

On the ratios and geometric boundaries of complex Horadam sequences

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Abstract

Horadam sequences are second-order linear recurrences in the complex plane which depend on two initial conditions and two recurrence coefficients which are complex numbers. Recently, numerous papers have been devoted to the periodicity of these sequences, as well as to generalizations and applications. In this paper we investigate aspects related to the sequence of ratios of consecutive terms and geometric bounds of Horadam sequences. We also propose some directions for further study.

Keywords: recurrent sequences, Horadam sequence, geometric patterns, generalised golden ratio, geometric bounds.

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1 Introduction

Horadam sequences are general second order recurrences defined by

$$w_{n+2} = pw_{n+1} + qw_n, \quad w_0 = a, w_1 = b, \quad (1)$$

where the parameters a, b, p and q are complex numbers. For convenience, the $\{w_n(a, b; p, q)\}_{n=0}^{\infty}$ is often written simply as $\{w_n\}_{n=0}^{\infty}$. Named in honor of A.F. Horadam who initiated the study of (1) in his 1960's papers [9] and [10], this recursion was studied in numerous recent papers [11].

The characteristic equation associated with the recurrence (1) is

$$P(x) = x^2 - px - q = 0, \quad (2)$$

whose roots termed *generators* are denoted by z_1 and z_2 and satisfy

$$p = z_1 + z_2, \quad q = -z_1z_2. \quad (3)$$

Periodic Horadam orbits were characterized in [1] and arise when zeros z_1 and z_2 of the characteristic equation (2) are roots of unity (i.e., $z_1 = e^{2\pi ip_1/k_1}$ and $z_2 = e^{2\pi ip_2/k_2}$ for some integers p_1, k_1, p_2, k_2). The enumeration and structure of self-repeating orbits was investigated in [2], [3] and [6]. Some applications of non-periodic orbits to random-number generation were given in [4].

Many important sequences are recovered as particular instances, and are indexed in the Online Encyclopedia of Integer Sequences (OEIS [12]):

- Fibonacci numbers for $(a, b) = (0, 1)$ and $(p, q) = (1, 1)$ ([A000045](#) in OEIS):

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, F_1 = 1.$$

- Lucas numbers for $(a, b) = (2, 1)$ and $(p, q) = (1, 1)$ ([A000032](#)):

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, L_1 = 1.$$

- Pell numbers for $(a, b) = (0, 1)$ and $(p, q) = (2, 1)$ ([A000129](#)):

$$P_{n+2} = 2P_{n+1} + P_n, \quad P_0 = 0, P_1 = 1.$$

Sequences of ratios of consecutive terms also present special interest:

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \frac{L_{n+1}}{L_n} = \frac{\sqrt{5} + 1}{2} = 1.618\dots = \varphi \text{ (golden ratio)} \quad (4)$$

$$\lim_{n \rightarrow \infty} \frac{P_{n+1}}{P_n} = \sqrt{2} \text{ (silver ratio)}. \quad (5)$$

Horadam sequences generate interesting geometries which can be visualised in the complex plane. In this paper we investigate ratios of Horadam sequences and explore parameter values producing geometric orbits matching an annulus of given radii $0 < R_1 < R_2$. We also analyse ratios of generalised Horadam sequences of third order, using explicit formulae derived in [8].

2 Preliminaries

We summarize key properties of Horadam sequences, like formulae for the general term, periodicity conditions and geometric bounds for stable orbits.

2.1 General sequence term

If the roots z_1, z_2 of (2) are distinct (non-degenerate case), then the general term of the sequence $\{w_n\}_{n=0}^{\infty}$ is (see, e.g., [1])

$$w_n = Az_1^n + Bz_2^n, \quad (6)$$

where constants A and B are obtained from the initial conditions as

$$A = \frac{az_2 - b}{z_2 - z_1}, \quad B = \frac{b - az_1}{z_2 - z_1}. \quad (7)$$

If the roots z_1, z_2 of (2) are equal (degenerate case), then the n -th term is

$$w_n = \left[a + \left(\frac{b}{z} - a \right) n \right] z^n = \left[az + (b - az)n \right] z^{n-1} = (C + Dn)z^n. \quad (8)$$

Notice that unless the sequence is identically zero ($A = B = 0, abz = 0$), orbits diverge whenever $\max\{|z_1|, |z_2|\} > 1$.

