UNIVERSITY OF DERBY

COMPUTATIONAL AND THEORETICAL ASPECTS OF ITERATED GENERATING FUNCTIONS

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Contents

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Abstract

The thesis offers an investigation into the analysis of so-called iterated generating functions and the schemes that produce them. Beginning with the study of some *ad hoc* scheme formulations, the notion of an iterated generating function is introduced and a mechanism to produce arbitrary finite sequences established. The development of schemes to accommodate infinite sequences leads—in the case of the Catalan sequence—to the discovery of what are termed Catalan polynomials whose properties are examined. Results are formulated for these polynomials through the algebraic adaptation of classical root-finding algorithms, serving as a basis for the synthesis of new generalised results for other infinite sequences and their associated polynomials.

Chapter 1

Introduction

1.1 Overview

In mathematics, and particularly the field of combinatorics, the concept of a generating function is a well-known and useful one. In its most basic form, it is defined as a univariate formal power series whose coefficients, taken in ascending order of power, are elements of a numerical sequence. Alternatively a generating function may be expressed in closed form, the Taylor series expansion of which represents the required power series.

Various types of generating function exist, two of the most well-documented being the ordinary and exponential varieties. *Iterated* generating functions, however, are generated from a recursive procedure and take the form of a series of polynomials which may be said to "converge" towards a power series, where the number of terms within each polynomial which correspond to terms of the power series increases with each iteration.

Compared to other types of generating functions, the iterated variety appears to be relatively obscure, with very little material on the subject known to exist. Therefore, it is the aim of this thesis to provide a better general understanding of the phenomenon through the presentation of a recent in-depth investigation (originally published as a series of articles by Clapperton *et al.* (2008a, 2008b, 2008c, 2009, 2010, 2011a)).

A convenient starting point is the relatively recent discovery of work conducted by a 17th-century Chinese mathematician which relates to the use of iterated generating functions to produce a popular integer sequence in discrete mathematics: the Catalan numbers. Recognising that the recursive procedure can be adapted for other known integer sequences, we begin by demonstrating its use in generating the Large Schröder and Motzkin numbers—these being related to the Catalan sequence—proceeding to other sequences with similar characteristics in a later chapter. The primary means of identifying suitable sequences is the "Online Encyclopedia of Integer Sequences" (O.E.I.S.), an electronic repository containing information on hundreds of thousands of known sequences in discrete mathematics.

Preceding the main focus of the thesis—iterated generating functions for infinite sequences—we first present a new result concerning the construction of iterative generating schemes for arbitrary *finite* sequences, which we use as motivation to begin the examination of infinite sequences.

The first area of study with regard to the infinite variety concerns a generalisation of the quadratic functional equation which governs the ordinary generating function (o.g.f.) of a "target" sequence, *i.e.*, a sequence whose elements are known. By imposing a recurrence scheme on the sequence and matching terms, the original governing equation specific to that sequence is shown to be recoverable. In the case of the Catalan numbers, methodically increasing the degree of the polynomial coefficients in the governing equation and re-applying the scheme results in the appearance of a series of "Catalan polynomials", whose role in generating finite Catalan subsequences is identified and formalised.

A further point of interest is the network of relationships between Catalan, Chebyshev and Dickson polynomials, continued fractions and Dyck paths—in particular the discovery that the Catalan polynomials are directly expressible in terms of Chebyshev polynomials enabling a catalogue of properties to be readily compiled for the former based on existing literature.

The algebraic adaptation of a suite of iterative numerical root-finding schemes formulated by Householder is found to produce pairs of Catalan polynomials (with associated non-linear identities), ratios of which are shown to form Padé approximants to the o.g.f. of the Catalan sequence. Generalisation of the Householder suite of algorithms allows similar identities for other integer sequences to be obtained.

Finally, previous results concerning recurrence properties of the Catalan polynomials allow for the formulation of a number of identities linking the polynomials (and subsequently, generalised polynomial families) with their derivatives.

1.2 Background and Related Literature

The Catalan numbers ($\{c_0, c_1, c_2, c_3, c_4, c_5, \ldots\} = \{1, 1, 2, 5, 14, 42, \ldots\}$) are a sequence of positive integers, probably most well-known in the field of combinatorics as forming the solutions to a variety of counting problems. Among the considerable number of these formulated over the years (see Stanley (1999) for an extensive compilation) are the following examples:

- Euler's polygon division problem, where the number of unique ways in which an n-sided convex polygon can be dissected into triangles by connecting its vertices with non-intersecting diagonals is the $(n-1)$th number from the Catalan sequence, or c_{n-2},

- the number of unique ways in which a string of n characters can be parenthesised, the solution being c_{n-1},

- the number of unique monotonic paths through a square grid of n-by-n cells, where no path may encroach beyond the diagonal of the grid; the solution to this problem is the Catalan number c_{n+1}.

Although named after Belgian mathematician Eugène Charles Catalan, the initial discovery of the sequence was, until relatively recently, attributed to Leonhard Euler. Whilst studying the problem of triangulated polygon division in the mid-1700s, Euler discovered that the solutions were elements of a sequence which would, following further work by Catalan (and others) almost a century later, come to be known as the Catalan sequence.

In 1988, historian Luo Jianjin published an article demonstrating an earlier awareness of the Catalan sequence by Mongolian scholar Ming Antu, who in the early 18th century had studied infinite series expansions of trigonometric functions in which the Catalan sequence appears. Luo also detailed (initially in the context of vector multiplication and, in a subsequent publication (Luo, 1993), in terms of polynomial algebra) a non-linear recursive method for creating a finite polynomial which acts as a generating function for Catalan subsequences, where with each recursion, one additional term is included within the new generating function whose coefficient is the next term of the Catalan sequence. It is this process which has been used as a starting point to develop the idea of an iterated generating function.

Due to the relative obscurity of Luo's work in the Western world, the result was reported by Larcombe (1999, 1999/2000), and inductively proven by Larcombe and Fennessey (1999).

Further analysis shows that the iterated generating functions produced by Luo's recursive algorithm can alternatively be obtained by discretisation of the quadratic functional equation which governs the o.g.f. of the Catalan sequence. Moreover, this method can also be adapted for other well-known integer sequences in combinatorics.

As previously stated, in contrast to the extensive availability of material on the more common types of generating functions (a particularly comprehensive reference being Wilf (1994)), it appears that very little has been written concerning iterative generating functions, the only other relevant literature found to date being a recent publication by Koepf (2010) concerning the algebraic adaptation of the Newton-Raphson method, a popular quadratically convergent numerical root-finding algorithm. When used to generate subsequences, the algebraic method is found to retain the quadratic convergence rate of its numerical predecessor.

Koepf's most recent work is an extension of two previous publications, the first of these being a prior discussion by von zur Gathen and Gerhard (1999) of an algebraic implementation of Newton-Raphson iteration, and the second being one by Koepf himself in 2006 in which von zur Gathen and Gerhard's previous findings are used to symbolically produce truncated power series as solutions to implicit equations. In this work, it is demonstrated that the derivation of the formula for the modified Newton-Raphson method can also be applied to higher-order schemes (where the benefit of accelerated convergence is typically offset by a proportional increase in the computational complexity of the schemes' implementation). The specific algorithms used comprise a suite known as "Householder" schemes, of which the Newton-Raphson second-order and Halley's third-order methods are both members.

As a starting point in our investigation of the phenomenon of iterated generating functions, we will introduce the classical methods of generating Catalan numbers, consider Ming's algorithm for producing an iterated generating function for the Catalan sequence, and look briefly at a more natural method of obtaining the same results which is readily applied to other sequences.

1.3 Catalan Sequence Generation and Iterated Generating Functions

A number of methods exist (see Koshy (2009, p. 106) for an overview) to calculate the general term of the Catalan sequence directly; for example, by the use of formulae involving binomial coefficients

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 0, \tag{1.1}$$

and

$$c_n = \binom{2n}{n} - \binom{2n}{n+1}, \qquad n \geq 0, \tag{1.2}$$

or sequentially, by the following recursive formulae

$$c_0 = 1, \quad c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i}, \qquad n \geq 0, \tag{1.3}$$

and

$$c_0 = 1, \quad c_{n+1} = \frac{2(2n+1)}{n+2} c_n, \qquad n \geq 0. \tag{1.4}$$

In contrast with the above purely numerical methods, the Catalan sequence can also be produced algebraically by expanding the well-known o.g.f. $C(x)$, say, as a Maclaurin series

$$\begin{aligned} C(x) = \frac{1-\sqrt{1-4x}}{2x} &= \sum_{n=0}^{\infty} c_n x^n \\ &= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \cdots ; \end{aligned} \tag{1.5}$$

note that $C(x)$ satisfies the quadratic equation $0 = 1 - C(x) + xC^2(x)$, of which the o.g.f. is one (and the appropriate) solution.

1.3.1 Discretisation: Ming's Method

Whilst the above are all well-established methods of generating the Catalan sequence, the use of algebraic iterative methods such as Ming's recursion is far less common. The algorithm begins by setting polynomials $A_1(x) = 1$ and $A_2(x) = x$, and defining subsequent values of $A_r(x)$ by the recursion[1]

$$A_{r+1}(x) = xA_r(x)\left(2\sum_{k=1}^{r-1} A_k(x) + A_r(x)\right), \qquad r \geq 2. \tag{1.6}$$

[1]The version presented here differs slightly from Luo's formulation in that the exponents in both the initial and resulting polynomials are offset by 1. However, the mechanism of recurrence remains unaffected.

Iterating using the above then produces the following series of polynomials:

$$
\begin{aligned}
A_1(x) &= 1 \\
A_2(x) &= x \\
A_3(x) &= 2x^2 + x^3 \\
A_4(x) &= 4x^3 + 6x^4 + 6x^5 + 4x^6 + x^7 \\
A_5(x) &= 8x^4 + 20x^5 + 40x^6 + 68x^7 + 94x^8 + \cdots + x^{15} \\
A_6(x) &= 16x^5 + 56x^6 + 152x^7 + 376x^8 + \cdots + x^{31} \\
&\vdots
\end{aligned} \tag{1.7}
$$

Setting $C_r(x) = \sum_{i=1}^{r} A_i(x)$, *i.e.* the sum of polynomials $A_1(x)$ to $A_r(x)$, we obtain:

$$
\begin{aligned}
C_1(x) &= \mathbf{1} \\
C_2(x) &= \mathbf{1} + \boldsymbol{x} \\
C_3(x) &= \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + x^3 \\
C_4(x) &= \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{5}\boldsymbol{x}^{\mathbf{3}} + 6x^4 + 6x^5 + 4x^6 + x^7 \\
C_5(x) &= \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{5}\boldsymbol{x}^{\mathbf{3}} + \mathbf{14}\boldsymbol{x}^{\mathbf{4}} + 26x^5 + 44x^6 + 69x^7 + 94x^8 + \cdots + x^{15} \\
C_6(x) &= \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{5}\boldsymbol{x}^{\mathbf{3}} + \mathbf{14}\boldsymbol{x}^{\mathbf{4}} + \mathbf{42}\boldsymbol{x}^{\mathbf{5}} + 100x^6 + 221x^7 + 470x^8 + \cdots + x^{31} \\
&\vdots
\end{aligned} \tag{1.8}
$$

another series of polynomials of exponentially increasing length. From the above, it is immediately apparent that $\lim_{r\to\infty}\{C_r(x)\} = C(x)$, or in other words, as r increases, the series of polynomials slowly converges towards (1.5), *i.e.* the power series that would result from the Maclaurin expansion of the Catalan sequence's o.g.f. Another observation is that the rate of convergence is linear; that is to say, with each iteration, one additional coefficient within the new polynomial is found to be a Catalan number. Hence, at any stage of the iterative process, the first r Catalan numbers are represented by the first r coefficients in $C_r(x)$.

1.3.2 Discretisation: A Natural Method

As demonstrated by Larcombe and Fennessey (1999), an alternative means of obtaining the series of polynomials resulting from Ming's recursion is by a simple, and entirely natural, algebraic adaptation of an elementary method used in numerical analysis to find the fixed points (or roots) of a function.

To summarise the original numerical method—a full derivation and analysis can be found in Burden and Faires (2010, Section 2.2, p. 56), for example—given a function $f(x) = 0$, we know that by setting $f(x) = x - g(x)$, a new function $x = g(x)$ is formed (the solutions of which are the roots of $f(x)$). A recursion scheme is then constructed by discretising this function to form $x_{n+1} = g(x_n)$. By using an initial approximation to the root (or one of the roots), x_0, successive values of x_n generated by this procedure may gradually converge towards the true value(s) of x; convergence in this sense is not an issue for us as we implement the method of discretisation algebraically.

If the full Catalan sequence as defined in (1.5), $C(x)$, is considered to be the root whose "value" in terms of x we wish to find, the construction of a suitable recursion scheme can begin by rearranging the Catalan sequence's o.g.f. into a governing equation, this being the aforementioned quadratic

$$0 = 1 - C(x) + xC^2(x). \tag{1.9}$$

Moving the linear term in $C(x)$ to the left-hand side and discretising then yields

$$C_{r+1}(x) = 1 + xC_r^2(x). \tag{1.10}$$

Finally, setting the initial value $C_0(x) = 0$ and iterating using the above formula produces a series of polynomials identical to those generated by Ming's recursion:

$$\begin{aligned}
C_1(x) &= \mathbf{1}\\
C_2(x) &= \mathbf{1} + \boldsymbol{x}\\
C_3(x) &= \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + x^3\\
C_4(x) &= \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{5}\boldsymbol{x}^{\mathbf{3}} + 6x^4 + 6x^5 + 4x^6 + x^7\\
C_5(x) &= \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{5}\boldsymbol{x}^{\mathbf{3}} + \mathbf{14}\boldsymbol{x}^{\mathbf{4}} + 26x^5 + 44x^6 + 69x^7 + 94x^8 + \cdots + x^{15}\\
C_6(x) &= \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{5}\boldsymbol{x}^{\mathbf{3}} + \mathbf{14}\boldsymbol{x}^{\mathbf{4}} + \mathbf{42}\boldsymbol{x}^{\mathbf{5}} + 100x^6 + 221x^7 + 470x^8 + \cdots + x^{31}\\
&\vdots
\end{aligned} \tag{1.11}$$

Whilst in the above example, the value of $C_0(x)$ is set to zero for simplicity, it should be noted that other initial values (either numerical or polynomial in x) will also produce subsequences which converge to the Catalan sequence with equal efficiency. For example, setting $C_0(x) = x$

instead yields the following:

$$
\begin{aligned}
&C_1(x) = \mathbf{1} + x^3 \\
&C_2(x) = \mathbf{1} + \boldsymbol{x} + 2x^4 + x^7 \\
&C_3(x) = \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + x^3 + 4x^5 + 4x^6 + 2x^8 + 6x^9 + 4x^{12} + x^{15} \\
&C_4(x) = \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{5}\boldsymbol{x}^{\mathbf{3}} + 6x^4 + 6x^5 + 12x^6 + 17x^7 + 24x^8 + \cdots + x^{31} \\
&C_5(x) = \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{5}\boldsymbol{x}^{\mathbf{3}} + \mathbf{14}\boldsymbol{x}^{\mathbf{4}} + 26x^5 + 44x^6 + 85x^7 + 142x^8 + \cdots + x^{63} \\
&C_6(x) = \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{5}\boldsymbol{x}^{\mathbf{3}} + \mathbf{14}\boldsymbol{x}^{\mathbf{4}} + \mathbf{42}\boldsymbol{x}^{\mathbf{5}} + 100x^6 + 221x^7 + 502x^8 + \cdots + x^{127} \\
&\quad\vdots
\end{aligned}
\tag{1.12}
$$

where once again, the first r terms of $C_r(x)$ contain Catalan numbers as polynomial coefficients.

Many other infinite sequences exist whose generating functions are of a form such that their governing equations can be discretised in the same manner, creating a recurrence relation of the form $G_{r+1}(x) = f(x, G_r(x))$ (f being polynomial in $G_r(x)$ with functional (polynomial) coefficients in x). In each case, the recursion produces a series of polynomials whose coefficients converge linearly in number towards the terms of the infinite sequence.

Additional examples can be found in the form of two integer sequences which are closely related to the Catalan numbers: the Large Schröder[2] and Motzkin sequences. As in the case of the Catalan numbers, the elements of both of these sequences represent the solutions to a range of counting problems (for a detailed introduction and historical background, refer to Stanley (1997) and Donaghey and Shapiro (1977) for the Schröder and Motzkin numbers, respectively). In this work, however, the main property of interest is the similarity in the composition of each sequence's o.g.f. to that of the Catalan sequence.

The Schröder sequence (no. A006318 in the O.E.I.S.) has the o.g.f.

$$
\begin{aligned}
S(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x} &= \sum_{n=0}^{\infty} s_n x^n \\
&= 1 + 2x + 6x^2 + 22x^3 + 90x^4 + 394x^5 + \cdots
\end{aligned}
\tag{1.13}
$$

[2]Named as such to distinguish it from the Small Schröder sequence, the elements of the Large Schröder sequence are, with the exception of the leading term, exactly double those of the Small Schröder sequence. As only the Large Schröder sequence will feature in this work, it will hereafter be referred to simply as the Schröder sequence.

which satisfies the quadratic governing equation

$$0 = 1 - (1-x)S(x) + xS^2(x). \tag{1.14}$$

Rearrangement of the linear term in $S(x)$ and discretisation forms the recurrence

$$S_{r+1}(x) = 1 + xS_r(x) + xS_r^2(x), \qquad r \geq 0, \tag{1.15}$$

which, when used in conjunction with the initial value $S_0(x) = 0$, results in the following series of polynomials:

$$\begin{aligned}
S_1(x) &= \mathbf{1}\\
S_2(x) &= \mathbf{1} + \boldsymbol{2x}\\
S_3(x) &= \mathbf{1} + \boldsymbol{2x} + \boldsymbol{6x^2} + 4x^3\\
S_4(x) &= \mathbf{1} + \boldsymbol{2x} + \boldsymbol{6x^2} + \boldsymbol{22x^3} + 36x^4 + 52x^5 + 48x^6 + 16x^7\\
S_5(x) &= \mathbf{1} + \boldsymbol{2x} + \boldsymbol{6x^2} + \boldsymbol{22x^3} + \boldsymbol{90x^4} + 232x^5 + 564x^6 + 1268x^7\\
&\qquad + 2448x^8 + \cdots + 256x^{15}\\
S_6(x) &= \mathbf{1} + \boldsymbol{2x} + \boldsymbol{6x^2} + \boldsymbol{22x^3} + \boldsymbol{90x^4} + \boldsymbol{394x^5} + 1320x^6 + 4184x^7\\
&\qquad + 12804x^8 + \cdots + 65536x^{31}\\
&\ \vdots
\end{aligned} \tag{1.16}$$

In a similar manner to the previous example for the Catalan numbers, each polynomial $S_r(x)$ incorporates Schröder numbers as its first r coefficients.

Finally, the Motzkin sequence (no. A001006 in the O.E.I.S.) has the o.g.f.

$$\begin{aligned}
M(x) = \frac{1 - x - \sqrt{1-2x-3x^2}}{2x^2} &= \sum_{n=0}^{\infty} m_n x^n\\
&= 1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + \cdots
\end{aligned} \tag{1.17}$$

which satisfies the quadratic governing equation

$$0 = 1 - (1-x)M(x) + x^2M^2(x). \tag{1.18}$$

Rearrangement and discretisation creates the recursion scheme

$$M_{r+1}(x) = 1 + xM_r(x) + x^2M_r^2(x), \qquad r \geq 0, \tag{1.19}$$

and produces the following series of polynomials when initialised at $M_0(x) = 0$:

$$\begin{aligned}
M_1(x) &= \mathbf{1} \\
M_2(x) &= \mathbf{1} + \boldsymbol{x} + x^2 \\
M_3(x) &= \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + 3x^3 + 3x^4 + 2x^5 + x^6 \\
M_4(x) &= \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{4}\boldsymbol{x}^{\mathbf{3}} + 8x^4 + 13x^5 + 18x^6 + 23x^7 + 27x^8 + \cdots + x^{14} \\
M_5(x) &= \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{4}\boldsymbol{x}^{\mathbf{3}} + \mathbf{9}\boldsymbol{x}^{\mathbf{4}} + 20x^5 + 41x^6 + 76x^7 + 133x^8 + \cdots + x^{30} \\
M_6(x) &= \mathbf{1} + \boldsymbol{x} + \mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{4}\boldsymbol{x}^{\mathbf{3}} + \mathbf{9}\boldsymbol{x}^{\mathbf{4}} + \mathbf{21}\boldsymbol{x}^{\mathbf{5}} + 50x^6 + 115x^7 + 250x^8 + \cdots + x^{62} \\
&\vdots
\end{aligned} \tag{1.20}$$

where once again, the same phenomenon of convergence can be observed.

The examples of Catalan, Schröder and Motzkin presented here serve only to illustrate, in algebraic fixed-point iteration, one of the most elementary methods of producing iterated generating functions. All three of these sequences possess governing equations with extremely similar characteristics—primarily that they are quadratic with monomial coefficients in x—with the consequence that the polynomials output by the schemes also exhibit a high degree of structural similarity.

Nevertheless, in the course of this study, it will be seen that there exist a great many additional known sequences with more disparate attributes, for which the same methodology can be applied with equal effect. In some cases, the degree of either the discretised functional equation or its polynomial coefficients is cubic or higher; in others, the values of the coefficients are such that the resulting polynomials contain more than one correct sequence term with each iteration. The form of iterated generating functions, and the properties they possess, will be further explored.

Chapter 2

Iterated Generating Functions for Finite Sequences

2.1 Introduction

Most sequences of interest are infinite ones. Here, however, a result (theorem) is presented concerning the construction of iterated generating function schemes for *finite* sequences, offering a standalone chapter in which a new result is given, proved and demonstrated for interest using a selection of sequence types. We then move on to the study of infinite sequences in the next, and subsequent, chapters.

Given a "target" finite sequence, whose terms may be either known (*i.e.*, which may already be included in a repository such as the O.E.I.S.), or entirely arbitrary, the objective is to establish a recurrence scheme designed to output iterated generating functions which gradually converge toward the target sequence in the usual manner through the coefficients of the terms they contain. Due to the potentially arbitrary nature of the sequence terms, the existence of either a closed form expression for generating them numerically or an alternative generating function in closed form is not a prerequisite.

Two different proofs are provided—one inductive, one from first principles—which, in conjunction with supporting examples, show the theorem to be consistent in a natural limiting context.

2.2 Result

The theorem is as follows:

Theorem 2.1. *Let, for integer $m \geq 2$, $\{v_1, v_2, v_3, \ldots, v_m\}$ be an arbitrary finite sequence of integers, and define polynomials $f(x) = f(x;m) = v_1 + (v_2 - v_1)x + [v_3 - (v_1 + v_2)]x^2 + \cdots + [v_m - (v_1 + \cdots + v_{m-1})]x^{m-1}$ and $g(x) = g(x;m) = 1 + x + x^2 + \cdots + x^{m-2}$, of degree $m-1, m-2$, respectively. Then, with $F_1(x) = v_1$, the first-order scheme*

$$F_{i+1}(x) = f(x) + xg(x)F_i(x), \qquad i \geq 1,$$

generates polynomials $F_2(x), F_3(x), F_4(x), \ldots,$ where, for some $\Delta_i(x) \in \mathbb{Z}[x]$,

$$F_i(x) = v_1 + v_2 x + v_3 x^2 + \cdots + v_i x^{i-1} + x^i \Delta_i(x), \qquad i = 1, \ldots, m.$$

In other words, for $i = 1, \ldots, m$, the polynomial $F_i(x)$ produced by the scheme is an o.g.f. for the subsequence $\{v_1, v_2, v_3, \ldots, v_i\}$, whilst for $i > m$, $F_i(x)$ is an o.g.f. for the complete sequence $\{v_1, v_2, v_3, \ldots, v_m\}$.

Noting that $\deg\{f(x)\} = m-1$ and $\deg\{g(x)\} = m-2$, it is evident that if $\deg\{F_i(x)\} = p \geq 0$, say, then $\deg\{F_{i+1}(x)\} = \max\{\deg\{f(x)\}, \deg\{xg(x)F_i(x)\}\} = \max\{m-1, 1+(m-2)+p\} = m-1+p$. Starting with the degree zero polynomial $F_1(x) = v_1$, the recursive procedure increases the degree of the next iterate by $m-1$ each time.

Before detailing the proofs of Theorem 2.1, it is necessary to make some remarks concerning the notion of "preservation" exhibited by any iterated generating function scheme, where polynomial terms whose coefficients match target sequence elements are retained without alteration in the next iterated polynomial as determined by the recurrence—in general one or more new "correct" terms are added and preserved at each step, obeying a linear convergence rate.

