A GENERATING FUNCTION APPROACH TO THE AUTOMATED EVALUATION OF SUMS OF EXPONENTIATED MULTIPLES OF GENERALIZED CATALAN NUMBER LINEAR COMBINATIONS

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Abstract. Based on a previous technique deployed in some specific low order cases, we develop an automated computational procedure to evaluate instances within a class of infinite series comprising exponentiated multiples of generalized linear combinations of Catalan numbers. The methodology is explained, and new results given.

1. Introduction

1.1. **Background.** Let $c_n = \frac{1}{n+1}\binom{2n}{n}$ denote the $(n+1)$th term $(n \geq 0)$ of the sequence $\{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{1, 1, 2, 5, 14, \ldots\}$ of Catalan numbers whose (ordinary) generating function is

$$G(x) = \frac{1}{2x}(1 - \sqrt{1-4x}) = \sum_{n\geq 0} c_n x^n. \tag{1.1}$$

This paper examines the evaluation of a class of series

$$\begin{aligned} I_p(\beta) &= (-1)^p \sum_{n\geq 1} \beta^n h_p(c_{n-1}, \ldots, c_{n+p-2}) \\ &= \sum_{n\geq 1} \beta^n [b_0^{(p)} c_{n-1} + b_1^{(p)} c_n + b_2^{(p)} c_{n+1} + \cdots + b_{p-1}^{(p)} c_{n+p-2}], \end{aligned} \tag{1.2}$$

the summand of which comprises, for integer $p \geq 1$, a particular p-term linear combination of Catalan numbers with an exponentiated multiplier $\beta \neq 0$. Values of the constants $b_0^{(p)}, \ldots, b_{p-1}^{(p)}$ are, from [2, Lemma 2, p. 216], known to have closed form

$$b_r^{(p)} = (-1)^r 16^{p-(r+1)} \binom{2p-(r+1)}{r}, \qquad r = 0, \ldots, p-1, \tag{1.3}$$

being those (up to a sign) of the linear function $h_p(c_{n-1}, \ldots, c_{n+p-2})$ which, for any fixed p, arises in a power series expansion of the trigonometric function $\sin(2p\alpha)$ according to

$$\sin(2p\alpha)/2 = \sum_{n=1}^{p} \alpha_n^{(p)} \sin^{2n-1}(\alpha) + \sum_{n=1}^{\infty} \frac{h_p(c_{n-1}, \ldots, c_{n+p-2})}{2^{2(n+p)-3}} \sin^{2(n+p)-1}(\alpha) \tag{1.4}$$

in odd powers of $\sin(\alpha)$.

This type of power series was first discussed in 2000 by Larcombe [1] (where historical context was provided), and more recently aspects of convergence have been examined [6] with the natural principal range of convergence extended from $|\alpha| < \frac{\pi}{2}$ (seen in [1]) to $|\alpha| \leq \frac{\pi}{2}$ analytically. Noting the parity of the sine function, it is found that the α interval $(0, \frac{\pi}{2}]$ in (1.4) translates to a range $\beta \in (0, \frac{1}{4}]$ in (1.2), and on this basis evaluations have been made for the series $S(\beta) = I_1(\beta)$ in [3], and for $T(\beta) = \frac{1}{2}I_2(\beta)$ and $U(\beta) = I_3(\beta)$ in [4], using a variety of methods (that is, via direct evaluation of (1.4) itself, by appeal to hypergeometric theory, and through a generating function route that we extend here). A purely hypergeometric approach to tackle the completely general series $I_p(\beta)$ for the first time can be seen in [5], where new evaluations of $I_4(\beta)$ and $I_5(\beta)$ have been given.

We now outline the remit of the paper, before moving on to further sections that detail our methodology and results.