2.2 Periodic orbits

Recall that [1, Theorem 3.2] states that a Horadam sequence is periodic if there is a positive integer k such that

$$A(z_1^k - 1)z_1 = 0, \quad B(z_2^k - 1)z_2 = 0.$$

with A, B given in (7). In non-trivial cases, z_1 and z_2 are distinct roots of unity and $AB \neq 0$. Periodic Horadam patterns include regular star polygons (Fig. 1 (a)), bipartite digraphs, or patterns with dual symmetry (Fig. 1 (b)).

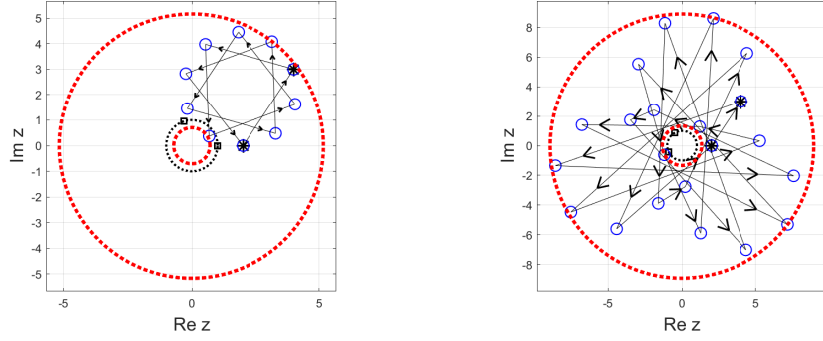


Fig. 1. Sequence $\{w_n\}_{n=0}^{100}$ (circles) from (6), for $a = 2$, $b = 4 + 3i$ (stars) and the generators (squares) (a) $z_1 = 1$, $z_2 = e^{2\pi i \frac{3}{10}}$; (b) $z_1 = e^{2\pi i \frac{1}{3}}$, $z_2 = e^{2\pi i \frac{4}{7}}$. Arrows indicate the orbit direction. Boundaries of annulus $U(0, |A| - |B|, |A| + |B|)$ (dotted line) with A, B from (7) and the unit circle (solid line) are also plotted.

2.3 Stable orbits and geometric bounds

If the (distinct) roots of (2) satisfy $|z_1| = |z_2| = 1$, then by the triangle inequality and (6), the Horadam orbits generated are located inside the annulus

$$U(0, R_1, R_2) = \{z \in \mathbb{C} : R_1 \leq |z| \leq R_2\},$$

where $R_1 = ||A| - |B||$ and $R_2 = |A| + |B|$ and A and B given by (7). Such (non)periodic orbits are called *stable* are bounded by annuli and exhibit complex patterns dense within the graph of 1D curves, or 2D annuli in the complex plane, as seen in Fig. 4.

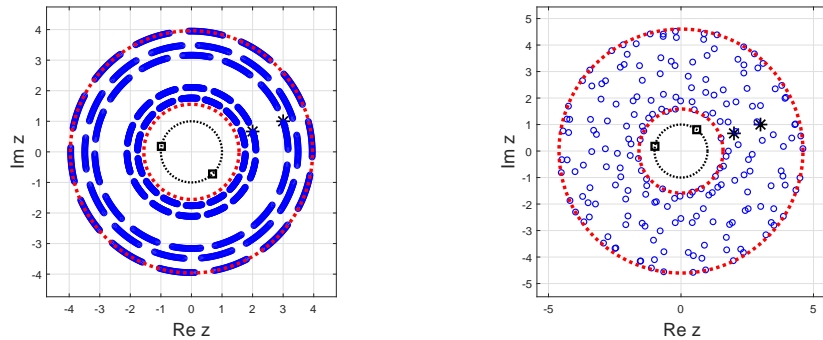


Fig. 2. Terms $\{w_n\}_{n=0}^N$ (circles) from (6), for $a = 2 + \frac{2}{3}i$, $b = 3 + i$ (stars) and (a) $z_1 = e^{2\pi i \frac{\sqrt{2}}{3}}$, $z_2 = e^{2\pi i \frac{\sqrt{2}}{3} + \frac{2}{5}}$, $N = 200$; (b) $z_1 = e^{2\pi i \frac{\sqrt{2}}{3}}$, $z_2 = e^{2\pi i \frac{\sqrt{5}}{15}}$, $N = 2000$.