Remark 2.2. Firstly, for the particular scheme under consideration it is easy to show that successive polynomials possess the required characteristic of preservation—adding after $F_2(x)$ (which would contain any lead string of zeros in the chosen target sequence; see Remark 2.5, and Footnote 1, p. 20) one and only one new term which is then preserved—by considering

$$\begin{aligned} F_{n+2}(x) - F_{n+1}(x) &= f(x) + xg(x)F_{n+1}(x) - [f(x) + xg(x)F_n(x)] \\ &= x[F_{n+1}(x) - F_n(x)]g(x). \end{aligned} \tag{2.1}$$

The observation is immediate since the lead term of $g(x)$ is non-zero. To see this, suppose that $F_n(x), F_{n+1}(x)$ agree up to and including terms in x^{r-1}, say. Then $F_{n+1}(x) - F_n(x) = a_r x^r + a_{r+1}x^{r+1} + a_{r+2}x^{r+2} + \cdots$ (for some integers $a_r \neq 0, a_{r+1}, a_{r+2}, \ldots$), so that by (2.1) $F_{n+2}(x) - F_{n+1}(x) = x[F_{n+1}(x) - F_n(x)]g(x) = (a_r x^{r+1} + a_{r+1}x^{r+2} + a_{r+2}x^{r+3} + \cdots)g(x)$, whence $F_{n+1}(x), F_{n+2}(x)$ agree up to and including terms in x^r.

Remark 2.3. Secondly, the actual level of agreement between successive polynomials is readily established by first re-writing (2.1) as

$$F_{i+1}(x) - F_i(x) = x[F_i(x) - F_{i-1}(x)]g(x), \tag{2.2}$$

and self-applying it $i-2$ times to give

$$F_{i+1}(x) - F_i(x) = x^{i-1}[F_2(x) - F_1(x)]g^{i-1}(x), \qquad i \geq 1. \tag{2.3}$$

Now denoting by $\Delta_f(x) \in \mathbb{Z}[x]$ the polynomial $v_2 - v_1 + [v_3 - (v_1 + v_2)]x + \cdots + [v_m - (v_1 + \cdots + v_{m-1})]x^{m-2}$ then $f(x) = v_1 + x\Delta_f(x)$. Since $F_1(x) = v_1$ the scheme gives $F_2(x) = f(x) + xg(x)F_1(x) = v_1 + x[\Delta_f(x) + v_1 g(x)]$, so that $F_2(x) - F_1(x) = x[\Delta_f(x) + v_1 g(x)]$ and in turn (2.3) reads, for $i \geq 1$,

$$\begin{aligned} F_{i+1}(x) - F_i(x) &= x^i[\Delta_f(x) + v_1 g(x)]g^{i-1}(x) \\ &= x^i\{v_2 + [v_3 + (i-2)v_2]x + \cdots\} \end{aligned} \tag{2.4}$$

after a little algebra. Thus, $F_i(x)$ and $F_{i+1}(x)$ agree in general up to and including terms in x^{i-1} at least (this would be increased by one term if v_2 were zero, and by two terms if additionally v_3 were zero).

2.3 Proofs of Theorem 2.1

Proof I (by induction). Theorem 2.1 holds for $i = 1$ since $F_1(x) = v_1 = v_1 + x^1\Delta_1(x)$ where $\Delta_1(x) = 0 \in \mathbb{Z}[x]$. Although it is not necessary to show that it is also valid for $i = 2, 3$, it is instructive to see this before showing that a general inductive step can be made. For $i = 2$,

$$\begin{aligned} F_2(x) &= f(x) + xg(x)F_1(x) \\ &= v_1 + (v_2 - v_1)x + [v_3 - (v_1 + v_2)]x^2 + \cdots + [v_m - (v_1 + \cdots + v_{m-1})]x^{m-1} \\ &\qquad + v_1 x(1 + x + x^2 + \cdots + x^{m-2}) \\ &= v_1 + v_2 x + (v_3 - v_2)x^2 + O(x^3) \\ &= v_1 + v_2 x + x^2\Delta_2(x), \end{aligned} \tag{2.5}$$

where $\Delta_2(x) = v_3 - v_2 + O(x) \in \mathbb{Z}[x]$. Taking $F_2(x) = v_1 + v_2 x + x^2\Delta_2(x)$ from (2.5), then for $i = 3$ a similar procedure yields

$$\begin{aligned} F_3(x) &= f(x) + xg(x)F_2(x) \\ &= v_1 + v_2 x + v_3 x^2 + [v_4 - v_3 + \Delta_2(x)]x^3 + O(x^4) \\ &= v_1 + v_2 x + v_3 x^2 + x^3\Delta_3(x), \end{aligned} \tag{2.6}$$

where $\Delta_3(x) = v_4 - v_2 + O(x) \in \mathbb{Z}[x]$. Suppose, therefore, the result is true for some $i = k \in [1, m-1]$, so that for some $\Delta_k(x) \in \mathbb{Z}[x]$,

$$F_k(x) = v_1 + v_2 x + v_3 x^2 + \cdots + v_k x^{k-1} + x^k \Delta_k(x), \qquad k \in [1, m-1], \tag{2.7}$$

and consider

$$\begin{aligned} F_{k+1}(x) &= f(x) + xg(x)F_k(x) \\ &= f(x) + x[v_1 + v_2 x + v_3 x^2 + \cdots + v_k x^{k-1} + x^k\Delta_k(x)] \\ &\qquad \times (1 + x + x^2 + \cdots + x^{m-2}), \end{aligned} \tag{2.8}$$

by assumption. Now we know by (2.4) that for $k \geq 1$ $F_{k+1}(x) - F_k(x) = x^k\Delta(x)$ ($\Delta(x) = \Delta(x;k) = v_2 + [v_3 + (k-2)v_2]x + \cdots \in \mathbb{Z}[x]$), so that

$$\begin{aligned} F_{k+1}(x) &= F_k(x) + x^k\Delta(x) \\ &= v_1 + v_2 x + v_3 x^2 + \cdots + v_k x^{k-1} + x^k\Delta_k(x) + x^k\Delta(x) \\ &= v_1 + v_2 x + v_3 x^2 + \cdots + v_k x^{k-1} + x^k\Delta^*(x) \end{aligned} \tag{2.9}$$

using (2.7) (the hypothesis), where $\Delta^*(x) = \Delta^*(x;k) = \Delta_k(x) + \Delta(x) \in \mathbb{Z}[x]$. Thus, from Remark 2.2 it suffices to show that

$$[x^k]\{F_{k+1}(x)\} = v_{k+1} \tag{2.10}$$

in order to complete the proof. Writing, from (2.8),

$$\begin{aligned} F_{k+1}(x) &= f(x) + [v_1 + v_2 x + v_3 x^2 + \cdots + v_k x^{k-1} + x^k\Delta_k(x)] \\ &\qquad \times (x + x^2 + x^3 + \cdots + x^{m-1}), \end{aligned} \tag{2.11}$$

then noting that $k \leq m-1$ the term in x^k is identified precisely as

$$\begin{aligned} &[v_{k+1} - (v_1 + \cdots + v_k)]x^k \\ &\qquad + v_1 \cdot x^k + v_2 x \cdot x^{k-1} + v_3 x^2 \cdot x^{k-2} + \cdots + v_k x^{k-1} \cdot x \\ &= v_{k+1}x^k. \ \square \end{aligned} \tag{2.12}$$

Proof II (from first principles). Consider first the given recursive scheme, with $F_1(x) = v_1$, which we write as

$$F_n(x) = f(x) + xg(x)F_{n-1}(x), \qquad n \geq 2, \tag{2.13}$$

and re-apply $n-2$ times to give, for $n \geq 2$,

$$\begin{aligned} F_n(x) &= f(x)[1 + xg(x) + x^2g^2(x) + \cdots + x^{n-2}g^{n-2}(x)] + x^{n-1}g^{n-1}(x)F_1(x) \\ &= f(x)\sum_{i=0}^{n-2}[xg(x)]^i + v_1[xg(x)]^{n-1} \\ &= f(x)\left(\frac{1-[xg(x)]^{n-1}}{1-xg(x)}\right) + v_1[xg(x)]^{n-1}. \end{aligned} \tag{2.14}$$

Note that (2.14) can also be obtained by first re-applying (2.2) $i-2$ times, so that

$$\begin{aligned} F_{i+1}(x) - F_i(x) &= x[F_i(x) - F_{i-1}(x)]g(x) \\ &= x^2[F_{i-1}(x) - F_{i-2}(x)]g^2(x) \\ &\vdots \\ &= x^{i-1}[F_2(x) - F_1(x)]g^{i-1}(x), \qquad i \geq 1 \end{aligned} \tag{2.15}$$

(this is (2.3)), and then summing both sides over the range $i = 1, \ldots, n-1$, which gives, for $n \geq 2$,

$$\begin{aligned} F_n(x) - F_1(x) &= [F_2(x) - F_1(x)]\sum_{i=1}^{n-1}[xg(x)]^{i-1} \\ &= [F_2(x) - F_1(x)]\left(\frac{1-[xg(x)]^{n-1}}{1-xg(x)}\right); \end{aligned} \tag{2.16}$$

now $F_1(x) = v_1$ by definition, and, from the given scheme, $F_2(x) = f(x) + xF_1(x)g(x) = f(x) + v_1xg(x)$, (2.16) duly yielding (2.14) after a little rearrangement.

Continuing, the function $g(x)$ is itself a geometric series which is summable as $g(x) = \sum_{i=0}^{m-2} x^i = (1-x^{m-1})/(1-x)$, whereupon $F_n(x)$ is expressed in the form

$$F_n(x) = f(x)\left(\frac{1-x}{1-2x+x^m}\right) + \{v_1 - f(x)(1-x)(1-2x+x^m)^{-1}\}[xg(x)]^{n-1}. \tag{2.17}$$

Since $g^{n-1}(x) = 1 + x\Delta_1(x)$, say (where $\Delta_1(x) = \Delta_1(x; m, n) \in \mathbb{Z}[x]$ is a finite polynomial with non-zero lead term $n-1$), and $(1-x)(1-2x+x^m)^{-1} = 1 + x\Delta_2(x)$ ($\Delta_2(x) = \Delta_2(x; m) \in \mathbb{Z}[[x]]$ being an infinite series in x), then (2.17) reduces to

$$F_n(x) = f(x)\left(\frac{1-x}{1-2x+x^m}\right) + x^{n-1}\{v_1[1 + x\Delta_1(x)] - f(x)[1 + x\Delta_3(x)]\}, \tag{2.18}$$

where $\Delta_3(x) = \Delta_3(x; m, n) = \Delta_1(x) + \Delta_2(x) + x\Delta_1(x)\Delta_2(x) \in \mathbb{Z}[[x]]$, and in turn, with $\Delta_4(x) = \Delta_4(x; m, n) = v_1\Delta_1(x) - f(x)\Delta_3(x) \in \mathbb{Z}[[x]]$,

$$F_n(x) = f(x)\left(\frac{1-x}{1-2x+x^m}\right) + [v_1 - f(x)]x^{n-1} + \Delta_4(x)x^n. \tag{2.19}$$

Now let

$$F(x) = F(x; m) = v_1 + v_2x + v_3x^2 + \cdots + v_mx^{m-1}, \tag{2.20}$$

containing $v_1, \ldots, v_m$ as coefficients. It is straightforward to show that

$$\begin{aligned}\left(\frac{1-2x+x^m}{1-x}\right)F(x) &= (1 - x - x^2 - \cdots - x^{m-1})F(x) \\ &= f(x) + x^m\Delta^\dagger(x),\end{aligned} \tag{2.21}$$

for some degree $m-2$ polynomial $\Delta^\dagger(x) = \Delta^\dagger(x; m) \in \mathbb{Z}[x]$, comprising the function $f(x)$ together with terms in $x^m, \ldots, x^{2(m-1)}$. This gives the first r.h.s. term of (2.19) as

$$\begin{aligned}\left(\frac{1-x}{1-2x+x^m}\right)f(x) &= F(x) - \left(\frac{1-x}{1-2x+x^m}\right)x^m\Delta^\dagger(x) \\ &= F(x) - [1 + x\Delta_2(x)]x^m\Delta^\dagger(x) \\ &= F(x) + x^m\Delta_5(x),\end{aligned} \tag{2.22}$$

where $\Delta_5(x) = \Delta_5(x; m) = -[1 + x\Delta_2(x)]\Delta^\dagger(x) \in \mathbb{Z}[[x]]$, so that, with $f(x) = v_1 + x\Delta_f(x)$ as before (*i.e.*, Remark 2.3), (2.19) reads

$$\begin{aligned}F_n(x) &= F(x) + x^m\Delta_5(x) + \{v_1 - [v_1 + x\Delta_f(x)]\}x^{n-1} + \Delta_4(x)x^n \\ &= F(x) + x^n\Delta^*(x) + x^m\Delta_5(x),\end{aligned} \tag{2.23}$$

where $\Delta^*(x) = \Delta^*(x; m, n) = \Delta_4(x) - \Delta_f(x) \in \mathbb{Z}[[x]]$. In other words, for $2 \le n \le m$, $F_n(x)$ agrees with $F(x)$ up to and including terms in x^{n-1}. Since $\Delta^*(x), \Delta_5(x) \in \mathbb{Z}[[x]]$, there must be cancellation in many terms within the r.h.s. of (2.23) in order that $F_n(x) \in \mathbb{Z}[x]$ for $n \ge 2$. The term $x^n\Delta^*(x)$ acts as a "curtain" which reveals more of the function $F(x)$ with each successive value of $n = 1, \ldots, m$ (it works for $n = 1$ since $F_1(x) = v_1$ by definition), that which is exposed being unaffected by the term $x^m\Delta_5(x)$ which simply adds terms in x whose powers are greater than the highest in $F(x)$. For $n \le m$ $F_n(x) = v_1 + v_2x + \cdots + v_nx^{n-1} + O(x^n)$, whilst for $n > m$ $F_n(x) = F(x) + O(x^m)$. This completes the proof. □

2.4 Examples

The following examples illustrate the use of Theorem 2.1 in constructing recurrence schemes for a range of finite sequences whose terms are arbitrary (with the exception of Example 1). In each case, the preserving nature of the scheme is clearly indicated, both up to and beyond the terms of the target sequence.

Example 1 ($m = 6$)

Consider the subsequence of Catalan numbers

$$\{1, 1, 2, 5, 14, 42\} = \{v_1, v_2, v_3, v_4, v_5, v_6\}, \tag{2.24}$$

for which Theorem 2.1 yields a recurrence scheme

$$F_{i+1}(x) = 1 + x^3 + 5x^4 + 19x^5 + x(1 + x + x^2 + x^3 + x^4)F_i(x), \qquad i \geq 1. \tag{2.25}$$

Initialising the scheme at $F_1(x) = v_1 = 1$ produces a series of polynomials $F_2(x), F_3(x), F_4(x)$, ..., with associated finite sequences of coefficients as follows:

$$\begin{aligned}
F_1(x):&\quad \{\mathbf{1}\},\\
F_2(x):&\quad \{\mathbf{1}, \mathbf{1}, 1, 2, 6, 20\},\\
F_3(x):&\quad \{\mathbf{1}, \mathbf{1}, \mathbf{2}, 4, 10, 30, 30, 29, 28, 26, 20\},\\
F_4(x):&\quad \{\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{5}, 13, 37, 47, 76, 103, 127, 143, 133, 103, 74, 46, 20\},\\
F_5(x):&\quad \{\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{14}, 41, 58, 104, 178, 276, 390, 496, 582, 609, 580, 499, \ldots\},\\
F_6(x):&\quad \{\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{14}, \mathbf{42}, 63, 120, 222, 395, 657, 1006, 1444, 1922, 2353, 2657, \ldots\},\\
F_7(x):&\quad \{\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{14}, \mathbf{42}, 64, 126, 244, 461, 842, 1457, 2400, 3724, 5424, 7382, \ldots\},\\
F_8(x):&\quad \{\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{14}, \mathbf{42}, 64, 127, 251, 490, 937, 1737, 3130, 5404, 8884, 13847, \ldots\},\\
F_9(x):&\quad \{\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{14}, \mathbf{42}, 64, 127, 252, 498, 974, 1869, 3542, 6545, 11698, 20092, \ldots\},\\
F_{10}(x):&\quad \{\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{5}, \mathbf{14}, \mathbf{42}, 64, 127, 252, 499, 983, 1915, 3720, 7135, 13428, 24628, \ldots\},\\
\vdots&
\end{aligned} \tag{2.26}$$

Example 2 ($m = 11$)

Consider the finite sequence

$$\{3, -5, 0, 0, 0, 0, 4, 6, 0, 0, -2\} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}, \tag{2.27}$$

for which the theorem yields a recurrence scheme ($i \geq 1$)

$$\begin{aligned}
F_{i+1}(x) = 3 &- 8x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 6x^6 + 4x^7 - 8x^8 - 8x^9 - 10x^{10}\\
&+ x(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9)F_i(x),
\end{aligned} \tag{2.28}$$

which, when initialised at $F_1(x) = 3$, produces the following series of sequences:

$$\begin{aligned}
F_1(x):&\quad \{\mathbf{3}\},\\
F_2(x):&\quad \{\mathbf{3}, \mathbf{-5}, 5, 5, 5, 5, 9, 7, -5, -5, -7\},\\
F_3(x):&\quad \{\mathbf{3}, \mathbf{-5}, \mathbf{0}, 5, 10, 15, 24, 31, 26, 21, 14, 14, 19, 14, 9, 4\ldots\},\\
F_4(x):&\quad \{\mathbf{3}, \mathbf{-5}, \mathbf{0}, \mathbf{0}, 5, 15, 34, 56, 75, 101, 120, 141, 160, 179, 188, 187, \ldots\},\\
F_5(x):&\quad \{\mathbf{3}, \mathbf{-5}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 5, 24, 56, 100, 175, 274, 401, 547, 707, 886, 1069, \ldots\},\\
F_6(x):&\quad \{\mathbf{3}, \mathbf{-5}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 9, 31, 75, 175, 348, 629, 1035, 1582, 2289, 3175, \ldots\},\\
F_7(x):&\quad \{\mathbf{3}, \mathbf{-5}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{4}, 11, 30, 105, 278, 633, 1267, 2302, 3884, 6173, \ldots\},\\
F_8(x):&\quad \{\mathbf{3}, \mathbf{-5}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{4}, \mathbf{6}, 5, 35, 138, 423, 1061, 2328, 4630, 8514, \ldots\},\\
F_9(x):&\quad \{\mathbf{3}, \mathbf{-5}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{4}, \mathbf{6}, \mathbf{0}, 5, 38, 183, 611, 1672, 4000, 8630, \ldots\},\\
F_{10}(x):&\quad \{\mathbf{3}, \mathbf{-5}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{4}, \mathbf{6}, \mathbf{0}, \mathbf{0}, 3, 48, 236, 847, 2519, 6519, \ldots\},\\
F_{11}(x):&\quad \{\mathbf{3}, \mathbf{-5}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{4}, \mathbf{6}, \mathbf{0}, \mathbf{0}, \mathbf{-2}, 8, 61, 297, 1144, 3663, \ldots\},\\
F_{12}(x):&\quad \{\mathbf{3}, \mathbf{-5}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{4}, \mathbf{6}, \mathbf{0}, \mathbf{0}, \mathbf{-2}, 3, 16, 77, 374, 1518, \ldots\},\\
\vdots&
\end{aligned} \tag{2.29}$$

Example 3 ($m = 4$)

The theorem also holds for target sequences drawn from rings other than $\mathbb{Z}$, as demonstrated in the following example in which $\upsilon_1, \ldots, \upsilon_m \in \mathbb{C}$. For the finite sequence

$$\{3 - 2\iota, 1 + \iota, 3 - 3\iota, -4\} = \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}, \tag{2.30}$$

the theorem yields the recurrence scheme ($i \geq 1$)

$$F_{i+1}(x) = 3 - 2\iota + (-2 + 3\iota)x - (1 + 2\iota)x^2 + (-11 + 4\iota)x^3 + x(1 + x + x^2)F_i(x). \tag{2.31}$$

Using the initial value $F_1(x) = 3 - 2\iota$ generates the following series of sequences:

$$\begin{aligned}
F_1(x):&\quad \{\mathbf{3-2\iota}\},\\
F_2(x):&\quad \{\mathbf{3-2\iota}, \mathbf{1+\iota}, 2-4\iota, -8+2\iota\},\\
F_3(x):&\quad \{\mathbf{3-2\iota}, \mathbf{1+\iota}, \mathbf{3-3\iota}, -5-\iota, -5-\iota, -6-2\iota, -8+2\iota\},\\
F_4(x):&\quad \{\mathbf{3-2\iota}, \mathbf{1+\iota}, \mathbf{3-3\iota}, \mathbf{-4}, -1-3\iota, -7-5\iota, -16-4\iota, -19-\iota, -14, -8+2\iota\},\\
F_5(x):&\quad \{\mathbf{3-2\iota}, \mathbf{1+\iota}, \mathbf{3-3\iota}, \mathbf{-4}, -2\iota, -2-6\iota, -12-8\iota, -24-12\iota, -42-10\iota, \ldots\},\\
F_6(x):&\quad \{\mathbf{3-2\iota}, \mathbf{1+\iota}, \mathbf{3-3\iota}, \mathbf{-4}, -2\iota, -1-5\iota, -6-8\iota, -14-16\iota, -38-26\iota, \ldots\},\\
F_7(x):&\quad \{\mathbf{3-2\iota}, \mathbf{1+\iota}, \mathbf{3-3\iota}, \mathbf{-4}, -2\iota, -1-5\iota, -5-7\iota, -7-15\iota, -21-29\iota, \ldots\},\\
F_8(x):&\quad \{\mathbf{3-2\iota}, \mathbf{1+\iota}, \mathbf{3-3\iota}, \mathbf{-4}, -2\iota, -1-5\iota, -5-7\iota, -6-14\iota, -13-27\iota, \ldots\},\\
F_9(x):&\quad \{\mathbf{3-2\iota}, \mathbf{1+\iota}, \mathbf{3-3\iota}, \mathbf{-4}, -2\iota, -1-5\iota, -5-7\iota, -6-14\iota, -12-26\iota, \ldots\},\\
F_{10}(x):&\quad \{\mathbf{3-2\iota}, \mathbf{1+\iota}, \mathbf{3-3\iota}, \mathbf{-4}, -2\iota, -1-5\iota, -5-7\iota, -6-14\iota, -12-26\iota, \ldots\},\\
\vdots&
\end{aligned} \tag{2.32}$$

<u>Example 4 ($m = 5$)</u>

In this final example, $v_1, \ldots, v_m \in \mathbb{Q}$. Consider the finite sequence

$$\left\{\frac{1}{2}, -\frac{2}{3}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{8}\right\} = \{v_1, v_2, v_3, v_4, v_5\}, \tag{2.33}$$

for which the theorem yields the recurrence scheme ($i \geq 1$)

$$F_{i+1}(x) = \frac{1}{2} - \frac{7}{6}x - \frac{1}{12}x^2 + \frac{11}{12}x^3 + \frac{1}{24}x^4 + x(1 + x + x^2 + x^3)F_i(x). \tag{2.34}$$