1.2. **This Paper.** It is appropriate to list the first few series described by (1.2), which are, using (1.3),

$$\begin{aligned}
I_1(\beta) &= \sum_{n\geq 1}\beta^n[c_{n-1}],\\
I_2(\beta) &= 2\sum_{n\geq 1}\beta^n[8c_{n-1}-c_n],\\
I_3(\beta) &= \sum_{n\geq 1}\beta^n[256c_{n-1}-64c_n+3c_{n+1}],\\
I_4(\beta) &= 4\sum_{n\geq 1}\beta^n[1024c_{n-1}-384c_n+40c_{n+1}-c_{n+2}],\\
I_5(\beta) &= \sum_{n\geq 1}\beta^n[65536c_{n-1}-32768c_n+5376c_{n+1}-320c_{n+2}+5c_{n+3}],\\
I_6(\beta) &= 2\sum_{n\geq 1}\beta^n[524288c_{n-1}-327680c_n+73728c_{n+1}-7168c_{n+2}+280c_{n+3}-3c_{n+4}],\\
I_7(\beta) &= \sum_{n\geq 1}\beta^n[16777216c_{n-1}-12582912c_n+3604480c_{n+1}-491520c_{n+2}\\
&\qquad +\ 32256c_{n+3}-896c_{n+4}+7c_{n+5}],\\
I_8(\beta) &= 8\sum_{n\geq 1}\beta^n[33554432c_{n-1}-29360128c_n+10223616c_{n+1}-1802240c_{n+2}\\
&\qquad +\ 168960c_{n+3}-8064c_{n+4}+168c_{n+5}-c_{n+6}], \qquad (1.5)
\end{aligned}$$

and so on. We develop one of the methods deployed in [4], which enables evaluations of those members of the series class (1.2) to be made easily in an automated fashion. While the hypergeometric formulation of [5] relies, using specialized algebraic software, on the symbolic calculation of a requisite (degree p) polynomial through which to deliver evaluations in any p instance, ours here—based on the Catalan sequence generating function $G(x)$ alone—is a much simpler one to design theoretically and execute computationally; we are thus able to realize, in a straightforward manner, evaluations of series not considered to date.

2. Methodology

For any $p \geq 1$ the series $I_p(\beta)$ can be represented as a function of β, $G(\beta)$ and the explicit $p-1$ Catalan numbers $c_0, \ldots, c_{p-2}$. This is evident from manipulation of the final summand term of (1.2) into an appropriate form for use. Ignoring the accompanying constant for convenience, we write

$$\begin{aligned}\sum_{n\geq 1}\beta^n c_{n+p-2} &= \beta^{-(p-2)}\sum_{n=p-1}^{\infty}\beta^n c_n\\ &= \beta^{-(p-2)}\left(\sum_{n=0}^{\infty}\beta^n c_n - \sum_{n=0}^{p-2}\beta^n c_n\right)\\ &= \beta^{-(p-2)}\left(G(\beta) - \sum_{n=0}^{p-2}\beta^n c_n\right), \qquad (2.1)\end{aligned}$$

giving, for particular values $p = 2, 3, 4, 5, \ldots,$

$$\begin{aligned}\sum_{n\geq 1}\beta^n c_n &= G(\beta) - 1,\\ \sum_{n\geq 1}\beta^n c_{n+1} &= \beta^{-1}[G(\beta) - (1+\beta)],\\ \sum_{n\geq 1}\beta^n c_{n+2} &= \beta^{-2}[G(\beta) - (1+\beta+2\beta^2)],\\ \sum_{n\geq 1}\beta^n c_{n+3} &= \beta^{-3}[G(\beta) - (1+\beta+2\beta^2+5\beta^3)], \qquad (2.2)\end{aligned}$$

whose continuing pattern is clear. Expressing $I_p(\beta)$ (1.2) concisely as

$$I_p(\beta) = \sum_{s=1}^{p}\left(\sum_{n\geq 1}\beta^n b_{s-1}^{(p)} c_{n+s-2}\right), \qquad (2.3)$$

then since (2.1) holds for $p = 1$ also (where, taking the sum $\sum_{n=0}^{-1}\beta^n c_n$ to be zero, it correctly reads $\sum_{n\geq 1}\beta^n c_{n-1} = \beta^{-(-1)}[G(\beta) - 0] = \beta G(\beta)$), we arrive from (2.3) at the following after a little work:

