On some results concerning generalized arithmetic triangles

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Abstract

In this paper we present theoretical and computational results regarding generalized arithmetic m-triangles. The numerical values recover well-known number sequences, indexed in the OEIS including binomial coefficients and their extensions. Some combinatorial interpretations, generating functions and also asymptotic formulae for these triangles are provided.

Keywords: recurrent sequences, pentanomial numbers, generalized binomial coefficients, asympotic formulae.

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1 Introduction

Let m and n be positive integers. The element of the m-arithmetic triangle located at the intersection of the ith row and jth column denoted by $p_{ij}^{(m)}$ is defined by the recurrence

$$p_{ij}^{(m)} = p_{i-1j}^{(m)} + p_{i-1j-1}^{(m)} + \cdots + p_{i-1j-m+1}^{(m)},$$

with the initial conditions

$$p_{0j}^{(m)} = \begin{cases} 0 \text{ if } j < 0, \\ 1 \text{ if } j = 0, \\ 0 \text{ if } j > 0. \end{cases}$$

The element in each cell is the sum of m elements: the element directly above the given element and the $m-1$ elements to the left of it. Hence, the matrix

$$P^{(m)}(n) = \left(p_{ij}^{(m)}\right), \quad 0 \le i, j \le n-1$$

of the elements of the m-arithmetic triangle is defined as follows

$$p_{ij}^{(m)} = \begin{cases} 0 & \text{if } i = 0,\ 1 \le j \le n-1, \\ 1 & \text{if } j = 0,\ 0 \le i \le n-1 \\ \sum\limits_{l=j-m+1}^{j} p_{i-1,l}^{(m)} & \text{if } 1 \le i, j \le n-1. \end{cases}$$

For $m = 2$ one obtains the elements in Pascal's triangle. Numerous OEIS sequences are obtained from particular columns of the m-triangle.

For example, for $p_{n3}^{(3)}$ one obtains the sequence indexed as A005581

$$0, 0, 2, 7, 16, 30, 40, 77, 112, 156, 210, \ldots,$$

in the Online Encyclopedia of Integer Sequences (OEIS) [7].

The sequence $p_{n4}^{(3)}$ whose terms are given by

$$0, 0, 1, 6, 19, 45, 90, 161, 266, 414, 615, \ldots$$

corresponds to sequence A005712.

2 Numerical computation of the m-triangles

For a fixed value of $m \geq 2$ and $n \geq 1$ one can compute the rows of the m-triangle by matrix iterations. Denoting by $p_i^{(m)}$ the ith row of the m-triangle, the following formula holds

$$p_{i+1}^{(m)} = p_i^{(m)} M_{m,n},$$

where the matrix $M_{m,n}$ has size $[n(m-1)+1] \times [n(m-1)+1]$ and has m diagonals whose entries are all equal to 1, as shown below

$$M_{m,n} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \quad (1)$$

One obtains recursively the following identities

$$p_i^{(m)} = p_{i-1}^{(m)} M_{m,n} = p_{i-2}^{(m)} M_{m,n}^2 = \cdots = p_0^{(m)} M_{m,n}^i.$$

Numerous integer sequences are obtained as particular cases:

- $m = 2$: binomial coefficients A007318;
- $m = 3$: trinomial coefficients A027907; The first four rows give the values:

$$1, 1, 1, 1, 1, 2, 3, 2, 1, 1, 3, 6, 7, 6, 3, 1, 1, 4, 10, 16, 19, 16, 10, 4, 1, \dots$$

- $m = 4$: quadrinomial coefficients A008287; First three rows give the values:

$$1, 1, 1, 1, 1, 1, 2, 3, 4, 3, 2, 1, 1, 3, 6, 10, 12, 12, 10, 6, 3, 1, \dots$$

- $m = 5$: pentanomial coefficients A035343.

The sequence of maximum values over each row gives the sequences A001405 ($m = 2$), A002426 ($m = 3$), A035343 ($m = 4$) and A005191 ($m = 5$).

Other related results can be found in [2], [3] and [4].

3 Some Explicit Formulae

Let $p_{ij}^{(m)}$ be the element located at the intersection of the ith row and jth column of the m-arithmetic triangle. The generating function of these numbers is given by [1]

$$(1 + x + x^2 + \ldots + x^{m-1})^i = \sum_{j=0}^{(m-1)i} p_{ij}^{(m)} x^j, \quad m \in \mathbb{N}, \quad i \in \mathbb{N} \cup \{0\}.$$

The element $p_{ij}^{(m)}$ is the coefficient of x^j in the formal expansion of

$$(1 + x + x^2 + \ldots + x^{m-1})^i.$$

We can formulate the following results.

Theorem 3.1 *Let* $l = \min\{i, j\}$. *Then for* $m \in \mathbb{N}$ *and* $i \in \mathbb{N} \cup \{0\}$

$$p_{ij}^{(m)} = \sum_{\substack{s_0+s_1+\ldots+s_{m-1}=i \\ s_1+2s_2+\ldots+(m-1)s_{m-1}=j \\ s_\nu=0,1,\ldots,l}} \frac{i!}{s_0! s_1! \ldots s_{m-1}!}, \quad j = 0, 1, \ldots, (m-1)i.$$

Theorem 3.2 *Let* $l = \min\{i, j\}$. *Then for* $m = 3$ *we have*

$$p_{ij}^{(3)} = \sum_{k=j-l}^{\lfloor j/2 \rfloor} \frac{i!}{k!(j-2k)!(i-j+k)!}, \quad j = 0, \ldots, 2i.$$

Example 3.3 As an example, we consider the case when $m = 3$, $i = 4$, $j = 2$. Then, according to Theorem 3.2, $l = 2$, $j - l = 0$, and $\lfloor j/2 \rfloor = 1$, and we get

$$p_{42}^{(3)} = \sum_{k=0}^{1} \frac{4!}{k!(2-2k)!(2+k)!} = \frac{4!}{0!2!2!} + \frac{4!}{1!0!3!} = 10$$

which is exactly the number positioned in the 4th row and 2nd column of the 3-arithmetic triangle (see the Table 1).