Using the initial value $F_1(x) = \frac{1}{2}$ generates the following series of sequences:

$$
\begin{array}{rl}
F_1(x): & \left\{\mathbf{\tfrac{1}{2}}\right\}, \\
F_2(x): & \left\{\mathbf{\tfrac{1}{2}}, \mathbf{-\tfrac{2}{3}}, \tfrac{5}{12}, \tfrac{17}{12}, \tfrac{13}{24}\right\}, \\
F_3(x): & \left\{\mathbf{\tfrac{1}{2}}, \mathbf{-\tfrac{2}{3}}, \mathbf{-\tfrac{1}{4}}, \tfrac{7}{6}, \tfrac{41}{24}, \tfrac{41}{24}, \tfrac{19}{8}, \tfrac{47}{24}, \tfrac{13}{24}\right\}, \\
F_4(x): & \left\{\mathbf{\tfrac{1}{2}}, \mathbf{-\tfrac{2}{3}}, \mathbf{-\tfrac{1}{4}}, \mathbf{\tfrac{1}{2}}, \tfrac{19}{24}, \tfrac{47}{24}, \tfrac{13}{3}, \tfrac{167}{24}, \tfrac{31}{4}, \tfrac{79}{12}, \tfrac{39}{8}, \tfrac{5}{2}, \tfrac{13}{24}\right\}, \\
F_5(x): & \left\{\mathbf{\tfrac{1}{2}}, \mathbf{-\tfrac{2}{3}}, \mathbf{-\tfrac{1}{4}}, \mathbf{\tfrac{1}{2}}, \mathbf{\tfrac{1}{8}}, \tfrac{3}{8}, 3, \tfrac{91}{12}, \tfrac{337}{24}, 21, \tfrac{205}{8}, \tfrac{157}{6}, \tfrac{521}{24}, \tfrac{29}{2}, \tfrac{95}{12}, \tfrac{73}{24}, \ldots\right\}, \\
F_6(x): & \left\{\mathbf{\tfrac{1}{2}}, \mathbf{-\tfrac{2}{3}}, \mathbf{-\tfrac{1}{4}}, \mathbf{\tfrac{1}{2}}, \mathbf{\tfrac{1}{8}}, -\tfrac{7}{24}, \tfrac{3}{4}, 4, \tfrac{133}{12}, 25, \tfrac{365}{8}, \tfrac{273}{4}, \tfrac{521}{6}, \tfrac{189}{2}, 88, \tfrac{1687}{24}, \ldots\right\}, \\
F_7(x): & \left\{\mathbf{\tfrac{1}{2}}, \mathbf{-\tfrac{2}{3}}, \mathbf{-\tfrac{1}{4}}, \mathbf{\tfrac{1}{2}}, \mathbf{\tfrac{1}{8}}, -\tfrac{7}{24}, \tfrac{1}{12}, \tfrac{13}{12}, \tfrac{55}{12}, \tfrac{373}{24}, \tfrac{245}{6}, \tfrac{2057}{24}, \tfrac{3599}{24}, \tfrac{5417}{24}, \tfrac{7085}{24}, \tfrac{4051}{12}, \ldots\right\}, \\
F_8(x): & \left\{\mathbf{\tfrac{1}{2}}, \mathbf{-\tfrac{2}{3}}, \mathbf{-\tfrac{1}{4}}, \mathbf{\tfrac{1}{2}}, \mathbf{\tfrac{1}{8}}, -\tfrac{7}{24}, \tfrac{1}{12}, \tfrac{5}{12}, 1, \tfrac{131}{24}, \tfrac{511}{24}, \tfrac{1489}{24}, \tfrac{440}{3}, \tfrac{7009}{24}, \tfrac{12053}{24}, \tfrac{9079}{12}, \ldots\right\}, \\
F_9(x): & \left\{\mathbf{\tfrac{1}{2}}, \mathbf{-\tfrac{2}{3}}, \mathbf{-\tfrac{1}{4}}, \mathbf{\tfrac{1}{2}}, \mathbf{\tfrac{1}{8}}, -\tfrac{7}{24}, \tfrac{1}{12}, \tfrac{5}{12}, \tfrac{1}{3}, \tfrac{29}{24}, \tfrac{167}{24}, \tfrac{169}{6}, \tfrac{2155}{24}, \tfrac{5651}{24}, \tfrac{12529}{24}, \tfrac{24071}{24}, \ldots\right\}, \\
F_{10}(x): & \left\{\mathbf{\tfrac{1}{2}}, \mathbf{-\tfrac{2}{3}}, \mathbf{-\tfrac{1}{4}}, \mathbf{\tfrac{1}{2}}, \mathbf{\tfrac{1}{8}}, -\tfrac{7}{24}, \tfrac{1}{12}, \tfrac{5}{12}, \tfrac{1}{3}, \tfrac{13}{24}, \tfrac{49}{24}, \tfrac{107}{12}, \tfrac{110}{3}, \tfrac{1009}{8}, \tfrac{2883}{8}, \tfrac{21011}{24}, \ldots\right\}, \\
\vdots &
\end{array}
\tag{2.35}
$$

Remark 2.4. The case $m = 2$ is sufficiently tractable to be dealt with in generality. For a target sequence $\{v_1, v_2\}$, say, then with $g(x) = 1$

$$
\begin{aligned}
F_n(x) &= f(x)\left(\frac{1 - x^{n-1}}{1 - x}\right) + v_1 x^{n-1} \\
&= f(x)(1 + x + x^2 + x^3 + \cdots + x^{n-2}) + v_1 x^{n-1}
\end{aligned}
\tag{2.36}
$$

for $n \geq 2$, via (2.14). Since $f(x) = v_1 + (v_2 - v_1)x$, (2.36) gives

$$
F_n(x) = v_1 + v_2(x + x^2 + x^3 + \cdots + x^{n-1}), \qquad n \geq 2, \tag{2.37}
$$

after some cancellation, yielding polynomials $F_2(x), F_3(x), F_4(x), F_5(x), \ldots$, with coefficient sequences $\{v_1, v_2\}, \{v_1, v_2, v_2\}, \{v_1, v_2, v_2, v_2\}, \{v_1, v_2, v_2, v_2,\ v_2\}$, *etc.*

Remark 2.5. We have already mentioned (in Remark 2.2) that should the target sequence contain a lead zero (or string of zeros), then this (these) will be accommodated in $F_2(x)$.[1] The polynomial $F_2(x)$ will also deliver additional terms if one or more zero(s) appear(s) in the target

[1]This is simply because the initial fixed polynomial $F_1(x) = v_1 = 0$ can be regarded as generating $\{0\}$, or $\{0, 0\}$, or $\{0, 0, 0\}, \ldots$, as appropriate. To illustrate, suppose, for instance, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} = \{0, 0, 0, 8, 5, 0, 0, -3\}$. Then $F_2(x) = 8x^3 - 3x^4 - 13x^5 - 13x^6 - 16x^7 \longmapsto \{v_1, v_2, v_3, v_4, -3, -13, -13, -16\}$, $F_3(x) \longmapsto \{v_1, v_2, v_3, v_4, v_5, -8, -21, -37, -37, -37, -37, -45, -42, -29, -16\}$, *etc.*, with $v_6 = v_7 = 0$, $v_8 = -3$ each added in *separate* iterations (*i.e.*, in $F_4(x), F_5(x), F_6(x)$).

sequence after $v_1 \neq 0$. For example, suppose $\{v_1, v_2, v_3, v_4, v_5\} = \{-2, 0, 0, 4, -6\}$. We find that $F_2(x) = -2 + 4x^3 - 10x^4 \longmapsto \{v_1, v_2, v_3, v_4, -10\}$, $F_3(x) \longmapsto \{v_1, v_2, v_3, v_4, v_5, -6, -6, -6, -10\}$, $F_4(x) \longmapsto \{v_1, v_2, v_3, v_4, v_5, -2, -8, -14, -24, -28, -22, -16, -10\}$, *etc.*

Remark 2.6. To finish, we show how Theorem 2.1 is consistent in the limit $m \to \infty$, where the target sequence is now an infinite one whose generating function is $F(x; \infty) = v_1 + v_2 x + v_3 x^2 + v_4 x^3 + \cdots$. Since, in the limit,

$$\begin{aligned} f(x;\infty) &= v_1 + (v_2 - v_1)x + [v_3 - (v_1 + v_2)]x^2 + [v_4 - (v_1 + v_2 + v_3)]x^3 + \cdots \\ &= (1 - x - x^2 - x^3 - \cdots)(v_1 + v_2 x + v_3 x^2 + v_4 x^3 + \cdots) \\ &= (1 - 2x)(1 - x)^{-1} F(x;\infty), \end{aligned} \tag{2.38}$$

and

$$g(x;\infty) = 1 + x + x^2 + x^3 + \cdots = (1 - x)^{-1}, \tag{2.39}$$

the scheme according to Theorem 2.1 reads, for $i \geq 1$,

$$\begin{aligned} F_{i+1}(x) &= f(x;\infty) + x g(x;\infty) F_i(x) \\ &= \frac{1}{1 - x}[(1 - 2x)F(x;\infty) + x F_i(x)]. \end{aligned} \tag{2.40}$$

Suppose the series $F(x; \infty)$ defining the infinite target sequence $\{v_m\}_1^\infty$ has an o.g.f. closed form $G_o(x)$, say. Then

$$F_{i+1}(x) = \frac{1}{1 - x}[(1 - 2x)G_o(x) + x F_i(x)]. \tag{2.41}$$

With reference to (1.9),(1.10) (relating to the Catalan sequence example), (2.41) must be a discretised version of an equation for the same function $G_o(x)$, and we see this trivially; suppressing the subscripts $i, i + 1$ it reads

$$F(x) = \frac{1}{1 - x}[(1 - 2x)G_o(x) + x F(x)], \tag{2.42}$$

which is satisfied by $F(x) = G_o(x)$.

2.5 Summary

Given a finite sequence of values, we have demonstrated in this chapter the formulation of a recursive scheme, which, through a simple mechanism, produces a series of iterated generating functions steadily converging to the full target sequence in the same manner exhibited by the schemes seen in Chapter 1. We conclude with a couple of general comments.

Firstly, it should be noted that the scheme presented in Theorem 2.1 is far from unique where the objective of reproducing a target finite sequence is concerned, as is evident from the following observation.

Given $v_1, v_2, \ldots, v_m$ (from any ring), we have a simple algorithm to construct a first-order scheme which sequentially produces terms in the target sequence $\{v_1, v_2, \ldots, v_m\}$, $m \geq 1$. Now, if, for $r \geq 1$, $\{u_1, u_2, \ldots, u_r\}$ is a second sequence, we can use the algorithm to construct a new first-order scheme—different to the first one—and iteratively generate the combined sequence $\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_r\}$ which necessarily includes $\{v_1, v_2, \ldots, v_m\}$. Thus, we see that there are an infinite number of distinct first-order schemes, whose basic structure is as given in Theorem 2.1 based on the defined functions $f(x), g(x)$, that will each generate $\{v_1, v_2, \ldots, v_m\}$; if $f(x; m), g(x; m)$ are such that the scheme generates $\{v_1, v_2, \ldots, v_m\}$, then increasing the target sequence to $\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_r\}$ merely adds r extra terms into both f, g.

It is also possible that there may be other first-order recurrences of a different fundamental nature, or indeed higher-order schemes, which are able to produce iterated generating functions converging towards an arbitrary finite sequence.

Chapter 3

Iterated Generating Functions for Infinite Sequences

3.1 Introduction

Returning to recurrence schemes for infinite integer sequences—a subject which was earlier briefly introduced in the context of the Catalan, Schröder and Motzkin sequences—the main focus of this chapter will be a methodology for recovering a recurrence scheme for an infinite sequence whose first few terms, at least, are known. Equating (or "matching") constants in a proposed recursive scheme (of general form) with terms from a sequence itself creates a system of equations which can be solved simultaneously to determine the values of the constants and so recover the expected recurrence available from natural o.g.f. discretisation as seen in Chapter 2. We look at some examples and the processes involved, from which automation of the procedures yield the discovery of so-called Catalan polynomials which, as well as being intimately connected to algorithms delivering Catalan number subsequences, have interesting properties which are examined in subsequent chapters.

Before introducing this procedure, however, it is first necessary to outline a number of additional sequences for which iterated generating function schemes can be constructed, thereby providing a more diverse range of sequences to be utilised as a basis for term-matching.

As previously noted, the recursive schemes for the three integer sequences discussed so far share a number of characteristics, with two being of particular interest. The first is that the equation governing the sequence's o.g.f. is quadratic with first- or second-order polynomial (or constant) coefficients. The subsequent discretisation of such an equation results in a recursive function of the form

$$G_{r+1}(x) = \Omega_1(x) + \Omega_2(x)G_r(x) + \Omega_3(x)G_r^2(x), \tag{3.1}$$

where $\Omega_1(x), \Omega_2(x), \Omega_3(x)$ are (at most) quadratic in x.

A second similarity between the recurrence schemes for the Catalan, Schröder and Motzkin sequences is that after each iteration, only a single correct sequence term is included within the new generating function.

However, using the O.E.I.S. as a reference, it is possible to locate many more instances of known integer sequences which can be generated by the same method, but which do not necessarily share these attributes, as the following examples demonstrate.

3.2 Discretisation: Further Examples

Fibonacci Sequence

The Fibonacci sequence (or sequence no. A000045 in the O.E.I.S.) has

$$\{0, 1, 1, 2, 3, 5, 8, 13, \ldots\} = \{f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7, \ldots\}, \tag{3.2}$$

say, as its first few terms. Unlike previous examples, its o.g.f.

$$F(x) = \frac{x}{1 - x - x^2} \tag{3.3}$$

does not contain a radical, and can be rearranged trivially and discretised as

$$F_{r+1}(x) = x + (x + x^2)F_r(x), \qquad r \geq 0. \tag{3.4}$$

When initialised at $F_0(x) = 0$, a series of polynomials $F_1(x), F_2(x), F_3(x), \ldots$ is produced with associated subsequences $\{f_0, f_1\}, \{f_0, f_1, f_2, 1\}, \{f_0, f_1, f_2, f_3, 2, 1\}, \{f_0, f_1, f_2, f_3, f_4, 4, 3, 1\}$, *etc.*

Sequence A052709

Sequence no. A052709 in the O.E.I.S. has

$$\{0, 1, 1, 3, 9, 31, 113, 431, \ldots\} = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots\}, \tag{3.5}$$

as its first few terms. Its o.g.f.

$$A(x) = \frac{1 - \sqrt{1 - 4x - 4x^2}}{2(1 + x)} \tag{3.6}$$

can be rearranged into a governing equation

$$0 = x - A(x) + (1 + x)A^2(x) \tag{3.7}$$

and discretised as

$$A_{r+1}(x) = x + (1 + x)A_r^2(x), \qquad r \geq 0. \tag{3.8}$$

When initialised at $A_0(x) = 0$, the scheme generates polynomials with associated subsequences as follows:

$$\begin{aligned}
&A_1(x): \quad \{a_0, a_1\},\\
&A_2(x): \quad \{a_0, a_1, a_2, 1\},\\
&A_3(x): \quad \{a_0, a_1, a_2, a_3, 5, 5, 3, 1\},\\
&A_4(x): \quad \{a_0, a_1, a_2, a_3, a_4, 23, 45, 75, 109, 133, 131, 101, 59, 25, 7, 1\},\\
&\quad\vdots
\end{aligned} \tag{3.9}$$

Similarly to previous examples, the scheme adds a single correct sequence term to each new generating function.

Sequence A077957

Sequence no. A077957 (or "powers of 2 alternating with zeros") in the O.E.I.S. begins

$$\{1, 0, 2, 0, 4, 0, 8, 0, \ldots\} = \{b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, \ldots\}. \tag{3.10}$$

Its o.g.f.

$$B(x) = \frac{1}{1 - 2x^2} \tag{3.11}$$

can be rearranged into a governing equation

$$0 = 1 - (1 - 2x^2)B(x) \tag{3.12}$$

and discretised as

$$B_{r+1}(x) = 1 + 2x^2 B_r(x), \qquad r \geq 0. \tag{3.13}$$

Using $B_0(x) = 0$, the following subsequences result:

$$\begin{aligned}
B_1(x): &\quad \{b_0\}, \\
B_2(x): &\quad \{b_0, b_1, b_2\}, \\
B_3(x): &\quad \{b_0, b_1, b_2, b_3, b_4\}, \\
B_4(x): &\quad \{b_0, b_1, b_2, b_3, b_4, b_5, b_6\}, \\
\vdots &
\end{aligned} \tag{3.14}$$

Two observations can be made regarding this particular result, both of which are in contrast with those from previous schemes: firstly, that *two* correct sequence terms are added with each iteration, as opposed to the more usual single term; and secondly, the resulting subsequences are entirely composed of terms from sequence A077957, with no extraneous terms following. By inference, as r increases, the degree of the resulting polynomials increases at a linear, rather than exponential, rate.

Sequence A025252

Sequence no. A025252 in the O.E.I.S. begins

$$\{0, 0, 1, 2, 1, 6, 9, 12, \ldots\} = \{d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7, \ldots\}. \tag{3.15}$$

Its o.g.f.

$$D(x) = \frac{1 - x^2 - 4x^3 - \sqrt{1 - 2x^2 - 8x^3 + x^4}}{4x^3} \tag{3.16}$$

can be rearranged into a governing equation

$$0 = x^2 + 2x^3 - (1 - x^2 - 4x^3)D(x) + 2x^3 D^2(x) \tag{3.17}$$

and discretised as

$$D_{r+1}(x) = x^2 + 2x^3 + (x^2 + 4x^3)D_r(x) + 2x^3 D_r^2(x), \qquad r \geq 0. \tag{3.18}$$

Using $D_0(x) = 0$, the following subsequences result:

$$\begin{aligned}
D_1(x): &\quad \{d_0, d_1, d_2, d_3\}, \\
D_2(x): &\quad \{d_0, d_1, d_2, d_3, d_4, d_5, 8, 2, 8, 8\}, \\
D_3(x): &\quad \{d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7, 40, 46, 48, 122, 128, 152, 296, 272, \\
&\qquad\qquad\qquad\qquad 288, 456, 320, 192, 256, 128\}, \\
\vdots &
\end{aligned} \tag{3.19}$$

As for the previous example, two correct sequence terms are added with each iteration.

A point of interest is that sequence A025252 is but one member of a particular family of 55 sequences which possess the common characteristic that sequence elements are in all cases defined by variations of a simple convolution formula. Each sequence in the family is also associated with an o.g.f. which satisfies a quadratic governing equation, thereby enabling a similar recurrence scheme to be constructed.

3.3 Recurrence Scheme Recovery

It is notable that all the sequences previously referenced are at least reasonably well-documented (either in existing literature or the O.E.I.S.) insofar that formulae for their o.g.fs can be found with little difficulty. However, there may be cases where although the terms of a sequence are known or calculable, the sequence's o.g.f. is not, precluding the usual method of constructing a recurrence scheme through discretisation. In such circumstances, the usefulness of a method of recovering the recurrence scheme using only the initial terms of the sequence (and a set of constraints placed on the scheme) becomes apparent.

In the methodology presented here, it is assumed that if a closed form expression for a sequence's o.g.f. exists, it satisfies a quadratic governing equation. From this assumption it follows that a recurrence scheme may be proposed which is of the form

$$G_{r+1}(x) = f(G_r(x); \Omega_1(x), \Omega_2(x), \Omega_3(x)) = \Omega_1(x) + \Omega_2(x)G_r(x) + \Omega_3(x)G_r^2(x), \tag{3.20}$$

which is a functional relation containing polynomial coefficients $\Omega_1(x), \Omega_2(x), \Omega_3(x)$ initially of unspecified degree. However, by setting constraints on the degrees of $\Omega_1(x), \Omega_2(x), \Omega_3(x)$, a set of constants is created whose members can be "tracked" through a series of recursions, combined and equated with a known sequence term at each stage. The consequence of this process is that a gradually-expanding system of equations is created which, at some stage, can be solved completely, yielding values for all constants and thereby reconstructing the original recurrence relation.

Catalan Sequence

This first example demonstrates how term-matching can be used to recover the natural recurrence scheme for the Catalan sequence.

Consider a recurrence scheme based on equation (3.20), with the constraint imposed that all polynomial coefficients are linear in x. Substituting $\Omega_1(x) = \alpha + \beta x$, $\Omega_2(x) = \gamma + \delta x$, $\Omega_3(x) = \varepsilon + \zeta x$, equation (3.20) now reads

$$G_{r+1}(x) = \alpha + \beta x + (\gamma + \delta x)G_r(x) + (\varepsilon + \zeta x)G_r^2(x). \tag{3.21}$$

Setting $G_0(x) = 0$, we first see that $G_1(x) = \alpha + \beta x$, giving trivially that $\alpha = [x^0]\{G_1(x)\} = c_0 = 1$. Thus, $G_1(x) = 1 + \beta x$ which is fed back into the updated recurrence $G_{r+1}(x) = 1 + \beta x + (\gamma + \delta x)G_r(x) + (\varepsilon + \zeta x)G_r^2(x)$, to give, with $r = 1$,

$$G_2(x) = 1 + \gamma + \varepsilon + [\delta + \zeta + \beta(1 + \gamma + 2\varepsilon)]x + \beta(\beta\varepsilon + \delta + 2\zeta)x^2 + \beta^2\zeta x^3. \tag{3.22}$$

We now have, therefore,

$$\begin{aligned} [x^0]\{G_2(x)\} &= c_0 = 1 = 1 + \gamma + \varepsilon, \\ [x^1]\{G_2(x)\} &= c_1 = 1 = \delta + \zeta + \beta(1 + \gamma + 2\varepsilon), \end{aligned} \tag{3.23}$$

from which $\varepsilon = -\gamma$ and

$$\delta + \zeta + \beta(1 - \gamma) = 1 \tag{3.24}$$

follow immediately. The recursion becomes now $G_{r+1}(x) = 1 + \beta x + (\gamma + \delta x)G_r(x) + (-\gamma + \zeta x)G_r^2(x)$ and duly yields, after a little work using $G_2(x) = 1 + x + \beta(-\beta\gamma + \delta + 2\zeta)x^2 + \beta^2\zeta x^3$,

$$G_3(x) = 1 + (\beta - \gamma + \delta + \zeta)x + [-\gamma + \delta + 2\zeta + \beta\gamma(\beta\gamma - \delta - 2\zeta)]x^2 + \cdots . \tag{3.25}$$

Since the lead term of $G_3(x)$ is correct, we write

$$\begin{aligned} [x^1]\{G_3(x)\} &= c_1 = 1 = \beta - \gamma + \delta + \zeta, \\ [x^2]\{G_3(x)\} &= c_2 = 2 = -\gamma + \delta + 2\zeta + \beta\gamma(\beta\gamma - \delta - 2\zeta). \end{aligned} \tag{3.26}$$

The first of these equations, when combined with (3.24), yields $0 = (\beta - 1)\gamma$, with two cases $\gamma = 0$ and $\beta = 1$ to consider.

Case 1: $\gamma = 0$

The equations in (3.26) reduce to

$$\begin{aligned} \beta + \delta + \zeta &= 1, \\ \delta + 2\zeta &= 2, \end{aligned} \tag{3.27}$$

so we require another equation involving at least one of β, δ and ζ. Since the recurrence reduces to $G_{r+1}(x) = 1 + \beta x + \delta x G_r(x) + \zeta x G_r^2(x)$, it suffices to use $G_3(x) = 1 + x + 2x^2 + \cdots$, without explicitly knowing the term in x^3, for then

$$\begin{aligned} G_4(x) &= 1 + \beta x + \delta x G_3(x) + \zeta x G_3^2(x) \\ &= 1 + (\beta + \delta + \zeta)x + (\delta + 2\zeta)x^2 + (2\delta + 5\zeta)x^3 + \cdots \\ &= 1 + x + 2x^2 + (2\delta + 5\zeta)x^3 + \cdots \end{aligned} \tag{3.28}$$

by (3.27), from which we can only take

$$[x^3]\{G_4(x)\} = c_3 = 5 = 2\delta + 5\zeta \tag{3.29}$$

to go with (3.27); these equations have the unique solution $\delta = \beta = 0$, $\zeta = 1$, and the solution scheme is

$$G_{r+1}(x) = 1 + xG_r^2(x), \tag{3.30}$$

as expected.

Case 2: $\beta = 1$

Despite the fact that this case results in a redundant solution scheme, the details are still worth mentioning. First, (3.26) simplifies to

$$-\gamma + \delta + \zeta = 0 \tag{3.31}$$

and

$$-\gamma + \delta + 2\zeta + \gamma(\gamma - \delta - 2\zeta) = 2. \tag{3.32}$$

Taking $G_3(x)$ to the same order as in Case 1, the recurrence formula at the corresponding point generates

$$\begin{aligned} G_4(x) &= 1 + (1 - \gamma + \delta + \zeta)x + (-3\gamma + \delta + 2\zeta)x^2 + \cdots \\ &= 1 + x + (-3\gamma + \delta + 2\zeta)x^2 + \cdots \end{aligned} \tag{3.33}$$

using (3.31), and we need only write down

$$[x^2]\{G_4(x)\} = c_2 = 2 = -3\gamma + \delta + 2\zeta. \tag{3.34}$$

From this, replacing $\delta + 2\zeta$ with $2 + 3\gamma$ then (3.32) contracts conveniently to $\gamma = 0$, leaving (3.31),(3.34) as simultaneous equations in δ, ζ and resulting in the scheme

$$G_{r+1}(x) = 1 + x - 2xG_r(x) + 2xG_r^2(x). \tag{3.35}$$

Since it was unnecessary to match $[x^3]\{G_4(x)\}$ to c_3, we have arrived at a scheme which will give generating functions containing c_0, c_1, c_2 in correct places as coefficients, but which may immediately break down in $G_4(x)$. This proves to be so, for we see that

$$\begin{aligned}
G_1(x) &= c_0 + c_1 x \longmapsto \{c_0, c_1\}, \\
G_2(x) &= c_0 + c_1 x + c_2 x^2 + 2x^3 \longmapsto \{c_0, c_1, c_2, 2\}, \\
G_3(x) &= c_0 + c_1 x + c_2 x^2 + 6x^3 + 12x^4 + \cdots + 8x^7 \\
&\longmapsto \{c_0, c_1, c_2, 6, 12, 16, 16, 8\}, \\
G_4(x) &= c_0 + c_1 x + c_2 x^2 + 6x^3 + 20x^4 + \cdots + 128x^{15} \\
&\longmapsto \{c_0, c_1, c_2, 6, 20, 56, 128, 264, 496, 832, 1216, 1472, 1408, 1024, 512, 128\}, \\
G_5(x) &\longmapsto \{c_0, c_1, c_2, 6, 20, 72, 240, 712, \ldots, 262144, 32768\},
\end{aligned} \tag{3.36}$$

and so on. This is equivalent to the discretisation of a governing equation $0 = 1 + x - (1 + 2x)G(x) + 2xG^2(x)$, whose solution

$$G(x) = \frac{1 + 2x - \sqrt{1 - 4x - 4x^2}}{4x} \tag{3.37}$$

is the o.g.f. of a sequence $\{1, 1, 2, 6, 20, 72, 272, 1064, \ldots\}$ (unlisted in the O.E.I.S.).