Theorem 2.1. *For $p \geq 1$,*

$$I_p(\beta) = \left(\sum_{i=0}^{p-1} b_i^{(p)}\beta^{1-i}\right)G(\beta) - \sum_{i=1}^{p-1} b_i^{(p)}\beta^{1-i}\left(\sum_{j=0}^{i-1} c_j\beta^j\right).$$

3. Results

3.1. **Closed Forms for $I_p(\beta)$.** The formula that is Theorem 2.1 is easily coded up, yielding closed forms (factored where required)

$$\begin{aligned}I_1(\beta) &= \beta G(\beta),\\ I_2(\beta) &= 2[(8\beta-1)G(\beta)+1],\\ I_3(\beta) &= (256\beta - 64 + 3\beta^{-1})G(\beta) + 61 - 3\beta^{-1},\\ I_4(\beta) &= 4[(1024\beta - 384 + 40\beta^{-1} - \beta^{-2})G(\beta) + 346 - 39\beta^{-1} + \beta^{-2}],\\ I_5(\beta) &= (65536\beta - 32768 + 5376\beta^{-1} - 320\beta^{-2} + 5\beta^{-3})G(\beta)\\ &\qquad + 28007 - 5066\beta^{-1} + 315\beta^{-2} - 5\beta^{-3},\\ I_6(\beta) &= 2[(524288\beta - 327680 + 73728\beta^{-1} - 7168\beta^{-2} + 280\beta^{-3} - 3\beta^{-4})G(\beta)\\ &\qquad + 266930 - 67105\beta^{-1} + 6894\beta^{-2} - 277\beta^{-3} + 3\beta^{-4}], \qquad (3.1)\end{aligned}$$

in the first six instances, though further ones are readily generated; this puts us in a position to give some new results.

3.2. **New Results.** In [5, Eqs. (1.8), (1.9), p. 261] evaluations were listed for $I_p(\beta)$ $(p = 1, 2, 3)$ at $\beta = \frac{1}{4}, \frac{3}{16}, \frac{1}{8}$ and $\frac{1}{16}$ (interest in these particular values is explained in Section 1.2 therein), and further ones generated correspondingly for $I_4(\beta)$ and $I_5(\beta)$ (see (4.4),(4.5) of Section 4, p. 268). All of these are a reproducible consequence of (3.1) above (see [3,4] for its lower order applications in more detail), from which we can add additional ones as follows. In line with the earlier work described, we find that

$$\begin{aligned} I_6(1/4) &= 186780, \\ I_6(3/16) &= 131076, \\ I_6(1/8) &= 83540, \\ I_6(1/16) &= 40260, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} I_7(1/4) &= 2723234, \\ I_7(3/16) &= 1412842090/729, \\ I_7(1/8) &= 2(575237 + 32768\sqrt{2}), \\ I_7(1/16) &= 2(3933277 - 2097152\sqrt{3}), \\ \\ I_8(1/4) &= 40311776, \\ I_8(3/16) &= 63323144416/2187, \\ I_8(1/8) &= 18663968, \\ I_8(1/16) &= 32(2097152\sqrt{3} - 3349019), \end{aligned} \tag{3.3}$$

all having been tested numerically—to a low tolerance—against those convergent values of the series of (1.5) that they describe in error-free form.

4. Summary

Some fairly routine analysis facilitates the automated evaluation of specific instances within a class of infinite series comprising exponentiated multiples of generalized Catalan number linear combinations, with some hitherto unseen results presented. The evaluating formula established (Theorem 2.1)—being easy to derive and straightforward to code up—means that the methodology has a clear advantage over the hypergeometric approach taken previously [5] in terms of its relative level of complexity.

This short article, together with [3–5], completes work undertaken on the evaluation of some unusual series that arise naturally within a topic whose historic origins are of mathematical interest [1,2] and have been recognized in a genealogy of the Catalan sequence (see the definitive account by Igor Pak in Richard Stanley's latest text [7, Appendix B, pp. 177–189]). It may be the case that certain linear combinations (with or without an exponentiated multiplier) of integers drawn from sequences other than the Catalan sequence are equally summable—either by an existing technique such as that described here, or else through a new one—which is a question posed and left open for the attention of any reader.

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