3 Main results

Here we explore ratios of consecutive terms of Horadam sequences, first considered earlier by [13]. We extend these with analysis of non-convergent cases. We also analyze parameter values producing orbits matching a given annulus of radii $0 < R_1 < R_2$.

3.1 Ratios of Horadam sequences

Extending the concept of Golden ratio for a general Horadam sequence $\{w_n\}_{n=0}^{\infty}$, we define the following sequence of ratios of consecutive terms:

$$q_n = \frac{w_{n+1}}{w_n}, \quad n \in \mathbb{N}, \quad (9)$$

well defined if $w_n \neq 0$ for $n \in \mathbb{N}$. We have two main cases to analyze.

Case 1. Non-degenerate case ($z_1 \neq z_2$): In this case we have

$$q_n = \frac{w_{n+1}}{w_n} = \frac{Az_1^{n+1} + Bz_2^{n+1}}{Az_1^n + Bz_2^n}. \quad (10)$$

The condition $w_n \neq 0$ gives $(az_2 - b)z_1^n \neq (az_1 - b)z_2^n$. The limit satisfies:

- (a) If $|z_1| < |z_2|$, then $\lim_{n \rightarrow \infty} q_n = z_2$
- (b) If $|z_1| > |z_2|$, then $\lim_{n \rightarrow \infty} q_n = z_1$
- (c) If $|z_1| = |z_2|$, then $\lim_{n \rightarrow \infty}$ is indeterminate.

While (a) and (b) have been investigated in the literature (see, e.g., [13]), item (c) also presents interest in itself. The orbit produced can be periodic if z_1 and z_2 are roots of unity (Fig. 3 (a)), or dense in a circle (Fig. 3 (b)). Indeed, if $|z_1| = |z_2|$, denoting by $z = z_2/z_1 = e^{ix}$, one can write

$$q_n = \frac{w_{n+1}}{w_n} = z_1 \frac{A + Be^{i(n+1)x}}{A + Be^{inx}}. \quad (11)$$

If $x \in \mathbb{Q}$, then sequence $\{q_n\}_{n=0}^{\infty}$ is periodic, while for $x \in \mathbb{R} \setminus \mathbb{Q}$ the orbit of $\{q_n\}_{n=0}^{\infty}$ is dense in a circle. Note that z_1 and z_2 must not be roots of unity!

Case 2. Degenerate case ($z_1 = z_2 = z$): In this case the condition $w_n \neq 0$ reduces here to $C \neq -nD$ for all $n \in \mathbb{N}$, and the limit is

$$\lim_{n \rightarrow \infty} q_n = \lim_{n \rightarrow \infty} \frac{w_{n+1}}{w_n} = \lim_{n \rightarrow \infty} \frac{(C + D(n+1))z^{n+1}}{(C + Dn)z^n} = z. \quad (12)$$

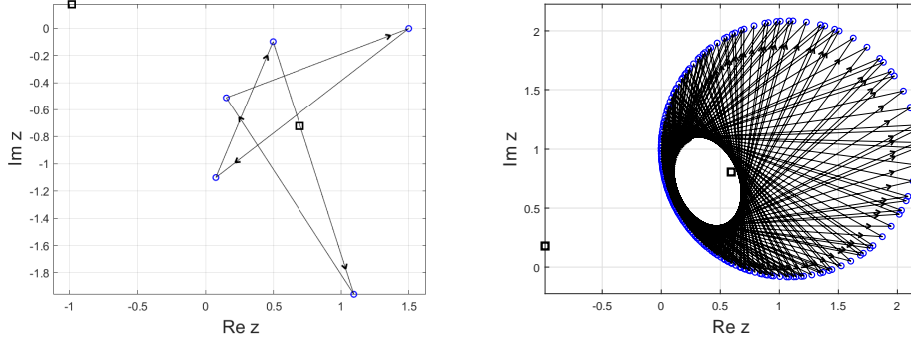


Fig. 3. Sequence $\{q_n\}_{n=0}^N = \left\{ \frac{w_{n+1}}{w_n} \right\}_{n=0}^N$ (circles) obtained from (6), for $a = 2 + \frac{2}{3}i$, $b = 3 + i$ (stars) and (a) $z_1 = e^{2\pi i \frac{\sqrt{2}}{3}}$, $z_2 = e^{2\pi i \frac{\sqrt{2}}{3} + \frac{2}{5}}$, $N = 200$; (b) $z_1 = e^{2\pi i \frac{\sqrt{2}}{3}}$, $z_2 = e^{2\pi i \frac{\sqrt{5}}{15}}$, $N = 2000$. Arrows indicate the direction of the orbit.