Having successfully recovered the original recurrence relation of the Catalan sequence (in Case 1), two observations can now be made. The first is that the polynomials generated in this process have a tendency to expand rapidly in length, excessively so given that higher-degree terms do not contribute in any useful way to subsequent iterations. As such, this can be severely prohibitive if the calculations are to be performed by hand, particularly in the case of more complex schemes where the degree of either the original recurrence relation or its functional coefficients is higher, thereby necessitating the evaluation of more unknowns to recover the scheme. A typical example here is the Motzkin sequence, whose original recurrence relation is quadratic with quadratic coefficients.

One method of removing this impediment is by truncating the polynomials at each point in the iterative process, having captured the requisite number of terms—a technique which can obviously be applied to any linearly convergent scheme whose order is known.

In the case of the Catalan sequence (see Appendix A), applying truncation not only simplifies the coefficient-matching procedure, but also has the effect of eliminating the element of redundancy (*i.e.*, Case 2, p. 29).

The second point of interest is to note that whilst in the above example, only one term from the Catalan sequence was matched at each stage of the iterative process, it is possible to attempt matching two or more terms with each recursion instead. However, when this is carried out, it is apparent that the whole algorithm degenerates. Using the Catalan sequence as an example once again, we firstly obtain $G_1(x) = \alpha + \beta x$ as before, giving trivially that $\alpha = [x^0]\{G_1(x)\} = c_0 = 1$, together with $\beta = [x^1]\{G_1(x)\} = c_1 = 1$. This then produces

$$G_2(x) = 1 + \gamma + \varepsilon + (1 + \gamma + \delta + 2\varepsilon + \zeta)x + (\delta + \varepsilon + 2\zeta)x^2 + \zeta x^3, \tag{3.38}$$

so that we can write

$$\begin{aligned}
[x^0]\{G_2(x)\} &= c_0 = 1 = 1 + \gamma + \varepsilon, \\
[x^1]\{G_2(x)\} &= c_1 = 1 = 1 + \gamma + \delta + 2\varepsilon + \zeta, \\
[x^2]\{G_2(x)\} &= c_2 = 2 = \delta + \varepsilon + 2\zeta, \\
[x^3]\{G_2(x)\} &= c_3 = 5 = \zeta,
\end{aligned} \tag{3.39}$$

with no solution. It is clear that permitting even more terms to be matched will generate only these same equations.

Schröder Sequence

Applying (3.21) and repeating the process of matching the coefficients of iterated generating functions to terms of the Schröder sequence produces a repeat scenario in that we find there are once more two solution strands. As hoped, the first recovers the natural recurrence relation of the sequence (*i.e.* equation (1.15)), whilst the (redundant) second strand offers a scheme which is a discretisation of the quadratic $0 = 1 + 2x - (1 + 3x)G(x) + 3xG^2(x)$ governing the sequence (again unlisted in the O.E.I.S.) $\{1, 2, 6, 30, 162, 954, 5886, \ldots\} = \{s_0, s_1, s_2, 30, 162, 954, 5886, \ldots\}$. As for the previous Catalan example, appropriate truncation of polynomials at each iteration eliminates the redundant solution.

Also as before, attempting to impose a recurrence scheme in which multiple sequence terms are matched per iteration results in an invalid solution set.

Sequence A052709

As before, we impose (3.21), for $r \geq 0$ ($G_0(x) = 0$), so that $G_1(x) = \alpha + \beta x$, and $\alpha = [x^0]\{G_1(x)\} = a_0 = 0$. Substituting $G_1(x) = \beta x$ into the modified recurrence $G_{r+1}(x) = \beta x + (\gamma + \delta x)G_r(x) + (\varepsilon + \zeta x)G_r^2(x)$ yields

$$G_2(x) = \beta(1+\gamma)x + \beta(\beta\varepsilon + \delta)x^2 + \beta^2\zeta x^3, \tag{3.40}$$

whence

$$[x^1]\{G_2(x)\} = a_1 = 1 = \beta(1+\gamma). \tag{3.41}$$

Continuing, with $G_2(x) = x + \beta(\beta\varepsilon + \delta)x^2 + \beta^2\zeta x^3$,

$$\begin{aligned} G_3(x) &= \beta x + (\gamma + \delta x)G_2(x) + (\varepsilon + \zeta x)G_2^2(x) \\ &= (\beta + \gamma)x + [\delta + \varepsilon + (\beta\varepsilon + \delta)\beta\gamma]x^2 + [\beta(\delta + 2\varepsilon)(\beta\varepsilon + \delta) + \beta^2\gamma\zeta + \zeta]x^3 + \cdots, \end{aligned} \tag{3.42}$$

from which we can write

$$[x^1]\{G_3(x)\} = a_1 = 1 = \beta + \gamma, \tag{3.43}$$

and

$$[x^2]\{G_3(x)\} = a_2 = 1 = \delta + \varepsilon + (\beta\varepsilon + \delta)\beta\gamma. \tag{3.44}$$

Writing, from (3.41), $\beta = \frac{1}{1+\gamma}$, then (3.43) reduces to $\gamma = 0, \Rightarrow \beta = 1$. We also have now that

$$\delta + \varepsilon = 1 \tag{3.45}$$

by (3.44), and a simplified form of $G_3(x)$,

$$G_3(x) = x + x^2 + (\delta + 2\varepsilon + \zeta)x^3 + \cdots. \tag{3.46}$$

We need only use $G_3(x) = x + x^2 + \cdots$ in the updated recurrence

$$G_{r+1}(x) = x + \delta x G_r(x) + (\varepsilon + \zeta x)G_r^2(x), \tag{3.47}$$

for we find that

$$\begin{aligned} G_4(x) &= x + \delta x G_3(x) + (\varepsilon + \zeta x)G_3^2(x) \\ &= x + (\delta + \varepsilon)x^2 + (\delta + 2\varepsilon + \zeta)x^3 + \cdots \\ &= x + x^2 + (\delta + 2\varepsilon + \zeta)x^3 + \cdots \end{aligned} \tag{3.48}$$

by (3.45), giving

$$[x^3]\{G_4(x)\} = a_3 = 3 = \delta + 2\varepsilon + \zeta. \tag{3.49}$$

One more iteration is required to complete the solution scheme. The polynomial $G_5(x)$ is obtained as

$$\begin{aligned} G_5(x) &= x + \delta x G_4(x) + (\varepsilon + \zeta x) G_4^2(x) \\ &= x + (\delta + \varepsilon)x^2 + (\delta + 2\varepsilon + \zeta)x^3 + (3\delta + 7\varepsilon + 2\zeta)x^4 + \cdots \\ &= x + x^2 + 3x^3 + (3\delta + 7\varepsilon + 2\zeta)x^4 + \cdots \end{aligned} \tag{3.50}$$

by (3.45),(3.49), so that

$$[x^4]\{G_5(x)\} = a_4 = 9 = 3\delta + 7\varepsilon + 2\zeta; \tag{3.51}$$

equations (3.45), (3.49) and (3.51) have unique solution $\delta = 0$, $\varepsilon = \zeta = 1$ and (3.47), with G replaced by A, reads as (3.8). Note that this time there is no element of redundancy in the process, with only the expected scheme delivered.

As before, no valid schemes are forthcoming when imposing a scheme in which multiple sequence terms are matched per iteration.

3.4 Concluding Remarks and a New Result

As previously mentioned, although successful, the formulations seen above are tedious and error-prone to accomplish by hand, and allow no easy experimentation to examine the possible relation(s) between the structure of an initial scheme and any final (convergent) form reached after term-matching. Fortunately, the relative simplicity of the methodology involved enables the automation of the procedure using a computer algebra system without too much difficulty. Development of a working algorithm allows us to experiment a little more with the form of the recurrence relation imposed on such schemes, and in the case of the Catalan numbers, leads to an interesting result.

If the degree of the initial recurrence relation is increased from quadratic (as in equation (3.21) used in the previous example for the Catalan sequence) to cubic, *i.e.*

$$G_{r+1}(x) = \alpha + \beta x + (\gamma + \delta x)G_r(x) + (\varepsilon + \zeta x)G_r^2(x) + (\eta + \theta x)G_r^3(x), \tag{3.52}$$

it is found that the familiar solution $G_{r+1}(x) = 1 + xG_r^2(x)$ is once again recovered, with the additional cubic term in $G_r(x)$ being eliminated during the process.

Increasing the degree of the function's polynomial coefficients from linear to quadratic (and higher) whilst retaining the quadratic functional degree is found to produce solutions which still contain one or more indeterminate unknown(s), with no resulting explicit solutions.

However, reducing the imposed scheme to a linear degree in $G_r(x)$ does, by contrast, provide a considerable number of solutions as the functional coefficients are altered, as can be seen in the following results (in which the implementation of the quadratic case is demonstrated).

Linear Functional Coefficients:

$$G_{r+1}(x) = 1 - x + 2xG_r(x)$$

Quadratic Functional Coefficients:

$$G_{r+1}(x) = 1 - 3x + x^2 + (4x - 3x^2)G_r(x)$$

Implementation (Initial Value $G_1(x) = c_0 = 1$):

$$\begin{aligned}
G_2(x) &: \quad \{c_0, c_1, -2\}, \\
G_3(x) &: \quad \{c_0, c_1, c_2, -11, 6\}, \\
G_4(x) &: \quad \{c_0, c_1, c_2, c_3, -50, 57, -18\}, \\
G_5(x) &: \quad \{c_0, c_1, c_2, c_3, c_4, -215, 378, -243, 54\}, \\
G_6(x) &: \quad \{c_0, c_1, c_2, c_3, c_4, 41, -902, 2157, -2106, 945, -162\}, \\
G_7(x) &: \quad \{c_0, c_1, c_2, c_3, c_4, 41, 122, -3731, 11334, -14895, 10098, -3483, 486\}, \\
G_8(x) &: \quad \{c_0, c_1, c_2, c_3, c_4, 41, 122, 365, -15290, 56529, -93582, 85077, -44226, \ldots\}, \\
&\vdots
\end{aligned}$$

Cubic Functional Coefficients:

$$G_{r+1}(x) = 1 - 5x + 6x^2 - x^3 + (6x - 10x^2 + 4x^3)G_r(x)$$

Quartic Functional Coefficients:

$$G_{r+1}(x) = 1 - 7x + 15x^2 - 10x^3 + x^4 + (8x - 21x^2 + 20x^3 - 5x^4)G_r(x) \tag{3.53}$$

Clearly, if the general form of the scheme possesses insufficient degrees of freedom as in the above cases, it is impossible to recover the natural discretisation of a sequence's o.g.f., implying that terms of the associated infinite sequence cannot be generated indefinitely either. However, it is apparent when executing each of these schemes iteratively that they generate Catalan subsequences of, respectively, 3, 5, 7 and 9 terms before "failing". From these, and further investigative computations (seeking schemes which give but an even number of Catalan terms), it has been possible to identify a class of polynomials—of which those appearing above are special cases—and both formalise and subsequently prove a general theorem in which they play an integral part in generating finite Catalan subsequences, the details of which are given in the next chapter together with a comprehensive description of the polynomials themselves.

Chapter 4

Catalan Polynomials

4.1 Introduction and Theorem

In the previous chapter, it was discovered that placing a specific set of constraints on an algorithm designed to recover a recurrence scheme for the Catalan sequence would, instead of recovering the sequence's natural recurrence relation, generate schemes which appear to produce a finite number of Catalan terms before "failing". In each case, the linear recurrence relation generated incorporates two polynomials of equal degree as functional coefficients, which are found to form part of a series of polynomials which will hereafter be referred to as Catalan polynomials.

The general form of the Catalan polynomial $P_n(x)$, say, is given by

$$P_n(x) = \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i} (-x)^i, \qquad n \geq 0. \tag{4.1}$$

It can also be expressed as a specialisation of the Gaussian hypergeometric function[1] ($(a)_i$, $(b)_i$, $(c)_i$ here denoting falling factorials)

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z \right) = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i} \frac{z^i}{i!}, \qquad |z| < 1. \tag{4.2}$$

With $a = -\frac{1}{2}n, b = -\frac{1}{2}(n-1), c = -n$ and $z = 4x$, it is found that

$$\begin{aligned} P_n(x) &= {}_2F_1 \left(\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}(n-1) \\ -n \end{matrix} \,\middle|\, 4x \right) \\ &= \sum_{i=0}^{\infty} \frac{\left(-\frac{1}{2}n\right)_i \left(-\frac{1}{2}(n-1)\right)_i}{(-n)_i} \frac{(4x)^i}{i!}. \end{aligned} \tag{4.3}$$

[1]Although hypergeometric functions will not be discussed further in this work, a comprehensive treatment of the subject can be found in Petkovšek *et al.* (1996).

The first few polynomials are:

$$
\begin{aligned}
&P_0(x) = 1\\
&P_1(x) = 1\\
&P_2(x) = 1 - x\\
&P_3(x) = 1 - 2x\\
&P_4(x) = 1 - 3x + x^2\\
&P_5(x) = 1 - 4x + 3x^2\\
&P_6(x) = 1 - 5x + 6x^2 - x^3\\
&P_7(x) = 1 - 6x + 10x^2 - 4x^3\\
&P_8(x) = 1 - 7x + 15x^2 - 10x^3 + x^4\\
&P_9(x) = 1 - 8x + 21x^2 - 20x^3 + 5x^4\\
&\quad\vdots
\end{aligned}
\tag{4.4}
$$

Based on the results outlined in the last chapter (3.53), we theorise and prove the following statement:

Theorem 4.1. *For any integer $n \geq 0$, the first-order scheme*

$$F_{i+1}(x) = P_n(x) + [1 - P_{n+1}(x)]F_i(x); \qquad F_1(x) = c_0,$$

produces polynomials $F_1(x) = c_0, F_2(x), F_3(x), \ldots,$ where $F_i(x)$ acts as an o.g.f. for the finite subsequence $\{c_0, c_1, c_2, \ldots, c_{i-1}\}$, $i = 1, \ldots, n+1$.

However, before presenting the proof, it is pertinent to outline the essential properties of the Catalan polynomials for reasons of both completeness and interest.

4.2 Mathematical Properties of the Polynomials

4.2.1 Linear Recurrence Property and Closed Form

From (4.1), it is a straightforward matter to verify that the Catalan polynomials satisfy the basic second-order linear recurrence

$$0 = xP_n(x) - P_{n+1}(x) + P_{n+2}(x); \qquad P_0(x) = P_1(x) = 1, \tag{4.5}$$

from which the closed form

$$P_n(x) = \frac{1}{2^{n+1}} \frac{\left(1+\sqrt{1-4x}\right)^{n+1} - \left(1-\sqrt{1-4x}\right)^{n+1}}{\sqrt{1-4x}} \tag{4.6}$$

is readily established.

Proof. The associated characteristic equation $0 = \lambda^2 - \lambda + x$ of (4.5) has roots $\lambda_1(x) = \frac{1}{2}(1 + r(x)), \lambda_2(x) = \frac{1}{2}(1 - r(x))$ (defining $r(x) = \sqrt{1-4x}$ for convenience). For $\lambda_1 \neq \lambda_2$, the general solution $P_n(x) = A(x)\lambda_1^n(x) + B(x)\lambda_2^n(x)$ yields simultaneous equations $1 = A(x) + B(x) = A(x)\lambda_1(x) + B(x)\lambda_2(x)$ from the initial values of $P_0(x), P_1(x)$, with solutions $A(x) = (1 - \lambda_2(x))/(\lambda_1(x) - \lambda_2(x)) = \lambda_1(x)/r(x)$ and $B(x) = 1 - A(x) = -\lambda_2(x)/r(x)$, from which (4.6) follows. □

Remark 4.2. The case $\lambda_1 = \lambda_2$ excluded from the proof corresponds to x taking (maximum) value $\frac{1}{4}$. It gives a general solution $P_n(\frac{1}{4}) = (Cn + D)(\frac{1}{2})^n$, and in turn a particular one $P_n(\frac{1}{4}) = (n+1)(\frac{1}{2})^n$ which can be checked for any $n \geq 0$. Note that an alternative means to find $P_n(\frac{1}{4})$ is to set it up as the limit $P_n(\frac{1}{4}) = 2^{-(n+1)} \lim_{r(x)\to 0^+}\{[(1+r(x))^{n+1} - (1-r(x))^{n+1}]/r(x)\}$, which is easily dealt with using L'Hôpital's Rule for limits with indeterminate forms.

4.2.2 Fibonacci Numbers and Cyclotomic Polynomials

The well-known Lucas polynomials $l_n(x)$, say, satisfy the recurrence $0 = l_n(x) + xl_{n+1}(x) - l_{n+2}(x)$ ($l_0(x) = 2$, $l_1(x) = x$), whilst the related and illustrious Fibonacci polynomials $f_n(x)$, say, satisfy the identical recurrence $0 = f_n(x) + xf_{n+1}(x) - f_{n+2}(x)$ ($f_0(x) = 0$, $f_1(x) = 1$), each of which is similar in structure to (4.5) (see Koshy (2001, pp. 459, 443, resp.)). We note, as a point of interest, that the Catalan polynomials are connected to the Fibonacci sequence for at $x = -1$ the Catalan polynomials replicate the Fibonacci numbers—that is to say, $\{P_0(-1), P_1(-1), P_2(-1), P_3(-1), P_4(-1), \ldots\} = \{1, 1, 2, 3, 5, \ldots\}$. The cyclic (period 6) sequence $\{P_n(1)\}_0^\infty = \{1, 1, 0, -1, -1, 0, \ldots\}$ arises from a cyclotomic polynomial (see O.E.I.S. sequence no. A010892).

4.2.3 Chebyshev and Dickson Polynomials

It should be noted that the Catalan polynomials, as they are called here, are not new polynomials, the r.h.s. of (4.1),(4.6) having been equated in a paper on number theory by Mandl (1891, p. 236). In 2006, the ordered sequence of coefficients $\{1; 1; 1, -1; 1, -2; 1, -3, 1; 1, -4, 3; \ldots\}$ was entered as sequence no. A115139 into the O.E.I.S. by W. Lang, who lists some properties in connection with work on integral and inverse powers of the Catalan sequence o.g.f. $C(x)$. In this (Lang, 2000), we see that the Catalan polynomials are related to Chebyshev polynomials of the second kind $U_n(x)$ according to

$$P_n(x) = (\sqrt{x})^n U_n\left(\frac{1}{2\sqrt{x}}\right), \qquad n \geq 0, \tag{4.7}$$

(4.6) being immediate from the corresponding closed form for $U_n(x)$ found in, *e.g.*, Rivlin (1990, Problem 1.2.20, p. 10)

$$U_n(x) = \frac{\left(x + \sqrt{x^2-1}\right)^{n+1} - \left(x - \sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}}. \tag{4.8}$$

Likewise we have the o.g.f.

$$\frac{1}{1-t+xt^2} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{4.9}$$

directly from that of these Chebyshev polynomials (or from (4.5) in standard combinatorics fashion). We can also appeal to the fact that $U_n(x) = E_n(2x, 1)$ is a special case of the general two-parameter Dickson polynomial of the second kind (Lidl *et al.* (1993, Definition 2.2, p. 9)),

$$E_n(x,a) = \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i} (-a)^i x^{n-2i}, \tag{4.10}$$

to obtain a succinct matrix format for our Catalan polynomials; since, for $n \geq 0$ (Lidl *et al.* (1993, (2.4), p. 11)),

$$E_n(x,a) = (1,x)\begin{pmatrix} 0 & -a \\ 1 & x \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.11}$$

then

$$\begin{aligned} P_n(x) &= \left(\sqrt{x}\right)^n E_n(1/\sqrt{x}, 1) \\ &= \left(\sqrt{x}\right)^n \left(1, 1/\sqrt{x}\right) \begin{pmatrix} 0 & -1 \\ 1 & \frac{1}{\sqrt{x}} \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n \geq 0, \end{aligned} \tag{4.12}$$

which we verify here for the first few values of n:

$$\begin{aligned} P_0(x) &= \left(1, 1/\sqrt{x}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \left(1, 1/\sqrt{x}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= 1; \end{aligned}$$

$$
\begin{aligned}
P_1(x) &= \sqrt{x}\left(1, 1/\sqrt{x}\right)\begin{pmatrix}0 & -1\\ 1 & \frac{1}{\sqrt{x}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
&= \left(\sqrt{x}, 1\right)\begin{pmatrix}0\\1\end{pmatrix}\\
&= 1;\\
P_2(x) &= x\left(1, 1/\sqrt{x}\right)\begin{pmatrix}-1 & -\frac{1}{\sqrt{x}}\\ \frac{1}{\sqrt{x}} & \frac{1}{x}-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
&= x\left(1, 1/\sqrt{x}\right)\begin{pmatrix}-1\\ \frac{1}{\sqrt{x}}\end{pmatrix}\\
&= x\left(-1+\frac{1}{x}\right)\\
&= 1-x;\\
P_3(x) &= x\sqrt{x}\left(1, 1/\sqrt{x}\right)\begin{pmatrix}-\frac{1}{\sqrt{x}} & 1-\frac{1}{x}\\ \frac{1}{x}-1 & \frac{1}{\sqrt{x}}\left(\frac{1}{x}-2\right)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
&= \left(x\sqrt{x}, x\right)\begin{pmatrix}-\frac{1}{\sqrt{x}}\\ \frac{1}{x}-1\end{pmatrix}\\
&= -x+x\left(\frac{1}{x}-1\right)\\
&= 1-2x;\\
P_4(x) &= x^2\left(1, 1/\sqrt{x}\right)\begin{pmatrix}1-\frac{1}{x} & \frac{1}{\sqrt{x}}\left(2-\frac{1}{x}\right)\\ \frac{1}{\sqrt{x}}\left(\frac{1}{x}-2\right) & \frac{1}{x^2}-\frac{3}{x}+1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
&= x^2\left(1, 1/\sqrt{x}\right)\begin{pmatrix}1-\frac{1}{x}\\ \frac{1}{\sqrt{x}}\left(\frac{1}{x}-2\right)\end{pmatrix}\\
&= x^2\left[1-\frac{1}{x}+\frac{1}{x}\left(\frac{1}{x}-2\right)\right]\\
&= 1-3x+x^2;\\
P_5(x) &= x^2\sqrt{x}\left(1, 1/\sqrt{x}\right)\begin{pmatrix}\frac{1}{\sqrt{x}}\left(2-\frac{1}{x}\right) & -\frac{1}{x^2}+\frac{3}{x}-1\\ \frac{1}{x^2}-\frac{3}{x}+1 & \frac{1}{\sqrt{x}}\left(\frac{1}{x^2}-\frac{4}{x}+3\right)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
&= \left(x^2\sqrt{x}, x^2\right)\begin{pmatrix}\frac{1}{\sqrt{x}}\left(2-\frac{1}{x}\right)\\ \frac{1}{x^2}-\frac{3}{x}+1\end{pmatrix}\\
&= x^2\left(2-\frac{1}{x}\right)+x^2\left(\frac{1}{x^2}-\frac{3}{x}+1\right)\\
&= 1-4x+3x^2, \qquad (4.13)
\end{aligned}
$$

and so on.

With a little work, an alternative matrix format can be established for $P_n(x)$. By defining the matrices

$$\mathbf{L}(x) = \begin{pmatrix} 1 & 1 \\ 0 & -\sqrt{x} \end{pmatrix} \quad \text{and} \quad \mathbf{M}(x) = \begin{pmatrix} 1 & x \\ -1 & 0 \end{pmatrix}, \tag{4.14}$$

which have the property that

$$\begin{pmatrix} 0 & -\sqrt{x} \\ \sqrt{x} & 1 \end{pmatrix} = \mathbf{L}(x)\mathbf{M}(x)\mathbf{L}^{-1}(x), \tag{4.15}$$

we can re-write (4.12) as

$$\begin{aligned} P_n(x) &= \left(\sqrt{x}\right)^n \left(1, 1/\sqrt{x}\right) \begin{pmatrix} 0 & -1 \\ 1 & \frac{1}{\sqrt{x}} \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \left(1, 1/\sqrt{x}\right) \begin{pmatrix} 0 & -\sqrt{x} \\ \sqrt{x} & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \left(1, 1/\sqrt{x}\right) \left[\mathbf{L}(x)\mathbf{M}(x)\mathbf{L}^{-1}(x)\right]^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \left(1, 1/\sqrt{x}\right) \mathbf{L}(x)\mathbf{M}^n(x)\mathbf{L}^{-1}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{4.16}$$

Noting that

$$\begin{aligned} \left(1, 1/\sqrt{x}\right) \mathbf{L}(x) &= (1, 0), \\ \mathbf{L}^{-1}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned} \tag{4.17}$$

(4.16) can now be written as

$$\begin{aligned} P_n(x) &= (1, 0)\mathbf{M}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= (1, 0) \begin{pmatrix} 1 & x \\ -1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{4.18}$$

The aforementioned text by Lidl *et al.* (1993, Lemma 2.17, p. 16) also permits $P_n(x)$ to be expressed in terms of an $n \times n$ tri-diagonal matrix as

$$P_n(x) = \left(\sqrt{x}\right)^n \det \left\{ \begin{pmatrix} \frac{1}{\sqrt{x}} & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \frac{1}{\sqrt{x}} & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\sqrt{x}} & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{x}} & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{\sqrt{x}} & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \frac{1}{\sqrt{x}} \end{pmatrix} \right\}, \tag{4.19}$$

and the closed form for $E_n(x, 1)$ from Lidl *et al.* (1993, p. 16) also recovers (4.6).

