.831	1726	1.15	3.07	1.899
10	1134	1.13	2.73	1.833	1150	1.15	3.06	1.899
$\sigma^2(x)$				0.023				$6.52 \cdot 10^{-7}$
$\mu(x)$				1.876				1.899
$T(x)$				56567.59				56950.76

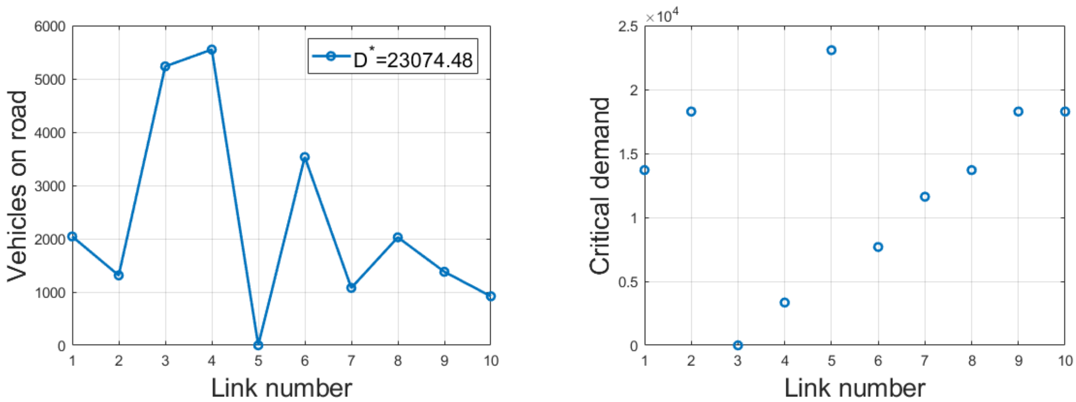
Table 5. Dynamic Programming solutions of the optimisation problems (4) and (27) for a demand $n = 30000$.

Other properties of the solution of (4) can be read from Table 5. The low power and high capacity routes take most traffic ($i = 3, 4$) and have the highest loading factor $\frac{x_i}{c_i} \sim 1.6$. Links 1 and 8 have same a_i, b_i and c_i values, but different p_i. The ratio $\frac{x_i}{c_i}$ is higher on road 1 (lower p_i value), thus the system can afford to overload this road; roads 9 and 10 have same a_i, b_i and p_i, but different capacity c_i. Here, the ratio x_9/x_{10} match the capacity ratio c_9/c_{10}, but the equal p_i values lead to equal loading factors.

By Proposition 4.1, the solution of the continuous equilibrium problem (18) exists whenever the demand is above the critical value given by $D^* = \sum_{i=1}^{10} f_i^{-1}(M_0)$, where $M_0 = \max_i\{f_i(0)\} = f_5(0) = 1.4$. It follows that (18) has a solution if and only if $x_1 + \cdots + x_{10} > D^* \sim 23074$, while the solution satisfies

$$x_i > f_i^{-1}(1.4) = c_i\sqrt{\frac{1.4 - a_i}{b_i}}, \quad i = 1, \ldots, 10, i \neq 5.$$

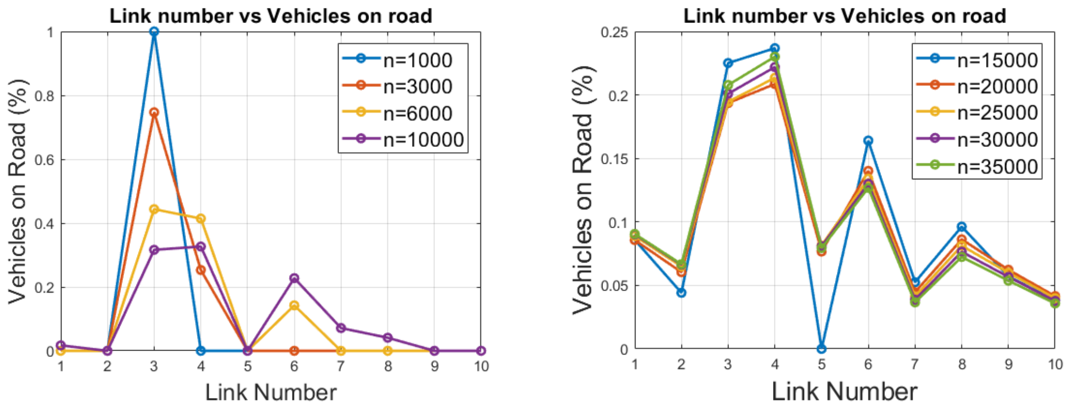
In Figure 2 we show some results concerning the continuous equilibrium problem.