Example 3.4 Consider the 4-arithmetic triangle and put $m = 4$, $i = 2$, and $j = 3$. Then $l = 2$, and by formula for $p_{ij}^{(m)}$ in Theorem 3.1, we have

$$p_{23}^{(4)} = \sum_{\substack{s_0+s_1+s_2+s_3=2 \\ s_1+2s_2+3s_3=3 \\ s_\nu=0,1,2}} \frac{2!}{s_0! s_1! s_2! s_3!}.$$

$R \setminus C$	-2	-1	0	1	2	3	4	5	6	7	8	9	10	...
0	0	0	1											
1	0	0	1	1	1									
2	0	0	1	2	3	2	1							
3	0	0	1	3	6	7	6	3	1					
4	0	0	1	4	10	16	19	16	10	4	1			
5	0	0	1	5	15	30	45	51	45	30	15	5	1	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Table 1
3-Arithmetic Triangle

From the conditions

$$s_1 + 2s_2 + 3s_3 = 3 \qquad \text{and} \qquad s_1 + s_2 + s_3 \leq 2,$$

we find two sets of solutions

$$\{s_1 = 0, s_2 = 0, s_3 = 1\} \qquad \text{and} \qquad \{s_1 = 1, s_2 = 1, s_3 = 0\}.$$

Then the value of $s_0 = 1$ for the first set and $s_0 = 0$ for the second set.

Hence, we obtain $p_{23}^{(4)} = \frac{2!}{1!0!0!1!} + \frac{2!}{0!1!1!0!} = 2+2 = 4$, which is the number positioned at the intersection of the 2th row and 3nd column of the 4-arithmetic triangle (see the Table 2).

$R \setminus C$	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	...
0	0	0	0	1												
1	0	0	0	1	1	1	1									
2	0	0	0	1	2	3	4	3	2	1						
3	0	0	0	1	3	6	10	12	12	10	6	3	1			
4	0	0	0	1	4	10	20	31	40	44	40	31	20	10	4	1

Table 2
4-Arithmetic Triangle

4 Future work

Future investigations will be dedicated to the identification of new integer sequences related to m-sequences and to establishing of asymptotic expansions for the numbers $p_{ij}^{(m)}$ when $i, j \to \infty$ and $m \in \mathbb{N}$, $m \geq 2$ fixed.

The key results in the proof are based on theory given in [5] and [6].

Proposition 4.1 *Let ξ be a random variable with the probability distribution*

$$\mathbf{P}\{\xi = \mathbf{k}\} = \frac{1}{m}, \quad k = 0, 1, \ldots, m-1.$$

Then the cumulants $\mathfrak{c}_n$ of a random variable ξ are defined by the formula

$$\mathfrak{c}_1 = \mathbf{E}[\xi] = \frac{m-1}{2}, \quad \mathfrak{c}_{2\nu} = \frac{B_{2\nu}}{2\nu}(m_{2\nu} - 1), \quad \mathfrak{c}_{2\nu+1} = 0,$$

where $B_{2\nu}$ are the Bernoulli numbers, $\nu = 1, 2, \ldots$.

Proposition 4.2 *Let $\xi_1, \ldots, \xi_i$ be independent random variables with the probability distribution of ξ. Then we have*

$$p_{ij}^{(m)} = m^i \, \mathbf{P}\{\xi_1 + \ldots + \xi_i = j\}, \quad j = 0, 1, \ldots, (m-1)i$$

The formula for $\mathfrak{c}_n$ above can be derived using the expression [5]

$$\ln \mathbf{E}\left[e^{z\xi}\right] = \frac{m-1}{2}z + \sum_{n=2}^{\infty} \frac{\mathfrak{c}_n}{n!} z^n, \quad |z| < \frac{2\pi}{m}.$$

Theorem 4.3 *Let $i \to \infty$, $m \geq 2$, $m \in \mathbb{N}$ and let $j \to \infty$, $j \in \mathbb{N}$, such that*

$$j = \frac{1}{2}\left((m-1)i + x\sqrt{\frac{i(m^2-1)}{3}}\right), \quad |x| \leq c, \ c = \text{const}.$$

Then, uniformly with respect to $x \in [-c, c]$, we have

$$p_{ij}^{(m)} = m^i \sqrt{\frac{6}{\pi i(m^2-1)}} \, e^{-\frac{x^2}{2}} \left(1 + \sum_{\nu=1}^{r} \frac{Q_{2\nu}(x)}{i^\nu} + O\left(i^{-r-1}\right)\right), \quad r = 1, 2, \ldots$$

where $Q_{2\nu}(x)$ are polynomials in x, $\nu = 1, 2, \ldots$, given by

$$Q_\nu(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{\substack{k_1+2k_2+\ldots+\nu k_\nu=0 \\ k_1+k_2+\ldots+k_\nu=s}} H_{\nu+2s}(x) \prod_{t=1}^{\nu} \frac{1}{k_t!} \left(\frac{\mathfrak{c}_{t+2}}{(t+2)!\sigma^{t+2}}\right)^{k_t}.$$

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