4.2.4 Continued Fractions and Dyck Paths

One further characteristic of the Catalan polynomials is that the ratio $P_n(x)/P_{n+1}(x)$ is, for $n \geq 0$, the nth continued fraction associated with the Catalan sequence (based on the rearrangement $C(x) = \frac{1}{1-xC(x)}$ of the sequence's governing equation (1.9)) which, when expanded as a Maclaurin series, describes an o.g.f. with $n+1$ Catalan numbers $c_0, \ldots, c_n$ as the coefficients of terms $x^0, \ldots, x^n$. Defining the 0th continued fraction as $P_0(x)/P_1(x) = 1 = c_0$, we observe that

$$\begin{aligned}
\frac{P_1(x)}{P_2(x)} &= \frac{1}{1-x} = c_0 + c_1x + \cdots \,, \\
\frac{P_2(x)}{P_3(x)} &= \frac{1}{1-\frac{x}{1-x}} = c_0 + c_1x + c_2x^2 + \cdots \,, \\
\frac{P_3(x)}{P_4(x)} &= \frac{1}{1-\frac{x}{1-\frac{x}{1-x}}} = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots \,,
\end{aligned} \tag{4.20}$$

etc.[2] Recall that a Dyck path (of length $2n$) is an even path in the x-y plane from the origin $(0,0)$ to the point $(2n,0)$, using steps $(1,1)$ (north-east) and $(1,-1)$ (south-east), which does not fall below the x-axis anywhere. Whilst c_n is the number of unconstrained Dyck paths of length $2n$ (and also by Peart and Woan (2001) the number of paths of length $2(n+1)$ with no peak at height 2), this set of continued fractions counts Dyck paths with bounded height: $1/(1-x)$ counts paths with height at most 1, $1/(1-x/(1-x))$ counts paths of height at most 2, and so on. A useful reference here is a paper by Flajolet (1980) on combinatorial interpretations of continued fractions, whose properties in relation to the Catalan numbers continue to be developed in a lattice path context and others such as ordered trees and sequences in permutations (see, for instance, Jani and Rieper (2000), or Brändén *et al.* (2002)).

4.2.5 Other Properties

We finish this subsection by listing some other properties of the Catalan polynomials.

Non-Linear Recurrences

Rather than (4.5) being used to derive the closed form (4.6) as shown, the latter could serve

[2]Let $CF_n(x)$ be the nth continued fraction. Since $CF_{n+1}(x) = 1/(1-xCF_n(x))$ by definition, it is a simple matter to prove inductively that $CF_n(x) = P_n(x)/P_{n+1}(x)$ for $n \geq 0$ using the Catalan polynomial recursion (4.5).

as a starting point to validate (or deduce) (4.5) by hand. Equally, (4.6) confirms the following *non-linear* recurrence (originally found through experimental computation)

$$0 = P_n^2(x) - P_{2n-1}(x) - x^2 P_{n-2}^2(x), \qquad n \geq 2. \tag{4.21}$$

This can be combined with (4.5) to yield the similar recurrence

$$0 = P_n^2(x) - P_{2n}(x) - x P_{n-1}^2(x), \qquad n \geq 1. \tag{4.22}$$

Neither (4.21) nor (4.22) appear in literature explicitly, and of course each can be re-written in terms of the aforementioned Chebyshev or Dickson polynomials (indeed, the Dickson polynomial version of (4.22) appears in Lidl *et al.* (1993, Lemma 2.16, p. 15)), where it is notable in giving the more general result

$$0 = P_{r+s}(x) - P_r(x)P_s(x) + xP_{r-1}(x)P_{s-1}(x), \qquad r, s \geq 1, \tag{4.23}$$

of which (4.22) is the instance $r = s = n$. Finally, the identity found in Lidl *et al.* (1993, p. 16) $\sum_{i=0}^{n} E_{2i}(x,1) = E_n^2(x,1)$ immediately yields

$$\sum_{i=0}^{n} x^{n-i} P_{2i}(x) = P_n^2(x), \qquad n \geq 0, \tag{4.24}$$

a result also available through "telescoping" by writing (4.22) as $P_{2i}(x) = P_i^2(x) - xP_{i-1}^2(x)$, and then multiplying both sides by x^{n-i} and summing over i (this requires $P_{-1}(x) = 0$; see the proof of the subsequent lemma).

Governing Differential Equation

The differential equation satisfied by the Catalan polynomials is, from Lidl *et al.* (1993, Theorem 2.15(ii), p. 15) with some work,

$$0 = n(n-1)P_n(x) + [n + 2(3-2n)x]P_n'(x) + x(4x-1)P_n''(x). \tag{4.25}$$

Divisibility Properties

It is a simple matter, combining (4.5) with (4.21), to show that $P_{2n+1}(x)$ is divisible by $P_n(x)$, with

$$\frac{P_{2n+1}(x)}{P_n(x)} = P_{n+1}(x) - xP_{n-1}(x), \qquad n \geq 1. \tag{4.26}$$

Computations affirm that for $n \geq 2$ the polynomial $P_{2n+1}(x)$ contains $P_n(x)$ as a factor, with those polynomials which are irreducible being $P_r(x)$ for $r = 4, 6, 10, 12, 16, 18, \ldots$ (each one less than the primes from 5 onwards). A corollary of (4.26) (also a consequence of Lidl *et al.* (1993, Lemma 2.19(ii), p. 16) listed in terms of Chebyshev polynomials) is that, defining $R_n(x)$ to be the ratio

$$R_n(x) = \frac{P_{2n+1}(x)}{(\sqrt{x})^{n+1}\, P_n(x)}, \tag{4.27}$$

then

$$R_n(x)|_{\sqrt{x}=\pm 1,\pm\frac{1}{2},\pm\frac{1}{3},\pm\frac{1}{4},\ldots} \in \mathbb{Z}, \qquad n \geq 0, \tag{4.28}$$

with the caveat that for specific values of $n = 2, 5, 8, 11, \ldots$, $R_n(x)|_{\sqrt{x}=\pm 1}$ is undefined since both $P_n(1)$ and $P_{2n+1}(1)$ are zero.[3] Considering those evaluations of $R_n(x)$ at $\sqrt{x} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, for a moment, another couple of observations are that (i) $R_n(x)|_{\sqrt{x}=\frac{1}{2}} = 2\ \forall n \geq 0$ (following trivially from the fact that (Remark 4.2) $P_n(\frac{1}{4}) = (n+1)(\frac{1}{2})^n$), and (ii) $\{R_0(x)|_{\sqrt{x}=1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots}\}$ generates the sequence of natural numbers $\{1, 2, 3, 4, \ldots\}$ (since $R_0(x) = \frac{1}{\sqrt{x}}$). Results for evaluations at $\sqrt{x} = -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots$, are immediate, for it is easy to see that

$$R_n(x)|_{\sqrt{x}=-1,-\frac{1}{2},-\frac{1}{3},-\frac{1}{4},\ldots} = \begin{cases} -R_n(x)|_{\sqrt{x}=1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots} & n \text{ (even)} = 0, 2, 4, 6, \ldots \\ R_n(x)|_{\sqrt{x}=1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots} & n \text{ (odd)} = 1, 3, 5, 7, \ldots \end{cases} \tag{4.29}$$

Another divisibility property to be noted is that (by Lidl *et al.* (1993, Lemma 2.18(ii), p. 16)) the ratio

$$Q_n(x) = \frac{\frac{P_{n-1}(x)}{(\sqrt{x})^{n-1}} - \left(\frac{1}{x} - 4\right)^{\frac{1}{2}(n-1)}}{n} \tag{4.30}$$

also satisfies the arithmetic relation

$$Q_n(x)|_{\sqrt{x}=\pm 1,\pm\frac{1}{2},\pm\frac{1}{3},\pm\frac{1}{4},\ldots} \in \mathbb{Z} \tag{4.31}$$

for all odd prime n, with the additional observations that (i) $Q_1(x)|_{\sqrt{x}=\pm 1,\pm\frac{1}{2},\pm\frac{1}{3},\pm\frac{1}{4},\ldots} = 0$, (ii) $Q_3(x)|_{\sqrt{x}=\pm 1,\pm\frac{1}{2},\pm\frac{1}{3},\pm\frac{1}{4},\ldots} = 1$, and (iii) for prime $n \geq 3$ then $Q_n(x)|_{\sqrt{x}=\pm\frac{1}{2}} = 1$.

Roots

Empirical evidence suggests that for $n \geq 2$, $P_n(x)$ possesses $\lfloor n/2 \rfloor$ roots which are all real and positive. In fact Lidl *et al.* (1993, p. 16) state a closed form for all roots of the Chebyshev polynomial $E_n(x, 1)$, based on its tri-diagonal matrix form, so that we can write

$$P_n(x) = \left(\sqrt{x}\right)^n \prod_{\lambda=1}^{n} \left(\frac{1}{\sqrt{x}} - 2\cos\left(\frac{\lambda\pi}{n+1}\right)\right), \qquad n \geq 1. \tag{4.32}$$

[3] See the cyclic sequence $\{P_n(1)\}_0^\infty$ in Section 4.2.2. These values of n are given by $n = 3m - 1$ (integer $m = 1, 2, 3, \ldots$), whence $2n + 1 = 3(2m) - 1$ is of the same form.

Initial cases are readily verified by hand, the first non-trivial one being the following:

$\underline{n=4}$: Noting that $\cos(\pi/5) = \frac{1}{4}(1+\sqrt{5})$, $\cos(2\pi/5) = \frac{1}{4}(-1+\sqrt{5})$, $\cos(3\pi/5) = -\cos(2\pi/5) = -\frac{1}{4}(-1+\sqrt{5})$ and $\cos(4\pi/5) = -\cos(\pi/5) = -\frac{1}{4}(1+\sqrt{5})$, (4.32) reads

$$
\begin{aligned}
P_4(x) &= \left(\sqrt{x}\right)^4 \prod_{\lambda=1}^{4} \left(\frac{1}{\sqrt{x}} - 2\cos\left(\frac{\lambda\pi}{5}\right)\right), \\
&= x^2 \left(\frac{1}{\sqrt{x}} - 2\cos\left(\frac{\pi}{5}\right)\right)\left(\frac{1}{\sqrt{x}} - 2\cos\left(\frac{2\pi}{5}\right)\right) \times \\
&\qquad\qquad \left(\frac{1}{\sqrt{x}} - 2\cos\left(\frac{3\pi}{5}\right)\right)\left(\frac{1}{\sqrt{x}} - 2\cos\left(\frac{4\pi}{5}\right)\right) \\
&= x^2 \left(\frac{1}{\sqrt{x}} - \frac{1}{2}\left(1+\sqrt{5}\right)\right)\left(\frac{1}{\sqrt{x}} - \frac{1}{2}\left(-1+\sqrt{5}\right)\right) \times \\
&\qquad\qquad \left(\frac{1}{\sqrt{x}} + \frac{1}{2}\left(-1+\sqrt{5}\right)\right)\left(\frac{1}{\sqrt{x}} + \frac{1}{2}\left(1+\sqrt{5}\right)\right) \\
&= x^2 \left(\frac{1}{x} - \frac{1}{4}\left(1+\sqrt{5}\right)^2\right)\left(\frac{1}{x} - \frac{1}{4}\left(-1+\sqrt{5}\right)^2\right) \\
&= x^2 \frac{\left[4-\left(1+\sqrt{5}\right)^2 x\right]}{4x}\frac{\left[4-\left(-1+\sqrt{5}\right)^2 x\right]}{4x} \\
&= \frac{1}{16}\left[4-\left(6+2\sqrt{5}\right)x\right]\left[4-\left(6-2\sqrt{5}\right)x\right] \\
&= \frac{1}{16}\left(16-48x+16x^2\right) \\
&= 1-3x+x^2. \qquad (4.33)
\end{aligned}
$$

In the absence of any other proof, we look ourselves at the roots of $P_n(x)$ from first principles using the closed form (4.6) as follows. We can assume that $x \neq \frac{1}{4}$ (for we know that $P_n(\frac{1}{4}) = 2^{-n}(n+1) \neq 0 \ \forall n \geq 0$), so solutions of the equation $P_n(x) = 0$ are those of the equation $(1+r(x))^{n+1} - (1-r(x))^{n+1} = 0$ where $r(x) = \sqrt{1-4x}$. Similarly, since $P_n(0) = 1 \neq 0$ $\forall n \geq 0$ we can further assume that $x \neq 0$, whence $r(x) \neq 1$ and we seek instead solutions of the equation

$$
0 = \left(\frac{1+r(x)}{1-r(x)}\right)^{n+1} - 1, \qquad (4.34)
$$

which is straightforward. Writing $1 = \exp(2\pi\lambda i)$ in standard fashion ($\lambda \in \mathbb{Z}$), then

$$
\frac{1+r(x)}{1-r(x)} = \exp(2\pi\lambda i/(n+1)), \qquad \lambda = 1, 2, 3, \ldots, n, \qquad (4.35)
$$

where $\lambda = 0$ is excluded because it clearly coincides with the discounted solution $x = \frac{1}{4}$. We also note that values of λ where $2\lambda = n+1$ are also excluded since the r.h.s. here is then $\exp(i\pi) = -1$, which is a contradiction (this will exclude one single potential solution for any given n odd). Rearranging gives, setting $\theta = \theta(\lambda) = 2\pi\lambda/(n+1)$,

$$r(x) = \frac{e^{i\theta} - 1}{e^{i\theta} + 1} \tag{4.36}$$

and, squaring both sides,

$$\begin{aligned} x &= \frac{1}{4}\left[1 - \left(\frac{e^{i\theta}-1}{e^{i\theta}+1}\right)^2\right] \\ &= \frac{e^{i\theta}}{\left(e^{i\theta}+1\right)^2} \\ &= \left(\frac{\left(e^{i\theta}+1\right)^2}{e^{i\theta}}\right)^{-1} \\ &= \left(e^{i\theta} + 2 + e^{-i\theta}\right)^{-1} \\ &= \frac{1}{2}\frac{1}{1+\cos(\theta)}. \end{aligned} \tag{4.37}$$

Thus, the equation $0 = P_n(x)$ has solutions $x_{\lambda(n)}$ from this line of argument, where, for $n \geq 2$,

$$x_{\lambda(n)} = \frac{1}{2}\frac{1}{1+\cos\left(\theta(\lambda)\right)} = \frac{1}{4\cos^2\left(\frac{1}{2}\theta(\lambda)\right)} = \frac{1}{4\cos^2\left(\frac{\lambda\pi}{n+1}\right)}, \tag{4.38}$$

although since the degree of $P_n(x)$ is known to be $\lfloor n/2 \rfloor$ some of those will necessarily be repeated as the index $\lambda \geq 1$ runs along its designated values. By way of example, consider the case $n = 5$, for which λ takes values 1,2,4 and 5 ($\lambda = 3$ is removed for here $2\lambda = n+1$ as mentioned above), yielding roots

$$\frac{1}{4\cos^2\left(\frac{\pi}{6}\right)}, \frac{1}{4\cos^2\left(\frac{\pi}{3}\right)}, \frac{1}{4\cos^2\left(\frac{2\pi}{3}\right)}, \frac{1}{4\cos^2\left(\frac{5\pi}{6}\right)} = \frac{1}{3}, 1, 1, \frac{1}{3}; \tag{4.39}$$

thus, the two distinct roots $1, \frac{1}{3}$ of $P_5(x) = 1 - 4x + 3x^2 = (1-x)(1-3x)$ are correctly identified.

4.3 Proof of Theorem 4.1

It is apparent that the basic form of the recurrence stated in Theorem 4.1 is

$$F_{r+1}(x) = f(x) + xg(x)F_r(x), \tag{4.40}$$

which was discussed in Chapter 2 in relation to a finite target sequence. Here, $f(x) = f(x;n) = P_n(x)$ and $g(x) = g(x;n) = [1 - P_{n+1}(x)]/x$ (note that for $n > 0$ the numerator $1 - P_{n+1}(x)$ always has a non-zero lead term in x so that $g(x)$ contains no inverse powers of x; $g(x) = 0$ identically for $n = 0$). In view of this we know in advance (by Remarks 2.2 and 2.3) that, for any fixed n, the resulting iterative scheme will yield a succession of associated sequences which are "preserving" and singly-linear (*i.e.*, one correct sequence term is added per iteration) in their convergence rate, although this is not necessary for the proof itself.

We will show that, for $n \geq 0$, $i = 1, \ldots, n+1$,

$$F_i(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{i-1} x^{i-1} + x^i \Delta_i(x) \tag{4.41}$$

for some $\Delta_i(x) = \Delta_i(x; n) \in \mathbb{Z}[x]$, arguing by induction on i.

The latter part of the proof relies on the use of the following lemma (see also Appendix B):

Lemma 4.3. *For integer $n \geq 1$,*

$$C^n(x) = \frac{C(x)P_{n-1}(x) - P_{n-2}(x)}{x^{n-1}}.$$

Note that in Lang (2000, (1), p. 408) the assertion is made that every positive integer power of the o.g.f. $C(x)$ has a form $C^n(x) = p_{n-1}(x) + q_{n-1}(x)C(x)$ for certain polynomials $p_{n-1}(x)$, $q_{n-1}(x)$ each of degree $n-1$ in $1/x$. In the proof of Proposition 1 therein (p. 411)—where a different (non-inductive) line of reasoning is made by Lang—they are shown to be related to Chebyshev polynomials of the second kind, and we see $q_{n-1}(x) = -xp_n(x)$, and further that $p_n(x) = -x^{-n}P_{n-1}(x)$, which reconstructs the lemma.

Proof of Lemma 4.3. By induction. Defining an additional polynomial $P_{-1}(x) = 0$ (also consistent with $n = -1$ in (4.12)), Lemma 4.3 clearly holds for $n = 1$. Suppose it is true for some $n = k \geq 1$, and consider

$$\begin{aligned} C^{k+1}(x) &= C(x)C^k(x) \\ &= C(x)\left(\frac{C(x)P_{k-1}(x) - P_{k-2}(x)}{x^{k-1}}\right) && \text{(by assumption)} \\ &= \frac{C^2(x)P_{k-1}(x) - C(x)P_{k-2}(x)}{x^{k-1}} \\ &= \frac{1}{x^{k-1}}\left[\left(\frac{C(x)-1}{x}\right)P_{k-1}(x) - C(x)P_{k-2}(x)\right] && \text{(by (1.9))} \\ &= \frac{1}{x^k}\left[C(x)\{P_{k-1}(x) - xP_{k-2}(x)\} - P_{k-1}(x)\right]. \end{aligned} \tag{4.42}$$

Setting $n \to k-2$ in the linear recursion (4.5), and rearranging, gives $P_{k-1}(x) - xP_{k-2}(x) = P_k(x)$ (which holds for $k \geq 1$), and so $C^{k+1}(x) = \frac{1}{x^k}[C(x)P_k(x) - P_{k-1}(x)]$ as required. □

As far as Theorem 4.1 is concerned, for $n = 0$ we require $F_1(x) = c_0 + x\Delta_1(x)$, which holds trivially by the initial value for $F_1(x)$ choosing $\Delta_1(x) = 0$. Thus, the result is valid for $i = 1$, so that we have by hypothesis, for some $i = k \in [1, n]$ $(n \geq 1)$, (4.41) as

$$F_k(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{k-1} x^{k-1} + x^k \Delta_k(x), \tag{4.43}$$

$\Delta_k(x) = \Delta_k(x; n) \in \mathbb{Z}[x]$. We can re-write this as

$$\begin{aligned} F_k(x) &= c_0 + c_1 x + c_2 x^2 + \cdots + c_{k-1} x^{k-1} + x^k \Delta_k(x) \\ &\qquad + (c_k x^k + c_{k+1} x^{k+1} + \cdots) - (c_k x^k + c_{k+1} x^{k+1} + \cdots) \\ &= C(x) - (c_k x^k + c_{k+1} x^{k+1} + \cdots) + x^k \Delta_k(x) \\ &= C(x) + x^k \Delta^*(x), \end{aligned} \tag{4.44}$$

where $\Delta^*(x) = \Delta^*(x; k, n) = \Delta_k(x) - c_k - c_{k+1} x - c_{k+2} x^2 - \cdots \in \mathbb{Z}[[x]]$ is an infinite series. Now, directly from the scheme, employing (4.44) gives

$$\begin{aligned} F_{k+1}(x) &= P_n(x) + [1 - P_{n+1}(x)] F_k(x) \\ &= P_n(x) + [1 - P_{n+1}(x)][C(x) + x^k \Delta^*(x)] \\ &= C(x) + P_n(x) - C(x) P_{n+1}(x) + [1 - P_{n+1}(x)] x^k \Delta^*(x), \end{aligned} \tag{4.45}$$

and since $[x]\{P_{n+1}(x) - 1\} = [x]\{P_{n+1}(x)\} = -n$ for $n \geq 1$, then at this point we can write $1 - P_{n+1}(x) = x\Delta_{(n)}(x)$ for some finite polynomial $\Delta_{(n)}(x) \in \mathbb{Z}[x]$, whence, with $\Delta^\dagger(x) = \Delta^\dagger(x; k, n) = \Delta_{(n)}(x) \Delta^*(x) \in \mathbb{Z}[[x]]$,

$$\begin{aligned} F_{k+1}(x) &= C(x) + P_n(x) - C(x) P_{n+1}(x) + x^{k+1} \Delta^\dagger(x) \\ &= C(x) - x^{n+1} C^{n+2}(x) + x^{k+1} \Delta^\dagger(x) \end{aligned} \tag{4.46}$$

by Lemma 4.3. Now since the lead term of $C^{n+2}(x)$ is $c_0^{n+2} \neq 0$ then $x^{n+1} C^{n+2}(x)$ contains terms in $x^{n+1}, x^{n+2}, x^{n+3}, \ldots$, whilst $x^{k+1} \Delta^\dagger(x)$ has terms in $x^{k+1}, x^{k+2}, x^{k+3}$, *etc.* This means, since $k \in [1, n]$, that $F_{k+1}(x)$ agrees with $C(x)$ up to and including the term in x^k, upholding the inductive step for (4.46) reads, equivalently,

$$F_{k+1}(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + x^{k+1} \Delta_{k+1}(x), \tag{4.47}$$

for some $\Delta_{k+1}(x) = \Delta_{k+1}(x; n) = \Delta^\dagger(x) - x^{n-k} C^{n+2}(x) + c_{k+1} + c_{k+2} x + c_{k+3} x^2 + \cdots \in \mathbb{Z}[x]$ (note that there must be cancellation between a host of terms for $\Delta_{k+1}(x) \in \mathbb{Z}[x]$). $\square$

Computer output confirms, for fixed $n \geq 0$, that the set of polynomials $F_1(x), F_2(x), F_3(x), F_4(x), \ldots, F_{n+1}(x)$ serve to generate Catalan subsequences $\{c_0\}, \{c_0, c_1\}, \{c_0, c_1, c_2\}, \{c_0, c_1, c_2, c_3\}, \ldots, \{c_0, c_1, c_2, \ldots, c_n\}$, and we give instances of the finite polynomials $\Delta_i(x) = \Delta_i(x; n)$ associated with them (for $n = 0, \ldots, 5$), together with an accompanying remark:

$n = 0$: Scheme is $F_{i+1}(x) = 1$

Since $F_1(x) = c_0$ by default, then $\Delta_1(x; 0) = 0$.

$n = 1$: Scheme is $F_{i+1}(x) = 1 + xF_i(x)$

Clearly $\Delta_1(x; 1) = 0$, and $F_2(x) = c_0 + c_1 x \Rightarrow \Delta_2(x; 1) = 0$.

$n = 2$: Scheme is $F_{i+1}(x) = 1 - x + 2xF_i(x)$

Again we have $\Delta_1(x; 2) = 0$, with $F_2(x) = c_0 + c_1 x \Rightarrow \Delta_2(x; 2) = 0$, and $F_3(x) = c_0 + c_1 x + c_2 x^2 \Rightarrow \Delta_3(x; 2) = 0$.