(a) Equilibrium solution

(b) Link number vs Critical demand

Figure 2. Critical demand values and equilibrium solution for 10 roads.



(a) Small demand

(b) Large demand

Figure 3. Optimal solution: Link Number vs % Vehicles on Road.

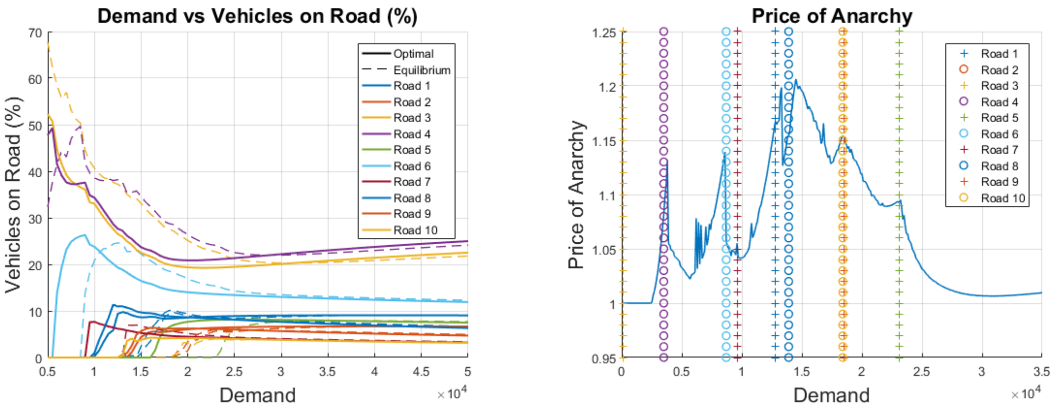
The solution of the continuous equilibrium problem (18) for the critical demand D^* is depicted in Figure 2 (a). Since $n = 30000 > D^*$, it was expected that the solution of problem (4) would involve traffic on all roads. Problem (18) can also be used to examine the demand levels required for involving traffic flows along new roads in the optimal solution. The ordered vector

$$0.8 = f_3(0) < f_4(0) < f_6(0) < f_7(0) < f_1(0) = f_8(0) < f_2(0) = f_9(0) = f_{10}(0) < f_5(0) = 1.4,$$

generates the ordered set $I = [3, 4, 6, 7, 1, 8, 2, 9, 10, 5]$. For each $j = 1, \ldots, 10$ we can define an equilibrium problem of type (18), involving the first j elements in set I, which has a solution, once the demand exceeds a critical value D_j. The results obtained for this model are illustrated in Figure 2 (b). At small demand only link 3 is used, while beyond $D = 3344$, link 4 starts to take traffic. Once the demand exceeds the critical value $D* = 23074$, all links will have some vehicles on them.

The optimal assignment of vehicles along individual roads is depicted in Figure 3. For small demand, vehicles are assigned to few roads, as seen in Figure 3 (a). For $n = 1000$ all cars choose link 3, while at $n = 3000$ about 25% of cars move along link 4. As the demand increases, other links are becoming cost-effective progressively (e.g., link 6 for $n = 6000$ and links 1, 7 and 8 for $n = 10000$). For larger demand, most roads are loaded, as in Figure 3 (b). For $n = 15000$ only link 5 is not yet occupied, while for $n \geq 20000$ all roads have traffic. However, with increasing demand, the vehicle density seems to stabilize towards an equilibrium state, which depends on the model parameters. In this model, the peaks are at links 3 (20%), 4 (23%) and 6 (12.5%).