3.2 Horadam orbits confined to the interior of an annulus

Consider the real numbers $0 < R_1 < R_2$. We find the condition required for a Horadam sequence to be confined to the region between an inner circle of radius R_1 and an outer circle of radius R_2 , both centered in the origin.

Theorem 3.1 *Let $\{w_n\}_{n=0}^\infty$ be a Horadam sequence with distinct generators satisfying $\max\{|z_1|, |z_2|\} = 1$. The orbit lies within $U(0, R_1, R_2)$ if and only if*

$$|az_2 - b| = \frac{R_1 + R_2}{2}|z_2 - z_1|, \quad |az_1 - b| = \frac{R_2 - R_1}{2}|z_2 - z_1|,$$

where a, b are initial conditions and $|z_2 - z_1|$ is the distance between z_1, z_2 .

4 Summary and future work

We have presented some basic results concerning ratios of Horadam sequences and parameter values producing orbits confined to an annulus. Future research could explore ratios of higher order, or non-homogeneous Horadam sequences.

4.1 Ratios of third order generalised Horadam sequences

For generalized Horadam sequences of third order (see [5,8])

$$w_{n+3} = pw_{n+2} + qw_{n+1} + rw_n, \quad w_0 = a, w_1 = b, w_2 = c, \quad (13)$$

the general sequence term formula when z_1, z_2, z_3 are distinct is

$$w_n = Az_1^n + Bz_2^n + Cz_3^n,$$

where A, B, C are computed from initial conditions. The sequence of ratios is

$$q_n = \frac{Az_1^{n+1} + Bz_2^{n+1} + Cz_3^{n+1}}{Az_1^n + Bz_2^n + Cz_3^n}. \quad (14)$$

Proposition 4.1 *We may assume that $|z_1| \leq |z_2| \leq |z_3|$. For the ratios sequence $\{q_n\}_{n=0}^{\infty}$ defined by (14) converges to z_3 if $|z_2| < |z_3|$, but it may be periodic, or dense in circles or unions of complex 1D curves otherwise.*

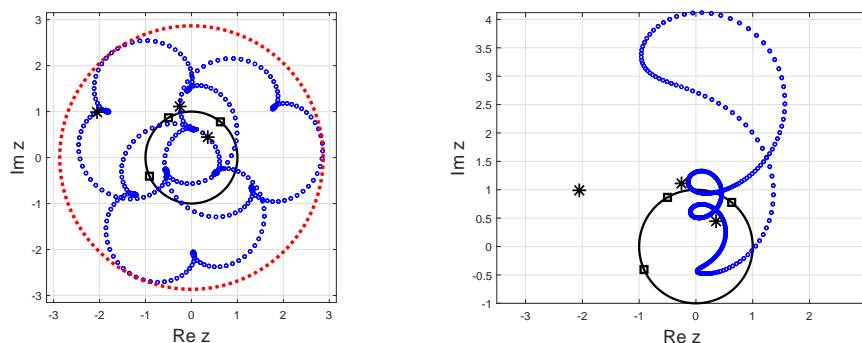


Fig. 4. Terms of sequence $\{w_n\}_{n=0}^N$ (circles) obtained from (6), for the initial conditions $a = 2 - 2i$, $b = 1 + 3i$ (stars) and the generator pairs (squares) (a) $z_1 = 1$ and $z_2 = e^{2\pi i \frac{2}{7}}$; (b) $z_1 = 1$ and $z_2 = e^{2\pi i \frac{3}{7}}$. Arrows indicate the direction of the orbit. Boundaries of annulus $U(0, ||A| - |B||, |A| + |B|)$ (dotted line) with A, B from (7) and the unit circle (solid line) are also plotted.

4.2 Ratios of non-homogeneous Horadam sequences

We have explored the periodicity of non-homogeneous Horadam sequences [7]

$$w_{n+2} = pw_{n+1} + qw_n + u_n, \quad w_0 = a, w_1 = b, \quad (15)$$

where parameters a, b, p and q are complex numbers, with constant or periodic perturbation $\{u_n\}_{n=0}^{\infty}$. When $\{u_n\}_{n=0}^{\infty}$ is self-repeating, the ratios sequence $\{q_n\}_{n=0}^{\infty}$ is periodic.

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