$n = 3$: Scheme is $F_{i+1}(x) = 1 - 2x + x(3 - x)F_i(x)$

We have $\Delta_1(x; 3) = 0$, with $F_2(x) = c_0 + c_1 x - x^2 \Rightarrow \Delta_2(x; 3) = -1$, $F_3(x) = c_0 + c_1 x + c_2 x^2 - 4x^3 + x^4 \Rightarrow \Delta_3(x; 3) = -4 + x$, and $F_4(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 - 14x^4 + 7x^5 - x^6 \Rightarrow \Delta_4(x; 3) = -14 + 7x - x^2$.

$n = 4$: Scheme is $F_{i+1}(x) = 1 - 3x + x^2 + x(4 - 3x)F_i(x)$

We have $\Delta_1(x; 4) = 0$, $\Delta_2(x; 4) = -2$, $\Delta_3(x; 4) = -11 + 6x$, $\Delta_4(x; 4) = -50 + 57x - 18x^2$ and $\Delta_5(x; 4) = -215 + 378x - 243x^2 + 54x^3$.

$n = 5$: Scheme is $F_{i+1}(x) = 1 - 4x + 3x^2 + x(5 - 6x + x^2)F_i(x)$

We have $\Delta_1(x; 5) = 0$, $\Delta_2(x; 5) = -3 + x$, $\Delta_3(x; 5) = -20 + 24x - 9x^2 + x^3$, $\Delta_4(x; 5) = -111 + 242x - 209x^2 + 83x^3 - 15x^4 + x^5$, $\Delta_5(x; 5) = -583 + 1881x - 2608x^2 + \cdots + 178x^5 - 21x^6 + x^7$ and $\Delta_6(x; 5) = -2994 + 12917x - 24909x^2 + \cdots + 309x^7 - 27x^8 + x^9$.

Remark 4.4. Computations also reveal that, for $n \geq 0$,

$$F_{n+2}(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + (c_{n+1} - 1)x^{n+1} + \cdots ; \tag{4.48}$$

in other words, when the iterative process first fails the first incorrect term is that in x^{n+1}, whose coefficient is out by precisely 1. This is straightforward to deduce from Theorem 4.1 and its proof. The actual scheme gives, with $i = n+1$,

$$\begin{aligned} F_{n+2}(x) &= P_n(x) + [1 - P_{n+1}(x)]F_{n+1}(x) \\ &= \cdots \\ &= C(x) - x^{n+1}C^{n+2}(x) + x^{n+2}\Delta^\dagger(x) \end{aligned} \tag{4.49}$$

by a repeat proof argument (here, $\Delta^\dagger(x) = \Delta^\dagger(x; n+1, n)$). Thus, $F_{n+2}(x)$ agrees with $C(x)$ up to and including the term in x^n, and we have trivially that $[x^{n+1}]\{F_{n+2}(x)\} = [x^{n+1}]\{C(x)\} - [x^0]\{C^{n+2}(x)\} = c_{n+1} - 1$, holding for $n \geq 0$. This observation is readily interpreted in terms of Dyck paths on setting a path height restriction of $y = h \geq 1$, say, for whilst there are c_n paths of length $2n$ for $n = 0, \ldots, h$, the number of paths of length $2(h+1)$ is evidently $c_{h+1} - 1$ since there is just a single path (consisting of $h+1$ consecutive north-east steps followed by $h+1$ consecutive south-east steps, peaking at the point $(h+1, h+1)$) which would violate the restriction; in this scenario $P_h(x)/P_{h+1}(x) = \sum_{i=0}^{h} c_i x^i + (c_{h+1} - 1)x^{h+1} + \cdots$ (by way of an example, when $h = 6$ we anticipate that $P_6(x)/P_7(x) = \sum_{i=0}^{6} c_i x^i + (c_7 - 1)x^7 + \cdots$, which is indeed correct).

To conclude this chapter, it will be shown that Theorem 4.1 is, as required, consistent in the limit $n \to \infty$. By consideration of the scheme as stated, we need to show that

$$\lim_{n\to\infty} \left\{ \frac{P_n(x)}{P_{n+1}(x)} \right\} = C(x). \tag{4.50}$$

Whilst evidently true from the continued fraction interpretation of the Catalan polynomials, we can make a formal argument using the definition of $P_n(x)$ (4.6). This is achieved by first noting that the bounds $1 < 1 + r(x) < 1 + \sqrt{2}$ and $1 - \sqrt{2} < 1 - r(x) < 1$ are easily determined for the function $r(x) = \sqrt{1 - 4x}$ over the interval $|x| < 1/4$. Consider, therefore, the ratio

$$\begin{aligned} \frac{P_n(x)}{P_{n+1}(x)} &= 2\frac{(1 + r(x))^{n+1} - (1 - r(x))^{n+1}}{(1 + r(x))^{n+2} - (1 - r(x))^{n+2}} \\ &= \frac{2}{1 + r(x)} \frac{1 - q^{n+1}(x)}{1 - q^{n+2}(x)}, \end{aligned} \tag{4.51}$$

where $q(x) = (1 - r(x))/(1 + r(x))$ is monotonic with bounds $-1 < q(x) \leq 1$ over its full domain $x \in (-\infty, \frac{1}{4}]$ (we find $dq/dx = -2(dr/dx)/(1 + r(x))^2 = 4/[r(x)(1 + r(x))^2] > 0$ for $x < 1/4$ since $r(x) > 0$). Thus, over the interval $|x| < 1/4$ in particular, $|q(x)| < 1$ (more precisely, the bounds on $r(x)$ yield[4] $(1 - \sqrt{2})/(1 + \sqrt{2}) = -0.1716 < q(x) < 1$), so $q^n(x) \to 0$ as $n \to \infty$, and

$$\frac{P_n(x)}{P_{n+1}(x)} \to \frac{2}{1 + r(x)} = C(x). \tag{4.52}$$

[4] Or, alternatively, having established the increasing monotonicity of $q(x)$ over $(-\frac{1}{4}, \frac{1}{4})$, we can write down the bounds on $q(x)$ here as $q(-1/4) < q(x) < q(1/4)$.

Whilst this has been formally argued pointwise over the interval $[-\frac{1}{4}, \frac{1}{4}]$, it is true that the ratio $P_n(x)/P_{n+1}(x)$ converges to $C(x)$ in the sense that the resulting power series of the ratio contains ever more terms in agreement as n increases (see Theorem 5.6 later, and Section 6.5).

4.4 Summary and Concluding Remarks

In this chapter, it has been shown that, as a result of the Catalan polynomials' association with Chebyshev and Dickson polynomials, an extensive catalogue of their properties can be readily compiled through adaptation of results found in existing literature.

Although first discovered in the course of *this* study through the mechanism of term-matching, the existence of more direct means of generating the Catalan polynomials (*i.e.*, the closed form (4.6) and matrix formulae (4.12) and (4.18)) raises the possibility of their generalisation to accommodate other infinite sequences such as the Schröder and Motzkin numbers—a subject which will be further explored in due course.

Another association of interest is that the Catalan polynomials are found to form so-called Padé approximants, firstly in the context of the algebraic implementation of the type of numerical root-finding algorithms examined by Koepf (2006 and 2010), and secondly in relation to the continued fractions they comprise, as discussed briefly in Section 4.2.4 (Lemma 4.3 here encapsulating the required criterion for successive ratios of neighbouring pairs of polynomials to form such approximants). This topic will be discussed in the next chapter.

Chapter 5

A Class of Non-Linear Identities for Catalan Polynomials

5.1 Introduction

In Chapter 1, the concept of iterated generating functions was first introduced in the context of the algebraic adaptation of fixed-point iteration, an elementary numerical root-finding method. The recursive schemes generated by this method were found to deliver generating functions which converge to the o.g.f. of the associated infinite sequence at a linear rate.

Extending this subject, the first section of this chapter will focus on the construction of iterative schemes specifically for the Catalan sequence via the algebraic adaptation of a more complex suite of root-finding algorithms known as Householder's methods, in which the well-known quadratically convergent Newton-Raphson method features as the Householder method of lowest order, followed immediately by the cubically convergent Halley's method.

The fact that the iterative schemes generated by the adapted methods produce generating functions (in the form of ratios of polynomials) which converge to the Catalan sequence at an accelerated rate, reflecting their original numerical counterparts, is not surprising—however, of particular interest is that in each case, the ratios delivered by the schemes consist of pairs of Catalan polynomials. This enables us to utilise some of the recurrence properties exhibited by Catalan polynomials (detailed in Chapter 4 and revisited in the next section) to formulate a new class of non-linear identities.

Additionally, it is found that isolated Catalan polynomial ratios form so-called Padé approximants to the o.g.f. of the Catalan sequence, leading to a new proof of a stronger result which states that all such polynomial ratios are Padé approximants.

5.2 Summary of Recurrence Properties

Before starting our exploration of Householder algorithms, it is first convenient to collate and summarise for later reference the three recurrences found to be satisfied by the Catalan polynomials in Chapter 4.

It was previously noted that the two non-linear identities

$$0 = P_n^2(x) - P_{2n-1}(x) - x^2 P_{n-2}^2(x) \tag{5.1}$$

and

$$0 = P_n^2(x) - P_{2n}(x) - x P_{n-1}^2(x) \tag{5.2}$$

hold, each of which can be deduced from the other in combination with the basic linear recurrence

$$0 = xP_n(x) - P_{n+1}(x) + P_{n+2}(x); \qquad P_0(x) = P_1(x) = 1. \tag{5.3}$$

5.3 Newton-Type Iteration

5.3.1 Newton-Raphson Method

Given a sufficiently differentiable univariate function $f(z) = 0$, say, and an initial approximation to a root of the function, z_0, the Newton-Raphson numerical method (see Burden and Faires (2010, Section 2.3, p. 67) for a comprehensive derivation and methodology) utilises the iterative scheme

$$z_{r+1} = z_r - \frac{f(z_r)}{f'(z_r)}, \qquad r = 0, 1, 2, \ldots \tag{5.4}$$

to generate a series of values $z_1, z_2, \ldots$, where as r increases, z_r represents a progressively better approximation of the true root, z.

Now, as before, let $C(x)$ be the o.g.f. for the Catalan sequence, satisfying the quadratic

$$0 = 1 - C(x) + xC^2(x). \tag{5.5}$$

By replacing $C(x)$ with z and considering x to be a parameter to z, (5.5) can be re-written as

$$0 = 1 - z + xz^2 = f(z) = f(z; x), \tag{5.6}$$

where $f'(z) = \frac{\partial f}{\partial z}$. This gives an algebraic scheme

$$\begin{aligned} z_{r+1} &= z_r - \frac{1 - z_r + xz_r^2}{-1 + 2xz_r} \\ &= \frac{1 - xz_r^2}{1 - 2xz_r}, \end{aligned} \tag{5.7}$$

which we execute as

$$F_{r+1}(x) = \frac{1 - xF_r^2(x)}{1 - 2xF_r(x)}, \qquad r \geq 0, \tag{5.8}$$

subject to $F_0(x) = 1$ (the first term of the Catalan sequence). The functions resulting from the scheme are found to be:

$$\begin{aligned} F_1(x) &= \frac{1 - x}{1 - 2x} = \frac{P_2(x)}{P_3(x)}, \\ F_2(x) &= \frac{1 - 5x + 6x^2 - x^3}{1 - 6x + 10x^2 - 4x^3} = \frac{P_6(x)}{P_7(x)}, \\ F_3(x) &= \cdots = \frac{P_{14}(x)}{P_{15}(x)}, \\ F_4(x) &= \cdots = \frac{P_{30}(x)}{P_{31}(x)}, \end{aligned} \tag{5.9}$$

and so on, suggesting that the rth iterate $F_r(x)$ is related to the Catalan polynomials as

$$F_r(x) = \frac{P_{\alpha(r)-1}(x)}{P_{\alpha(r)}(x)}, \qquad r \geq 0, \tag{5.10}$$

where $\alpha(r) = 2^{r+1} - 1$ and reflects the quadratic convergence rate of the Newton-Raphson scheme (equation (5.10) is also consistent with the 0th iterate: $F_0(x) = 1 = P_0(x)/P_1(x) = P_{\alpha(0)-1}(x)/P_{\alpha(0)}(x)$). Noting that the function $\alpha(r)$ has the property that $\alpha(r+1) = 2\alpha(r)+1$, then as an assertion (5.10) would, if true, mean that the algorithm produces polynomial ratios $F_1(x), F_2(x), F_3(x), \ldots$, such that

$$\begin{aligned} F_{r+1}(x) &= \frac{1 - x[P_{\alpha(r)-1}(x)/P_{\alpha(r)}(x)]^2}{1 - 2x[P_{\alpha(r)-1}(x)/P_{\alpha(r)}(x)]} \\ &= \frac{P_{\alpha(r+1)-1}(x)}{P_{\alpha(r+1)}(x)} \\ &= \frac{P_{2\alpha(r)}(x)}{P_{2\alpha(r)+1}(x)}. \end{aligned} \tag{5.11}$$

Thus, in order to establish (5.10) we need to show that, for integer $n \geq 0$,

$$\frac{1 - x[P_{n-1}(x)/P_n(x)]^2}{1 - 2x[P_{n-1}(x)/P_n(x)]} = \frac{P_{2n}(x)}{P_{2n+1}(x)}. \tag{5.12}$$

This can be achieved by making use of identities (5.1) to (5.3), writing:

$$\begin{aligned}
\frac{1-x[P_{n-1}(x)/P_n(x)]^2}{1-2x[P_{n-1}(x)/P_n(x)]} &= \frac{P_n^2(x)-xP_{n-1}^2(x)}{P_n^2(x)-2xP_{n-1}(x)P_n(x)} \\
&= \frac{P_{2n}(x)}{P_n^2(x)-2xP_{n-1}(x)P_n(x)} && \text{(by (5.2))} \\
&= \frac{P_{2n}(x)}{P_n^2(x)-2xP_{n-1}(x)P_n(x)+x^2P_{n-1}^2(x)-x^2P_{n-1}^2(x)} \\
&= \frac{P_{2n}(x)}{[P_n(x)-xP_{n-1}(x)]^2-x^2P_{n-1}^2(x)} \\
&= \frac{P_{2n}(x)}{P_{n+1}^2(x)-x^2P_{n-1}^2(x)} && \text{(by (5.3))} \\
&= \frac{P_{2n}(x)}{P_{2n+1}(x)}, && (5.13)
\end{aligned}$$

by (5.1). Equating the denominator of the first and last r.h.s. expressions in the lines above yields the new identity

$$P_n(x)[P_n(x)-2xP_{n-1}(x)] = P_{2n+1}(x), \tag{5.14}$$

which is verifiable directly using the closed form

$$P_n(x) = \frac{1}{2^{n+1}}\frac{\left(1+\sqrt{1-4x}\right)^{n+1}-\left(1-\sqrt{1-4x}\right)^{n+1}}{\sqrt{1-4x}}. \tag{5.15}$$

Once again, the relative simplicity of the processes involved allows the straightforward implementation using computer algebra software of an algorithm designed to generate pairs of identities. Having empirically determined (5.10) as a plausible relation, the program would assume that it holds, equate the first and last r.h.s. ratios of (5.13) and declare the pair in this case as (5.2),(5.14).

We can, therefore, summarise the results as follows. By executing the classic Newton-Raphson algorithm symbolically, clearly identifiable ratios of Catalan polynomials emerge at each stage of the iterative process. Exhibiting the anticipated quadratic convergence rate, the scheme leads naturally to a pair of polynomial identities (5.2),(5.14) of corresponding degree 2. As will be demonstrated in the next section, this phenomenon is repeated when the order of the scheme is increased. Before moving on to the third-order (Halley's) method, however, two remarks can be made regarding this second-order algorithm.

Remark 5.1. From the initial value $F_0(x) = 1$ we find that for $r \geq 0$, $F_r(x)$ has, in Maclaurin series form, Catalan numbers as coefficients of its first $2^{r+1}-1$ terms, these being certain instances of the continued fractions associated with the Catalan sequence o.g.f. previously discussed in Section 4.2.4 in relation to Dyck paths. With a starting value $F_0(x)$ of either 0 or 2, only $2^r - 1$ first coefficients of $F_r(x)$ are Catalan numbers, both schemes operating identically whilst iterating exactly one step behind that detailed above. This is simple to explain, for imposing $F_1(x)$ (as opposed to $F_0(x)$) $= 1$ on (5.8) gives

$$F_1(x) = 1 = \frac{1 - xF_0^2(x)}{1 - 2xF_0(x)}, \tag{5.16}$$

rearrangement of which gives $0 = xF_0(x)[2 - F_0(x)]$ with solutions $F_0(x) = 0, 2$. Note that if $F_0(x)$ (constant) ≥ 3 then $F_r(x)$, as delivered by (5.8), contains *no* Catalan polynomial as either a numerator or denominator (although its series form continues to display $2^r - 1$ Catalan numbers as first coefficients, as expected).

Remark 5.2. We confirm that the Newton-Raphson iterations set up are consistent in the limit $r \to \infty$. If we regard (5.8) as a discretised version of the equation

$$F(x) = \frac{1 - xF^2(x)}{1 - 2xF(x)}, \tag{5.17}$$

then a rearrangement trivially recovers (5.5) with $F(x)$ replacing $C(x)$; in other words, $\lim_{r\to\infty}\{F_r(x)\} = C(x)$.

5.3.2 Halley's Method

Halley's method (a simple extension of Newton-Raphson) uses the recurrence scheme

$$z_{r+1} = z_r - \frac{2f(z_r)f'(z_r)}{2f'^2(z_r) - f(z_r)f''(z_r)}, \qquad r = 0, 1, 2, \ldots, \tag{5.18}$$

a repeat implementation of which, with $f(z)$ as in (5.6), gives rise to the particular algorithm

$$F_{r+1}(x) = \frac{1 - 3xF_r(x) + x^2F_r^3(x)}{1 - x - 3xF_r(x) + 3x^2F_r^2(x)}. \tag{5.19}$$

Using the initial value $F_0(x) = 1$ as before and executing the above scheme for $r \geq 0$, we find that

$$\begin{aligned} F_1(x) &= \frac{P_4(x)}{P_5(x)}, \\ F_2(x) &= \frac{P_{16}(x)}{P_{17}(x)}, \\ F_3(x) &= \frac{P_{52}(x)}{P_{53}(x)}, \\ F_4(x) &= \frac{P_{160}(x)}{P_{161}(x)}, \end{aligned} \tag{5.20}$$

etc., suggesting this time that

$$F_r(x) = \frac{P_{\beta(r)-1}(x)}{P_{\beta(r)}(x)}, \qquad r \geq 0, \tag{5.21}$$

where $\beta(r) = 2 \cdot 3^r - 1$ so that the expected cubic convergence of the scheme is captured. Since $\beta(r+1) = 3\beta(r) + 2$ then (5.21) is sufficient for the following to hold for integer $n \geq 0$:

$$\frac{1 - 3x[P_{n-1}(x)/P_n(x)] + x^2[P_{n-1}(x)/P_n(x)]^3}{1 - x - 3x[P_{n-1}(x)/P_n(x)] + 3x^2[P_{n-1}(x)/P_n(x)]^2} = \frac{P_{3n+1}(x)}{P_{3n+2}(x)}. \tag{5.22}$$

That is to say,

$$\begin{aligned} P_n^3(x) - 3xP_{n-1}(x)P_n^2(x) + x^2P_{n-1}^3(x) &= P_{3n+1}(x), \\ P_n(x)[(1-x)P_n^2(x) - 3xP_{n-1}(x)P_n(x) + 3x^2P_{n-1}^2(x)] &= P_{3n+2}(x), \end{aligned} \tag{5.23}$$

both degree 3 identities once again being verifiable by computer with no difficulty.

Remark 5.3. As in Remark 5.2, we can remove the subscripts throughout (5.19), then re-write (and this time factorise) the resulting equation to arrive at $0 = [xF^2(x) - F(x) + 1][1 - 2xF(x)]$. Since the solution $F(x) = \frac{1}{2x}$ cannot be expanded as a power series in x, we take the other solution as the correct one which confirms that Halley's scheme, applied in this symbolic context, holds in the limit $r \to \infty$.

5.4 Higher-Order Results

Recalling that the rate of convergence exhibited by the second-order Newton-Raphson method is characterised by the function $\alpha(r) = 2^{r+1} - 1 = 2 \cdot 2^r - 1$, and similarly for Halley's method, $\beta(r) = 2 \cdot 3^r - 1$, a potential pattern is already discernable, which, by extension, permits a fully automated approach to the generation of identities for Catalan polynomials. In this section, we present a selection of results for higher-order schemes based on a suite of numeric algorithms described by an elegant and compact formulation.

5.4.1 Householder Iteration

As previously mentioned, the Newton-Raphson method and Halley's method are both to be found as members of a general family of iterative algorithms due to Householder (1970, (14), p. 169), these being specialisations of a method earlier devised by Schröder to find roots of non-linear univariate functions. For a $p + 2$ times continuously differentiable function $f(z)$, let

$z = a$ be a zero of f (but not df/dz). Then, given an initial value z_0 sufficiently close to a, successive iterates z_r, z_{r+1} delivered by the scheme

$$z_{r+1} = z_r + (p+1) \frac{\left. \frac{d^p}{dz^p} \left\{ \frac{1}{f(z)} \right\} \right|_{z=z_r}}{\left. \frac{d^{p+1}}{dz^{p+1}} \left\{ \frac{1}{f(z)} \right\} \right|_{z=z_r}} \tag{5.24}$$

will, for some constant $K > 0$, satisfy the inequality $|z_{r+1} - a| \leq K|z_r - a|^{p+2}$ in a neighbourhood of a, meaning that the recursive process will converge to the zero $z = a$. We call (5.24) the Householder scheme of $O(p)$, with an order $p+2$ convergence rate, noting that the cases $p = 0, 1$ recover (5.4) and (5.18), respectively.

5.4.2 Householder Quartic Scheme

Setting $p = 2$ in (5.24) results in the scheme

$$z_{r+1} = z_r - \frac{3f(z_r)[2f'^2(z_r) - f(z_r)f''(z_r)]}{6f'^3(z_r) + f^2(z_r)f'''(z_r) - 6f(z_r)f'(z_r)f''(z_r)}. \tag{5.25}$$

Subsequent computations affirm the conjecture that this quartic algorithm, applied to (5.6) (with z_r replaced by $F_r(x)$ again), generates polynomial ratios obeying the rule

$$F_r(x) = \frac{P_{\gamma(r)-1}(x)}{P_{\gamma(r)}(x)}, \qquad r \geq 0, \tag{5.26}$$

with $\gamma(r) = 2 \cdot 4^r - 1$. The same line of reasoning as made in the other cases above then leads to two further identities, of commensurate degree 4, thus:

$$\begin{aligned} &(1-x)P_n^4(x) - 4xP_{n-1}(x)P_n^3(x) \\ &\qquad + 6x^2P_{n-1}^2(x)P_n^2(x) - x^3P_{n-1}^4(x) = P_{4n+2}(x), \\ &P_n(x)[P_n(x) - 2xP_{n-1}(x)] \times \\ &\quad [(1-2x)P_n^2(x) - 2xP_{n-1}(x)P_n(x) + 2x^2P_{n-1}^2(x)] \\ &\qquad\qquad = P_{4n+3}(x). \end{aligned} \tag{5.27}$$

5.4.3 Further Identities

The degree 5 identities resulting from the $p = 3$ instance of (5.24) are

$$\begin{aligned}
&(1-2x)P_n^5(x) - 5x(1-x)P_{n-1}(x)P_n^4(x) \\
&\qquad + 10x^2P_{n-1}^2(x)P_n^3(x) \\
&\qquad - 10x^3P_{n-1}^3(x)P_n^2(x) + x^4P_{n-1}^5(x) = P_{5n+3}(x), \\
&P_n(x)[(1-3x+x^2)P_n^4(x) - 5x(1-2x)P_{n-1}(x)P_n^3(x) \\
&\qquad + 10x^2(1-x)P_{n-1}^2(x)P_n^2(x) \\
&\qquad - 10x^3P_{n-1}^3(x)P_n(x) + 5x^4P_{n-1}^4(x)] = P_{5n+4}(x).
\end{aligned} \tag{5.28}$$

The degree 6 identities yielded by the $p = 4$ instance of (5.24) are

$$\begin{aligned}
&(1-3x+x^2)P_n^6(x) - 6x(1-2x)P_{n-1}(x)P_n^5(x) \\
&\qquad + 15x^2(1-x)P_{n-1}^2(x)P_n^4(x) \\
&\qquad - 20x^3P_{n-1}^3(x)P_n^3(x) \\
&\qquad + 15x^4P_{n-1}^4(x)P_n^2(x) - x^5P_{n-1}^6(x) = P_{6n+4}(x), \\
&P_n(x)[P_n(x) - 2xP_{n-1}(x)] \times \\
&\quad [(1-x)(1-3x)P_n^4(x) - 2x(2-5x)P_{n-1}(x)P_n^3(x) \\
&\qquad + x^2(7-10x)P_{n-1}^2(x)P_n^2(x) \\
&\qquad - 6x^3P_{n-1}^3(x)P_n(x) + 3x^4P_{n-1}^4(x)] = P_{6n+5}(x).
\end{aligned} \tag{5.29}$$

5.5 Padé Approximants

An interesting observation is that the ratios of Catalan polynomials resulting from the various Householder schemes comprise "Padé approximants" to the Catalan sequence o.g.f. $C(x)$.