In Figure 4 we compare the solution of problem (4) obtained by means of dynamic programming, against the solution of problem (6) (equilibrium counterpart) obtained in [5] by a Tabu Search heuristic. Figure 4 (a) shows the percentage of cars (of the total demand) allocated to each road (x_i/n). Once a road starts to be used, the traffic increases to a certain peak, until another road becomes more time efficient. The price of anarchy displayed in Figure 4 (b) shows vertical lines at the moment when a new road is used in the equilibrium solution. The results match well with those depicted in Figure 4 (a), the vertical lines corresponding to the position of the peaks. The noise within the graphs is related to the use of the heuristic method in [5].



(a) Demand vs % Vehicles on Road

(b) Price of Anarchy

Figure 4. Results for an example with 10 roads [5].

5. Conclusions and further work

In this paper we have considered a simplified traffic model, in which a source and a destination are connected via a number of parallel links, while various discrete and continuous optimisation and equilibrium problems have been formulated. The close interplay between discrete versions of the optimisation and equilibrium problems was explored theoretically, and illustrated by two models with 3 and 10 roads respectively, where solutions were given by means of dynamic programming.

In the future, comparisons with other TAP models [2, Chapter 3] could be considered, such as Traffic Paradoxes, whereby implementation of control structures on a network can result in an improvement to the network flow. For example, the well known Braess Paradox [10] is a phenomenon that can result in better overall travel time from the shutting down of roads (removal of network links).

Another interesting research direction for solving the problem (9) effectively, would be to develop a Lagrangian duality approach, which also employs the separability property of the convex objective function T (a finite sum of convex functions on $\mathbb{R}^m_+$).

Multi-criteria optimisation could also be used as an alternative methodology (see, e.g., [11], [12], [13], [14], and the references therein). Indeed, problem (4) may be seen as a particular linear scalarisation (with equal weights) of the multi-objective optimisation problem

$$\begin{cases} \text{Minimise } \ (g_1(x_1), \dots, g_m(x_m)) \\ \text{subject to} \\ x_1 + \cdots + x_m = n \\ x_1, \dots, x_m \in \mathbb{N}. \end{cases} \tag{30}$$

Then every optimal solution $x^0 = (x_1^0, \dots, x_m^0)$ of problem (4) is a Pareto-optimal (efficient) solution of the multi-objective optimisation problem (30), which means that for any feasible point $x = (x_1, \dots, x_m) \in \mathbb{N}^m$ with $x_1 + \cdots + x_m = n$, we have

$$\begin{cases} g_1(x_1) & \leq g_1(x_1^0) \\ \vdots \\ g_m(x_m) \leq g_m(x_m^0) \end{cases} \quad \Longrightarrow \quad \begin{cases} g_1(x_1) & = g_1(x_1^0) \\ \vdots \\ g_m(x_m) = g_m(x_m^0). \end{cases}$$

Similarly, the optimisation (equilibrium counterpart) problem (6) can be seen as a particular (ℓ_2-type) nonlinear scalarization of the multi-objective optimisation problem

$$\begin{cases} \text{Minimise } \ (e_1(x_1), \dots, e_m(x_m)) \\ \text{subject to} \\ x_1 + \cdots + x_m = n \\ x_1, \dots, x_m \in \mathbb{N}, \end{cases} \tag{31}$$

where the scalar functions $e_1, \dots, e_m$ are defined by

$$e_i(x_i) = |f_i(x_i) - \mu(x_1, \dots, x_m)|.$$

This means that every optimal solution of (6) can be found among the Pareto-optimal solutions of the multi-objective optimisation problem (31).

Realistic complex network geometries may also be considered, where the source and the destination are connected by multiple routes which share certain road segments, or scenarios involving multiple sources and sinks (i.e., Sioux Falls model [15]).

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