The two primary applications of Padé approximants are: (i) in providing efficient rational approximations to a host of functions (including many so-called "special" mathematical functions), and (ii) in yielding quantitative information about functions known only from their power series form and qualitative behaviour. It is in the first context that our interest lies. Although variations exist (see, for example, von zur Gathen and Gerhard (1999, (22), p. 112)), there seems to be a general consensus as to the definition of such an approximant which we state formally here (with reference to Baker and Graves-Morris (1996, Section 1.1, p. 1) and

Gil *et al.* (2007, Section 9.2, p. 276)):

Definition 5.4. *Given a function $f(x)$ and integers $m, p \geq 0$, the order (m, p) Padé approximant of $f(x)$ is the rational function*

$$\frac{u(x)}{v(x)} = \frac{u_0 + u_1 x + u_2 x^2 + \cdots + u_m x^m}{1 + v_1 x + v_2 x^2 + \cdots + v_p x^p}$$

which, when expanded as a Maclaurin series, has cancellation strictly in its first $m + p + 1$ terms with the corresponding series form of $f(x)$. In other words, for some $\Delta(x) \in \mathbb{Z}[[x]]$ with non-zero lead term,

$$u(x) - f(x)v(x) = O(x^{m+p+1}) = x^{m+p+1}\Delta(x).$$

In an informative summary paper by Gragg (1972, Theorem 3.1, p. 10), a result due to Frobenius—recognised for the original concept of the so-called Padé Table and his development of basic aspects of the theory—is noted which states that there always exists a reduced order (m, p) approximant to the series form of $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$ (for which $u(x)$ and $v(x)$ are relatively prime, with $u(0) = f_0$, $v(0) = 1$).

5.5.1 Padé Approximants via Newton-Raphson Method

We write an nth approximant of $C(x)$ as the rational $p_n(x) = u_n(x)/v_n(x)$, $n \geq 1$. Let the initial (first) approximant be $p_1(x) = c_0 = 1$, so that $u_1(x) = v_1(x) = 1$. It is immediate from (5.9) that the Newton-Raphson scheme yields successive approximants with $u_2(x) = P_2(x) = 1 - x$, $v_2(x) = P_3(x) = 1 - 2x$, then $u_3(x) = P_6(x)$, $v_3(x) = P_7(x)$, and so on, where $\deg\{u_n(x)\} = \deg\{v_n(x)\} = 2^{n-1} - 1$. Now from the algorithm itself we also have

$$\begin{aligned}\frac{u_{n+1}(x)}{v_{n+1}(x)} = p_{n+1}(x) &= \frac{1 - xp_n^2(x)}{1 - 2xp_n(x)} \\ &= \frac{1 - x[u_n(x)/v_n(x)]^2}{1 - 2x[u_n(x)/v_n(x)]} \\ &= \frac{v_n^2(x) - xu_n^2(x)}{v_n(x)[v_n(x) - 2xu_n(x)]},\end{aligned} \tag{5.30}$$

giving relations

$$\begin{aligned}u_{n+1}(x) &= v_n^2(x) - xu_n^2(x), \\ v_{n+1}(x) &= v_n(x)[v_n(x) - 2xu_n(x)],\end{aligned} \tag{5.31}$$

so that in turn

$$\begin{aligned}
u_{n+1}(x) - C(x)v_{n+1}(x) &= v_n^2(x) - xu_n^2(x) - C(x)v_n(x)[v_n(x) - 2xu_n(x)] \\
&= [1 - C(x)]v_n^2(x) - xu_n^2(x) + 2xC(x)u_n(x)v_n(x) \\
&= [-xC^2(x)]v_n^2(x) - xu_n^2(x) + 2xC(x)u_n(x)v_n(x) \qquad \text{(by (5.5))} \\
&= -x[u_n(x) - C(x)v_n(x)]^2. \qquad (5.32)
\end{aligned}$$

If $u_1(x) - C(x)v_1(x)$ is of general order x^d, say (that is, $O(x^d)$ w.r.t. its lead term), it is straightforward to see, via (5.32), the order of $u_n(x) - C(x)v_n(x)$ is $x^{(d+1)2^{n-1}-1}$. Thus, since by (5.5) $u_1(x) - C(x)v_1(x) = 1 - C(x) = -xC^2(x)$, then $d = 1$ and $u_n(x) - C(x)v_n(x) = O(x^{2^n-1})$, whence we conclude that the general rational $p_n(x)$ is in fact precisely an order $(2^{n-1} - 1, 2^{n-1} - 1)$ Padé approximant of $C(x)$ because $2^n - 1 = (2^{n-1} - 1) + (2^{n-1} - 1) + 1 = \deg\{u_n(x)\} + \deg\{v_n(x)\} + 1$.

This result is perhaps not that surprising for the Newton-Raphson root-finding scheme is known to be particularly efficient (see, for example, Burden and Faires (2010, Section 2.3, p. 67)) and Padé approximants offer, in some sense, the best (local) approximation of a given function. In view of this it is even less surprising to see the phenomenon repeated as a consequence of executing algorithms of Halley and higher order.

5.5.2 Padé Approximants via Halley's Method

The correlation between equations (5.31) and the identities (5.2),(5.14) is clear. Adopting the same notation as above we are, therefore, able to write down the recurrences

$$\begin{aligned}
u_{n+1}(x) &= v_n^3(x) - 3xu_n(x)v_n^2(x) + x^2u_n^3(x), \\
v_{n+1}(x) &= v_n(x)[(1 - x)v_n^2(x) - 3xu_n(x)v_n(x) + 3x^2u_n^2(x)], \qquad (5.33)
\end{aligned}$$

directly from (5.23), noting that, with $u_1(x) = v_1(x) = 1$ again, the Halley scheme yields an nth approximant $p_n(x) = u_n(x)/v_n(x)$ (comprising, as we have seen, ratios of certain neighbouring pairs of Catalan polynomials) for which $\deg\{u_n(x)\} = \deg\{v_n(x)\} = 3^{n-1} - 1$. This time we have

$$\begin{aligned}
&u_{n+1}(x) - C(x)v_{n+1}(x) \\
&\quad = x^2u_n^3(x) - 3x^2C(x)u_n^2(x)v_n(x) \\
&\qquad - 3x[1 - C(x)]u_n(x)v_n^2(x) + [1 - (1 - x)C(x)]v_n^3(x), \qquad (5.34)
\end{aligned}$$

and, once more applying (5.5) to replace $1 - C(x)$ with $-xC^2(x)$ in the third r.h.s. term, and similarly (in regard to the last term) writing $1 - (1 - x)C(x) = 1 - C(x) + xC(x) = -xC^2(x) + xC(x) = xC(x)[1 - C(x)] = xC(x)[-xC^2(x)] = -x^2C^3(x)$, (5.34) contracts to

$$\begin{aligned} u_{n+1}(x) - C(x)v_{n+1}(x) \\ &= x^2u_n^3(x) - 3x^2C(x)u_n^2(x)v_n(x) \\ &\qquad + 3x^2C^2(x)u_n(x)v_n^2(x) - x^2C^3(x)v_n^3(x) \\ &= x^2[u_n(x) - C(x)v_n(x)]^3. \end{aligned} \tag{5.35}$$

We readily see from this result that $u_n(x) - C(x)v_n(x)$ is $O(x^{2\cdot 3^{n-1}-1})$, and (since $2 \cdot 3^{n-1} - 1 = (3^{n-1} - 1) + (3^{n-1} - 1) + 1 = \deg\{u_n(x)\} + \deg\{v_n(x)\} + 1$) establish that in this case for $n \geq 1$ the general rational $p_n(x)$ is an order $(3^{n-1} - 1, 3^{n-1} - 1)$ Padé approximant of $C(x)$.

Although not detailed here, higher-order schemes (Householder quartic and beyond) have been implemented computationally and found to output Padé approximants commensurate with their order. This leads to the following conjecture which as yet remains unproven (although supporting results are presented in the next chapter):

Conjecture 5.5. *From an initial first approximant* $u_1(x)/v_1(x) = 1$ *of the Catalan o.g.f.* $C(x)$*, the* $O(p)$ *Householder scheme delivers a series of order* $((p + 2)^{n-1} - 1, (p + 2)^{n-1} - 1)$ *Padé approximants* $u_n(x)/v_n(x)$ *of* $C(x)$ $(n = 2, 3, 4, \ldots)$.

The degrees of $u_n(x), v_n(x)$ are the same by virtue of the fact that, based on the specific cases examined computationally, we anticipate that the schemes output rationals $P_{\Omega_p(r)-1}(x)/P_{\Omega_p(r)}(x)$ for the rth iterate $(r \geq 0)$, where $\Omega_p(r) = 2(p + 2)^r - 1$ ($\Omega_0(r), \Omega_1(r), \Omega_2(r)$ corresponding to $\alpha(r), \beta(r), \gamma(r)$, resp.); we would then have $\deg\{P_{\Omega_p(r)-1}(x)\} = \lfloor \frac{1}{2}(\Omega_p(r) - 1) \rfloor = (p + 2)^r - 1 = \lfloor \frac{1}{2}\Omega_p(r) \rfloor = \deg\{P_{\Omega_p(r)}(x)\}$.

In fact, all Catalan polynomial ratios $P_0(x)/P_1(x), P_1(x)/P_2(x), P_2(x)/P_3(x)$, *etc.*, are Padé approximants, and we explore this idea further in Chapter 6 in relation to additional polynomials associated with other sequences (namely, Schröder and Motzkin).

5.5.3 Theorem

Having observed the emergence of Padé approximants from the Newton-type schemes described generally by (5.24), we are in a position to theorise the order of each successive approximation within any given Householder algorithm. It has long been established (as noted by Gragg (1972)) that certain sequences of rational functions within a Padé Table[1] can correspond to those convergents of a particular continued fraction, and it is indeed the case that ratios of Catalan polynomials fulfil this role in the context of approximating the Catalan sequence o.g.f.[2] Accordingly, drawing on the previously formulated result of Theorem 4.1 (with associated statement (4.41)), we can construct a first principles proof of the following statement:

Theorem 5.6. *The ratio $P_n(x)/P_{n+1}(x)$ of Catalan polynomials is, for $n \geq 0$, an order $(\lfloor \frac{1}{2}n \rfloor, \lfloor \frac{1}{2}(n+1) \rfloor)$ Padé approximant of the Catalan sequence o.g.f. $C(x)$.*

Proof. Noting that each denominator function $P_1(x), P_2(x), P_3(x), \ldots,$ has a (non-zero) lead term of 1 (as is required), we need to show that, for $n \geq 0$, there exists a power series $\Delta_{(n)}(x) \in \mathbb{Z}[[x]]$ for which

$$\begin{aligned} P_n(x) - C(x)P_{n+1}(x) &= x^{\lfloor \frac{1}{2}n \rfloor + \lfloor \frac{1}{2}(n+1) \rfloor + 1}\Delta_{(n)}(x) \\ &= x^{n+1}\Delta_{(n)}(x). \end{aligned} \tag{5.36}$$

Consider Theorem 4.1, and the first-order scheme given. Rather than appealing to the form of the general Catalan polynomial $P_n(x)$ as a binomial coefficient sum (4.1), we will instead use our knowledge of this result to construct an appropriate argument in a succinct manner. For $n \geq 0$, $i = 1, \ldots, n+1$, we know that (see (4.41))

$$F_i(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{i-1}x^{i-1} + x^i\Delta_i(x) \tag{5.37}$$

for some $\Delta_i(x) = \Delta_i(x; n) \in \mathbb{Z}[x]$, and so for $i = n+1$ in particular,

$$F_{n+1}(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + x^{n+1}\delta_1(x), \tag{5.38}$$

[1]A table of order (α, β) Padé approximants in which $\alpha, \beta \geq 0$ vary. Mechanisms for constructing such tables are well documented.

[2]Note that because ratios of Catalan polynomials form sequential continued fraction approximations of $C(x)$ which are readily interpreted combinatorially in terms of Dyck paths (previously discussed in Section 4.2.4), it is immediate that $\lim_{n\to\infty} \{P_n(x)/P_{n+1}(x)\} = C(x)$ (a formal argument having been made in the previous chapter, (4.50) to (4.52)).

where $\delta_1(x) = \delta_1(x;n) = \Delta_{n+1}(x;n) \in \mathbb{Z}[x]$. Furthermore, by the nature of the linear scheme (adding, and subsequently preserving, but one new term per iteration[3]) it follows that

$$F_{n+2}(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + x^{n+1}\delta_2(x), \tag{5.39}$$

for which $\delta_2(x) = \delta_2(x;n) \in \mathbb{Z}[x]$ must necessarily possess the same lead term as $\delta_1(x)$. Writing $L(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$, we thus have

$$\begin{aligned} F_{n+1}(x) &= L(x) + x^{n+1}\delta_1(x), \\ F_{n+2}(x) &= L(x) + x^{n+1}\delta_2(x). \end{aligned} \tag{5.40}$$

Now, directly from the scheme itself,

$$F_{n+2}(x) = P_n(x) + [1 - P_{n+1}(x)]F_{n+1}(x), \tag{5.41}$$

into which substitution of equations (5.40) yields

$$x^{n+1}\delta_3(x) = P_n(x) - P_{n+1}(x)[L(x) + x^{n+1}\delta_1(x)] \tag{5.42}$$

after a little rearrangement ($\delta_3(x) = \delta_3(x;n) = \delta_2(x) - \delta_1(x) \in \mathbb{Z}[x]$, and has no constant (lead) term). Of course, writing $\delta_4(x) = \delta_4(x;n) = c_{n+1} + c_{n+2}x + c_{n+3}x^2 + \cdots \in \mathbb{Z}[[x]]$, we can express $C(x)$ as

$$C(x) = L(x) + x^{n+1}\delta_4(x), \tag{5.43}$$

whence (5.42) reads

$$x^{n+1}\delta_3(x) = P_n(x) - P_{n+1}(x)[C(x) + x^{n+1}\delta_5(x)], \tag{5.44}$$

with $\delta_5(x) = \delta_5(x;n) = \delta_1(x) - \delta_4(x) \in \mathbb{Z}[[x]]$. Finally, defining $\Delta_{(n)}(x) = \delta_3(x) + P_{n+1}(x)\delta_5(x) \in \mathbb{Z}[[x]]$, this can be written as

$$P_n(x) - C(x)P_{n+1}(x) = x^{n+1}\Delta_{(n)}(x), \tag{5.45}$$

as required. □

[3]This observation is dealt with at the start of Section 4.3.

Some quick computations confirm Theorem 5.6, where we see that

$$\begin{aligned}
\Delta_{(0)}(x) &= -(1 + 2x + 5x^2 + 14x^3 + 42x^4 + \cdots), \\
\Delta_{(1)}(x) &= -(1 + 3x + 9x^2 + 28x^3 + 90x^4 + \cdots), \\
\Delta_{(2)}(x) &= -(1 + 4x + 14x^2 + 48x^3 + 165x^4 + \cdots), \\
\Delta_{(3)}(x) &= -(1 + 5x + 20x^2 + 75x^3 + 275x^4 + \cdots), \\
\Delta_{(4)}(x) &= -(1 + 6x + 27x^2 + 110x^3 + 429x^4 + \cdots), \\
\Delta_{(5)}(x) &= -(1 + 7x + 35x^2 + 154x^3 + 637x^4 + \cdots), \\
\Delta_{(6)}(x) &= -(1 + 8x + 44x^2 + 208x^3 + 910x^4 + \cdots), \qquad (5.46)
\end{aligned}$$

etc.

5.5.4 Remarks

To conclude this section on Padé approximants, we make a few pertinent remarks, the first of which concerns a particular type of continued fraction known as a c-continued fraction (or simply C-fraction). In a publication by Perron (1977, Theorem 3.2, p. 108), a *general* association between C-fractions and power series representations of a function is stated, of which we have here but one instance. This theorem is quoted thus:[4]

> *For every c-continued fraction there is a power series* $\mathcal{B}_0(x) = 1 + d_1x + d_2x^2 + d_3x^3 + \cdots$ *which, in the case of an infinite continued fraction, is defined uniquely in such a way that the successive ratios (continuands)* $A_\lambda(x)/B_\lambda(x)$ *have Taylor series which with increasing* λ *have more and more coefficients in common with* $\mathcal{B}_0(x)$*, at least up to* x^λ*. If the continued fraction is finite with depth* n*, then the latter is true for the Taylor series of* $A_n(x)/B_n(x)$ *for every* $\lambda \leq n$*.*♣
>
> ♣*Obviously the initial term 1 of the continued fraction and series could be replaced by any other constant* c_0*. However, for uniformity it is generally convenient to set* $c_0 = 1$.

[4] With thanks to Prof. Dr. Wolfram Koepf for translation from the original German.

As alluded to in Footnote 2, p. 63, we have already seen in (4.20) that the respective 0th, 1st, 2nd, 3rd, ..., continued fractions for $C(x)$ are

$$\begin{aligned} \frac{P_0(x)}{P_1(x)} &= \frac{1}{1} = c_0, \\ \frac{P_1(x)}{P_2(x)} &= \frac{1}{1-x} = c_0 + c_1 x + \cdots, \\ \frac{P_2(x)}{P_3(x)} &= \frac{1}{1-\frac{x}{1-x}} = c_0 + c_1 x + c_2 x^2 + \cdots, \\ \frac{P_3(x)}{P_4(x)} &= \frac{1}{1-\frac{x}{1-\frac{x}{1-x}}} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots, \end{aligned} \tag{5.47}$$

etc. Meanwhile, a C-fraction has a clear association with a formal complex power series at the origin and is defined in the Handbook of Cuyt *et al.* (2008, (2.3.1), p. 35) as having general form

$$\begin{aligned} CF(z) &= b_0 + \mathrm{K}_{m=1}^{\infty}\left(\frac{a_m z^{\alpha_m}}{1}\right) \\ &= b_0 + \frac{a_1 z^{\alpha_1}}{1+\frac{a_2 z^{\alpha_2}}{1+\frac{a_3 z^{\alpha_3}}{1+\cdots}}} \end{aligned} \tag{5.48}$$

for $b_0 \in \mathbb{C}$, $a_m \in \mathbb{C} \setminus \{0\}$, $\alpha_m \in \mathbb{N}$ (K in (5.48) denoting an infinite fraction, being taken from the German word "Kettenbruch", and analogous to the sum Σ). Thus, our series of approximants (5.47) are described by $\frac{1}{x}CF(x)$ on setting $b_0 = 0$, $a_1 = 1$, $a_2 = a_3 = a_4 = \cdots = -1$, $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = 1$, where in fact $CF(x)$ here is what is known as a *regular* C-fraction (because each term in x appearing is a linear one). The correspondence between the approximants[5] and the Catalan numbers they contain in series form (5.47) is then given by Cuyt *et al.* (2008, Theorems 2.4.1 (A),(B), p. 39), and a later connection with regular C-fractions and specifically Padé approximants is covered by a couple of results in Section 4.3 thereof, pp. 64–65.

Remark 5.7. It is worth mentioning that in fact the series $\Delta_{(n)}(x)$ has a clearly identifiable closed form $\Delta_{(n)}(x) = -C^{n+2}(x)$ $(= -\sum_{i=0}^{\infty} \frac{n+2}{2i+n+2}\binom{2i+n+2}{i} x^i$ as a binomial coefficient series). This follows trivially from Lemma 4.3 (originally due to Lang (2000)), which is used to prove Theorem 4.1. Although the said lemma establishes (5.36) immediately, the proof provided here is felt to be instructive in its own right.

[5] Considered in Cuyt *et al.* (2008) to be the 1st - 4th approximants, which appears to be a standard notation.

5.6 Summary and Concluding Remarks

In this chapter, it has been observed that the algebraic adaptations of the Householder suite of algorithms deliver ratios of Catalan polynomials which in all cases are found to be Padé approximants. Through an automated process based on the recurrence properties known to hold for Catalan polynomials, these ratios can be equated to form identities which are believed to be quite unusual in terms of their structure and indigenous levels of non-linearality.

A few additional observations can now be made regarding these results. Firstly, although the algebraicised Householder schemes developed so far have focused exclusively on the Catalan sequence, thereby producing the immediately recognisable Catalan polynomial ratios, subsequent analysis has shown that Householder schemes specialised for other sequences, including the Schröder and Motzkin sequences, yield polynomials which are structurally and functionally similar to the Catalan polynomials. Furthermore, it is apparent that the processes involved can, to some extent, be generalised. This subject will be explored in detail in the next chapter.

Secondly, it is worth noting that where the symbolic adaptation of numerical root-finding methods is concerned, no higher-order algorithms other than the class of Householder methods have been investigated or implemented in the course of this work, although a considerable number are known to exist. Therefore, although believed to be unusual, it is not known if the phenomenon of Catalan (or generalised) polynomial ratios resulting from the execution of such schemes is indeed commonplace.

Finally, another result can be detailed concerning the Catalan polynomial identities. Noting that the identities are readily expressible in terms of the aforementioned Chebyshev and (lesser-known) Dickson polynomials, we can now also describe a new suite of identities for these classes of polynomials. Since, for $n \geq 0$, the general Chebyshev polynomial of the second kind $U_n(x)$ is related to $P_n(x)$ as $U_n(x) = (2x)^n P_n\left(\frac{1}{4x^2}\right)$ (see (4.7)), then, for instance, equations (5.2),(5.14) become

$$[U_n(x) - U_{n-1}(x)][U_n(x) + U_{n-1}(x)] = U_{2n}(x),$$
$$2U_n(x)[xU_n(x) - U_{n-1}(x)] = U_{2n+1}(x), \tag{5.49}$$

whilst (5.23) yields

$$
\begin{aligned}
2xU_n^3(x) - 3U_{n-1}(x)U_n^2(x) + U_{n-1}^3(x) &= U_{3n+1}(x), \\
U_n(x)[(2x-1)(2x+1)U_n^2(x) - 6xU_{n-1}(x)U_n(x) + 3U_{n-1}^2(x)] &= U_{3n+2}(x),
\end{aligned} \tag{5.50}
$$

and so on.

Remark 5.8. It is an immediate consequence of the two identities (5.49) that the general Chebyshev polynomial $U_n(x)$ is reducible over the ring $\mathbb{Z}[x]$ for $n \geq 2$, a fact noted by Rayes and Trevisan (2006, Corollary 2, p. 509) before the first identity, for $U_{2n}(x)$, is used to formulate a primality criterion (Corollary 3, p. 510). Other than in the text of Rivlin (1990, p. 229) (who gives the formula for $U_{2n+1}(x)$ in a slightly different form and also remarks on the reducibility of $U_n(x)$), the equations (5.49) are not to be found explicitly in the literature. It is interesting to see that, as in (5.49),(5.50) shown here, Chebyshev identities are always available as an "even/odd" pair from their Catalan polynomial counterpart pair.

Chapter 6

Householder-Derived Identities for Generalised Polynomials

6.1 Introduction

In the previous chapter, a result was described whereby the (specialised) symbolic adaptation of a suite of numerical root-finding methods due to Householder led to the formation of ratios of Catalan polynomials, combinations of which were shown to satisfy non-linear identities. Although derivation of low-level results was performed manually, it was found that higher-order results could, without too much difficulty, be obtained and verified by computer automation of the processes involved.

A natural question arising from these results, and briefly mentioned in the concluding remarks to the discussion, was whether specialisations (or indeed, a generalisation) of the Householder algorithm could be developed for other integer sequences in which similar phenomena might be observed. By conducting a thorough analysis of the Householder methodology, the results of which will be presented in this chapter, this question will be answered (in the affirmative).

A second observation which also will be expanded upon was that the ratios produced constitute Padé approximants to the o.g.f. of the Catalan sequence, again raising the possibility that other specialisations might also produce Padé approximants to the o.g.fs of their associated sequences. However, as we will see in due course, this does not always prove to be the case.

6.2 The Householder Function and a Generalised Polynomial

6.2.1 Preliminary Results

To begin, some new results concerning a generalisation of the Householder function will be derived. Recounting the numerical method for reference, we define (suppressing the z dependency for convenience) the function

$$\mathcal{H}_p(f) = \mathcal{H}_p(f(z); z) = z + (p+1)\frac{\frac{d^p}{dz^p}\left\{\frac{1}{f(z)}\right\}}{\frac{d^{p+1}}{dz^{p+1}}\left\{\frac{1}{f(z)}\right\}} \tag{6.1}$$

in terms of a $p+1$ continuously differentiable function $f(z)$, and let $z = a$ be a zero of f (but not df/dz). Then, given an initial value z_0 sufficiently close to a, successive iterates z_r, z_{r+1} delivered by the scheme

$$z_{r+1} = \mathcal{H}_p(f)|_{z=z_r} \tag{6.2}$$

will, for some constant $K > 0$, satisfy the inequality $|z_{r+1}-a| \leq K|z_r-a|^{p+2}$ in a neighbourhood of a, meaning that the recursive process will converge to the zero $z = a$. We call (6.2) the Householder scheme of $O(p)$, with an order $p+2$ convergence rate, noting that the respective cases $p = 0, 1$ recover the well-known quadratically convergent Newton-Raphson and cubically convergent Halley root-finding algorithms for which

$$\begin{aligned}\mathcal{H}_0(f) &= z - \frac{f(z)}{f'(z)},\\ \mathcal{H}_1(f) &= z - \frac{2f(z)f'(z)}{2f'^2(z) - f(z)f''(z)};\end{aligned} \tag{6.3}$$

the next two schemes are based on the functions

$$\begin{aligned}\mathcal{H}_2(f) &= z - \frac{3f(z)[2f'^2(z) - f(z)f''(z)]}{6f'^3(z) + f^2(z)f'''(z) - 6f(z)f'(z)f''(z)},\\ \mathcal{H}_3(f) &= z - \frac{4f(z)[6f'^3(z) - 6f(z)f'(z)f''(z) + f^2(z)f'''(z)]}{24f'^4(z) - 36f(z)f'^2(z)f''(z) + 6f^2(z)f''^2(z) + 8f^2(z)f'(z)f'''(z) - f^3(z)f^{(4)}(z)}.\end{aligned} \tag{6.4}$$

When implemented algebraically for the equation governing the o.g.f. of the Catalan sequence

$$0 = 1 - G(x) + xG^2(x), \tag{6.5}$$

the $O(p)$ Householder scheme—using

$$f(z) = f(z; x) = 1 - z + xz^2 \tag{6.6}$$

pertaining to (6.5)—will output from some initial value (degree zero polynomial) ratios of polynomials whose Maclaurin series expansions successively agree more closely with $G(x)$. Moreover, when the initial value is $1 = c_0$ (or, in this case, equivalently 0 or 2 (by Remark 5.1)) these ratios are found to be isolated ratios of neighbouring (*i.e.*, adjacent) Catalan polynomials whose separation reflects the exponential convergence rate, base $p+2$, commensurate with the $O(p)$ algorithm.

It is evident from (6.1) that in order to construct a generalised Householder scheme, an expression for the nth derivative of the reciprocal function $1/f(z)$ is a prerequisite. The manner in which this is achieved is somewhat unusual, the general Catalan polynomial $P_n(x)$ being included explicitly in the resulting expression.

Theorem 6.1. *Let $f(z) = f(z;x)$ be a quadratic in z of the form $f(z) = A(x)z^2+B(x)z+C(x)$, where $A(x), B(x), C(x) \in \mathbb{Z}[x]$. Then, for $n \geq 0$,*

$$\frac{d^n}{dz^n}\left\{\frac{1}{f(z)}\right\} = (-1)^n n! \frac{f'^n(z)}{f^{n+1}(z)} P_n\left(\frac{A(x)f(z)}{f'^2(z)}\right).$$

Proof. This is done inductively. For $n = 0$ both sides are $1/f(z)$. Suppose the result holds for some $n = k \geq 0$, and consider

$$\begin{aligned}
\frac{d^{k+1}}{dz^{k+1}}\left\{\frac{1}{f(z)}\right\} &= \frac{d}{dz}\frac{d^k}{dz^k}\left\{\frac{1}{f(z)}\right\} \\
&= \frac{d}{dz}\left\{(-1)^k k! \frac{f'^k(z)}{f^{k+1}(z)} P_k\left(\frac{A(x)f(z)}{f'^2(z)}\right)\right\} \qquad \text{(by assumption)} \\
&= (-1)^k k! \frac{d}{dz}\left\{\frac{f'^k(z)}{f^{k+1}(z)} \sum_{i=0}^{\lfloor \frac{1}{2}k \rfloor} \binom{k-i}{i} \left(-\frac{A(x)f(z)}{f'^2(z)}\right)^i\right\} \qquad \text{by (4.1)} \\
&= (-1)^k k! \sum_{i=0}^{\lfloor \frac{1}{2}k \rfloor} \binom{k-i}{i} [-A(x)]^i F(z;i,k), \qquad (6.7)
\end{aligned}$$

where

$$F(z;i,k) = \frac{d}{dz}\left\{\frac{f'^{(k-2i)}(z)}{f^{k+1-i}(z)}\right\}. \qquad (6.8)$$

Applying the Quotient Rule to (6.8), we obtain the derivative function $F(z;i,k)$ in the form

$$F(z;i,k) = \frac{f'^{(k+1)}(z)}{f^{k+2}(z)}\left[2(k-2i)A(x)\left(\frac{f(z)}{f'^2(z)}\right)^{i+1} - (k+1-i)\left(\frac{f(z)}{f'^2(z)}\right)^i\right], \qquad (6.9)$$

where $f''(z)$ has been replaced by $2A(x)$, substitution of which into (6.7) yields, writing

$$Q(z;x) = -\frac{A(x)f(z)}{f'^2(z)}, \qquad (6.10)$$

$$\frac{d^{k+1}}{dz^{k+1}}\left\{\frac{1}{f(z)}\right\} = (-1)^{k+1}(k+1)!\frac{f'^{(k+1)}(z)}{f^{k+2}(z)}\sum_{i=0}^{\lfloor\frac{1}{2}k\rfloor}\binom{k-i}{i}\times$$
$$\left[\frac{k+1-i}{k+1}Q^i(z;x) + 2\frac{k-2i}{k+1}Q^{i+1}(z;x)\right]. \tag{6.11}$$

Thus, the inductive step is upheld if we can show that

$$\begin{aligned}G_k(z;x) &= \sum_{i=0}^{\lfloor\frac{1}{2}k\rfloor}\binom{k-i}{i}\left[\frac{k+1-i}{k+1}Q^i(z;x) + 2\frac{k-2i}{k+1}Q^{i+1}(z;x)\right] \\ &= P_{k+1}(-Q(z;x)) \\ &= \sum_{i=0}^{\lfloor\frac{1}{2}(k+1)\rfloor}\binom{k+1-i}{i}Q^i(z;x).\end{aligned} \tag{6.12}$$

There are two cases to consider: Case A (k even) and Case B (k odd). For brevity, only Case A is set out here, Case B being arguable along similar lines.

Case A: Let k (even) $= 2m$ ($m = 0, 1, 2, \ldots$). Then since $\lfloor\frac{1}{2}k\rfloor = \lfloor m\rfloor = \lfloor m+\frac{1}{2}\rfloor = \lfloor\frac{1}{2}(2m+1)\rfloor = \lfloor\frac{1}{2}(k+1)\rfloor$, consider

$$\begin{aligned}G_k(z;x) &= \frac{1}{k+1}\left[\sum_{i=0}^{\lfloor\frac{1}{2}(k+1)\rfloor}\binom{k-i}{i}(k+1-i)Q^i(z;x)\right. \\ &\qquad\left. + 2\sum_{i=0}^{\lfloor\frac{1}{2}(k+1)\rfloor}\binom{k-i}{i}(k-2i)Q^{i+1}(z;x)\right] \\ &= \frac{1}{k+1}\left[\sum_{i=0}^{\lfloor\frac{1}{2}(k+1)\rfloor}\binom{k-i}{i}(k+1-i)\right. \\ &\qquad\left. + 2\sum_{i=1}^{\lfloor\frac{1}{2}(k+1)\rfloor+1}\binom{k-i+1}{i-1}(k-2i+2)\right]Q^i(z;x).\end{aligned} \tag{6.13}$$

The summand of the second bracketed sum vanishes (i) at the upper limit $i = \lfloor\frac{1}{2}(k+1)\rfloor + 1 = \frac{1}{2}k + 1$ (since here the term $k - 2i + 2 = 0$), and (ii) for an additional lower index value $i = 0$ (since it would contain the binomial coefficient $\binom{k+1}{-1} = 0$), so that, as required,

$$\begin{aligned}G_k(z;x) &= \frac{1}{k+1}\sum_{i=0}^{\lfloor\frac{1}{2}(k+1)\rfloor}\left[\binom{k-i}{i}(k+1-i) + 2\binom{k-i+1}{i-1}(k-2i+2)\right]Q^i(z;x) \\ &= \sum_{i=0}^{\lfloor\frac{1}{2}(k+1)\rfloor}\binom{k+1-i}{i}Q^i(z;x). \ \square\end{aligned} \tag{6.14}$$

Note that it is possible, with some work, to obtain Theorem 6.1 from Schwatt (1962, (12), p. 3), this being a comprehensive repository of all manner of exotic formulae. An alternative proof, using analytic function theory, is offered in Appendix C for interest, together with a first principles constructive proof.

As an immediate consequence of Theorem 6.1 we have the following corollary whereby we can re-state the form of $\mathcal{H}_p(f)$, for the general quadratic $f(z) = A(x)z^2 + B(x)z + C(x)$, in terms of Catalan polynomials:

Corollary 6.2. *For* $p \geq 0$*,*

$$\mathcal{H}_p(f) = z - \frac{f(z)}{f'(z)} \frac{P_p(A(x)f(z)/f'^2(z))}{P_{p+1}(A(x)f(z)/f'^2(z))}.$$

Moving on, let $f(z)$ satisfy the given quadratic, and define from it a polynomial

$$\begin{aligned} \alpha_n(x) &= \alpha_n(A(x), B(x), C(x)) \\ &= (1,0) \begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{6.15}$$

Lemma 6.3. *For* $n \geq 0$*,*

$$0 = A(x)C(x)\alpha_n(x) + B(x)\alpha_{n+1}(x) + \alpha_{n+2}(x).$$

Lemma 6.4. *For* $p, q \geq 1$*,*

$$0 = A(x)C(x)\alpha_{p-1}(x)\alpha_{q-1}(x) - \alpha_p(x)\alpha_q(x) + \alpha_{p+q}(x).$$

In the special case $A(x) = x$, $B(x) = -1$, $C(x) = 1$, $f(z)$ reads as in (6.6) (in line with the governing equation (6.5) for the Catalan sequence o.g.f. $G(x)$) and $\alpha_n(x) = \alpha_n(x, -1, 1)$ (6.15) coincides with the Catalan polynomial $P_n(x)$ as described by (4.18). Lemma 6.3 then recovers the linear recurrence (4.5), and likewise the non-linear recursion $0 = xP_{p-1}(x)P_{q-1}(x) - P_p(x)P_q(x) + P_{p+q}(x)$ given by (4.23) is the same specialisation of Lemma 6.4.

Proof of Lemma 6.3. Write

$$\begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix} = \mathbf{M}(x), \tag{6.16}$$

and define polynomials $\alpha_n(x) = \alpha_n(A(x), B(x), C(x))$, $\beta_n(x) = \beta_n(A(x), B(x), C(x)) \in \mathbb{Z}[x]$ according to the power law

$$\begin{pmatrix} \alpha_n(x) \\ \beta_n(x) \end{pmatrix} = \mathbf{M}^n(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n \geq 0, \tag{6.17}$$

noting that $\alpha_0(x) = 1$, $\beta_0(x) = 0$. Then

$$\begin{aligned}\begin{pmatrix}\alpha_{n+1}(x)\\ \beta_{n+1}(x)\end{pmatrix} &= \mathbf{M}^{n+1}(x)\begin{pmatrix}1\\0\end{pmatrix}\\ &= \mathbf{M}(x)\mathbf{M}^{n}(x)\begin{pmatrix}1\\0\end{pmatrix}\\ &= \mathbf{M}(x)\begin{pmatrix}\alpha_{n}(x)\\ \beta_{n}(x)\end{pmatrix},\end{aligned} \tag{6.18}$$

with component equations

$$\begin{aligned}\alpha_{n+1}(x) &= -B(x)\alpha_n(x) + A(x)\beta_n(x),\\ \beta_{n+1}(x) &= -C(x)\alpha_n(x),\end{aligned} \tag{6.19}$$

which combine trivially to give Lemma 6.3. □

Proof of Lemma 6.4. We first note that (6.17) can now be written as

$$\begin{pmatrix}\alpha_n(x)\\ -C(x)\alpha_{n-1}(x)\end{pmatrix} = \mathbf{M}^n(x)\begin{pmatrix}1\\0\end{pmatrix} \tag{6.20}$$

for $n \geq 0$ (where it is understood that $\alpha_{-1}(x) = 0$), and we consider the polynomial

$$\begin{aligned}-\alpha_{p+q}(x) &= (-1, 0)\begin{pmatrix}\alpha_{p+q}(x)\\ -C(x)\alpha_{p+q-1}(x)\end{pmatrix}\\ &= \frac{1}{C(x)}(1,0)\mathbf{D}(x)\begin{pmatrix}\alpha_{p+q}(x)\\ -C(x)\alpha_{p+q-1}(x)\end{pmatrix},\end{aligned} \tag{6.21}$$

where $\mathbf{D}(x)$ is the diagonal matrix

$$\mathbf{D}(x) = \begin{pmatrix}-C(x) & 0\\ 0 & A(x)\end{pmatrix}. \tag{6.22}$$

Noting further that $\mathbf{D}(x), \mathbf{M}(x)$ have the property that

$$\mathbf{D}(x)\mathbf{M}(x) = [\mathbf{M}(x)]^T\mathbf{D}(x), \tag{6.23}$$

we can continue from (6.21) as follows (using (6.20),(6.23) as appropriate):

$$
\begin{aligned}
-\alpha_{p+q}(x) &= \frac{1}{C(x)}(1,0)\mathbf{D}(x)\mathbf{M}^{p+q}(x)\begin{pmatrix}1\\0\end{pmatrix} \\
&= \frac{1}{C(x)}\begin{pmatrix}1\\0\end{pmatrix}^T \mathbf{D}(x)\mathbf{M}^{p}(x)\mathbf{M}^{q}(x)\begin{pmatrix}1\\0\end{pmatrix} \\
&= \frac{1}{C(x)}\begin{pmatrix}1\\0\end{pmatrix}^T [\mathbf{M}^{p}(x)]^T\mathbf{D}(x)\mathbf{M}^{q}(x)\begin{pmatrix}1\\0\end{pmatrix} \\
&= \frac{1}{C(x)}\left[\mathbf{M}^{p}(x)\begin{pmatrix}1\\0\end{pmatrix}\right]^T \mathbf{D}(x)\mathbf{M}^{q}(x)\begin{pmatrix}1\\0\end{pmatrix} \\
&= \frac{1}{C(x)}\begin{pmatrix}\alpha_p(x)\\-C(x)\alpha_{p-1}(x)\end{pmatrix}^T \mathbf{D}(x)\begin{pmatrix}\alpha_q(x)\\-C(x)\alpha_{q-1}(x)\end{pmatrix} \\
&= \frac{1}{C(x)}(\alpha_p(x),-C(x)\alpha_{p-1}(x))\begin{pmatrix}-C(x)\alpha_q(x)\\-A(x)C(x)\alpha_{q-1}(x)\end{pmatrix} \\
&= A(x)C(x)\alpha_{p-1}(x)\alpha_{q-1}(x)-\alpha_p(x)\alpha_q(x). \ \square
\end{aligned} \tag{6.24}
$$

Remark 6.5. Although Lemmas 6.3, 6.4 differ qualitatively we remark, noting $\alpha_0(x) = 1$, $\alpha_1(x) = -B(x)$ by definition (6.15), that Lemma 6.4 contains Lemma 6.3 as the instance $p = n+1$, $q = 1$.

6.2.2 Further Results

Lemma 6.6. *Let $f(z) = A(x)z^2 + B(x)z + C(x)$, where $A(x), B(x), C(x) \in \mathbb{Z}[x]$. Then the function $\mathcal{H}_p(f)$ satisfies, for $p \geq 0$, the recurrence*

$$\mathcal{H}_{p+1}(f) = z - \frac{f(z)}{f'(z) - A(x)[z - \mathcal{H}_p(f)]}.$$

Proof. From Corollary 6.2 we have

$$\mathcal{H}_p(f) = z - \frac{f(z)}{f'(z)}\frac{P_p(\hat{Q}(z;x))}{P_{p+1}(\hat{Q}(z;x))}, \tag{6.25}$$

where $\hat{Q}(z;x) = -Q(z;x) = A(x)f(z)/f'^2(z)$, which, when rearranged, reads

$$\frac{P_p(\hat{Q}(z;x))}{P_{p+1}(\hat{Q}(z;x))} = \frac{f'(z)}{f(z)}[z - \mathcal{H}_p(f)]. \tag{6.26}$$

Re-writing the linear recurrence (4.5) (with $x = z$, $n = p$) gives

$$z\frac{P_p(z)}{P_{p+1}(z)} = 1 - \frac{P_{p+2}(z)}{P_{p+1}(z)}, \tag{6.27}$$

so that

$$\hat{Q}(z;x)\frac{P_p(\hat{Q}(z;x))}{P_{p+1}(\hat{Q}(z;x))} = 1 - \frac{P_{p+2}(\hat{Q}(z;x))}{P_{p+1}(\hat{Q}(z;x))}. \tag{6.28}$$

Consider, now,

$$\begin{aligned}
\frac{f(z) - \hat{Q}(z;x)f'(z)[z - \mathcal{H}_p(f)]}{f(z)} &= 1 - \hat{Q}(z;x)\frac{f'(z)}{f(z)}[z - \mathcal{H}_p(f)] \\
&= 1 - \hat{Q}(z;x)\frac{P_p(\hat{Q}(z;x))}{P_{p+1}(\hat{Q}(z;x))} \\
&= 1 - \left(1 - \frac{P_{p+2}(\hat{Q}(z;x))}{P_{p+1}(\hat{Q}(z;x))}\right) \\
&= \frac{P_{p+2}(\hat{Q}(z;x))}{P_{p+1}(\hat{Q}(z;x))} \\
&= \frac{f(z)}{f'(z)[z - \mathcal{H}_{p+1}(f)]},
\end{aligned} \tag{6.29}$$

having employed both (6.26),(6.28) as needed. On replacing $\hat{Q}(z;x)$ with $A(x)f(z)/f'^2(z)$ in (6.29), Lemma 6.6 follows after some simple algebraic manipulation. □

Defining $\delta_p(f) = z - \mathcal{H}_p(f)$, Corollary 6.7 below is an obvious deduction from Lemma 6.6.

Corollary 6.7. *Let $f(z) = A(x)z^2 + B(x)z + C(x)$, where $A(x), B(x), C(x) \in \mathbb{Z}[x]$. Then the function $\mathcal{H}_p(f)$ has, for $p \geq 0$, a continued fraction representation given by*

$$\mathcal{H}_p(f) = z - \delta_p(f),$$

where

$$\delta_{p+1}(f) = \frac{f(z)}{f'(z) - A(x)\delta_p(f)}.$$

Each representation involves $f(z), f'(z)$ only; for example, with $\delta_0(f) = z - \mathcal{H}_0(f) = z - [z - f(z)/f'(z)] = f(z)/f'(z)$, then

$$\begin{aligned}
\mathcal{H}_0(f) &= z - \delta_0(f) = z - \frac{f(z)}{f'(z)}, \\
\mathcal{H}_1(f) &= z - \delta_1(f) = z - \frac{f(z)}{f'(z) - \frac{A(x)f(z)}{f'(z)}}, \\
\mathcal{H}_2(f) &= z - \delta_2(f) = z - \frac{f(z)}{f'(z) - \frac{A(x)f(z)}{f'(z) - \frac{A(x)f(z)}{f'(z)}}},
\end{aligned} \tag{6.30}$$

etc.

6.2.3 Generalised Polynomial Ratios

Motivated by the work in Chapter 5, in which algebraic Householder schemes specialised for the Catalan sequence were found to output pairs of Catalan polynomials with associated non-linear identities, we are now in a position to apply Lemmas 6.3, 6.4 and 6.6 in examining the output from the $O(n)$ Householder scheme when executed at a "point" of the form $z(x) = \alpha_k(x)/\alpha_{k+1}(x)$. In particular, we suppose that for some integer ℓ, say,

$$\mathcal{H}_n(f)|_{z(x)=\alpha_k(x)/\alpha_{k+1}(x)} = \frac{\alpha_\ell(x)}{\alpha_{\ell+1}(x)}, \tag{6.31}$$

and we identify explicitly the r.h.s. polynomial ratio output (in other words, $\ell = \ell(k,n)$). Directly from Lemma 6.6,

$$\begin{aligned}
\mathcal{H}_{n+1}(f)|_{z(x)=\alpha_k(x)/\alpha_{k+1}(x)} &= \frac{\alpha_k(x)}{\alpha_{k+1}(x)} - \frac{A(x)[\frac{\alpha_k(x)}{\alpha_{k+1}(x)}]^2 + B(x)[\frac{\alpha_k(x)}{\alpha_{k+1}(x)}] + C(x)}{2A(x)[\frac{\alpha_k(x)}{\alpha_{k+1}(x)}] + B(x) - A(x)\{\frac{\alpha_k(x)}{\alpha_{k+1}(x)} - \frac{\alpha_\ell(x)}{\alpha_{\ell+1}(x)}\}} \\
&= \frac{1}{\alpha_{k+1}(x)} \times \\
&\qquad \left[\alpha_k(x) - \frac{A(x)\alpha_k^2(x) + B(x)\alpha_k(x)\alpha_{k+1}(x) + C(x)\alpha_{k+1}^2(x)}{A(x)\alpha_k(x) + B(x)\alpha_{k+1}(x) + A(x)\alpha_{k+1}(x)[\frac{\alpha_\ell(x)}{\alpha_{\ell+1}(x)}]}\right] \\
&= \frac{A(x)\alpha_k(x)[\frac{\alpha_\ell(x)}{\alpha_{\ell+1}(x)}] - C(x)\alpha_{k+1}(x)}{A(x)\alpha_k(x) + B(x)\alpha_{k+1}(x) + A(x)\alpha_{k+1}(x)[\frac{\alpha_\ell(x)}{\alpha_{\ell+1}(x)}]} \\
&= \frac{A(x)\alpha_k(x)\alpha_\ell(x) - C(x)\alpha_{k+1}(x)\alpha_{\ell+1}(x)}{A(x)\alpha_k(x)\alpha_{\ell+1}(x) + B(x)\alpha_{k+1}(x)\alpha_{\ell+1}(x) + A(x)\alpha_{k+1}(x)\alpha_\ell(x)} \\
&= \frac{A(x)\alpha_k(x)\alpha_\ell(x) - C(x)\alpha_{k+1}(x)\alpha_{\ell+1}(x)}{A(x)\alpha_k(x)\alpha_{\ell+1}(x) + \alpha_{k+1}(x)[A(x)\alpha_\ell(x) + B(x)\alpha_{\ell+1}(x)]} \\
&= \frac{A(x)\alpha_k(x)\alpha_\ell(x) - \alpha_{k+1}(x)\alpha_{\ell+1}(x)}{A(x)\alpha_k(x)\alpha_{\ell+1}(x) - \alpha_{k+1}(x)\alpha_{\ell+2}(x)}
\end{aligned} \tag{6.32}$$

if we set $C(x) = 1$ in Lemma 6.3 which gives $A(x)\alpha_\ell(x) + B(x)\alpha_{\ell+1}(x) = -\alpha_{\ell+2}(x)$. Furthermore, for this same specialisation of $C(x)$ Lemma 6.4 can be written as $A(x)\alpha_k(x)\alpha_{q-1}(x) - \alpha_{k+1}(x)\alpha_q(x) = -\alpha_{k+q+1}(x)$, and when used for separate values $q = \ell+1, \ell+2$, (6.32) simplifies to merely

$$\mathcal{H}_{n+1}(f)|_{z(x)=\alpha_k(x)/\alpha_{k+1}(x)} = \frac{\alpha_{k+\ell+2}(x)}{\alpha_{k+\ell+3}(x)}. \tag{6.33}$$

Now with $\mathcal{H}_0(f) = z - f(z)/f'(z)$, then

$$\begin{aligned}
\mathcal{H}_0(f)|_{z(x)=\alpha_k(x)/\alpha_{k+1}(x)} &= \frac{\alpha_k(x)}{\alpha_{k+1}(x)} - \frac{A(x)[\frac{\alpha_k(x)}{\alpha_{k+1}(x)}]^2 + B(x)[\frac{\alpha_k(x)}{\alpha_{k+1}(x)}] + 1}{2A(x)[\frac{\alpha_k(x)}{\alpha_{k+1}(x)}] + B(x)} \\
&= \\
&\vdots \\
&= \frac{\alpha_{2k+2}(x)}{\alpha_{2k+3}(x)}
\end{aligned} \tag{6.34}$$

if we repeat, albeit in a simpler manner, the essential steps which produced (6.33). Thus, we have provided—noting that the imposition of $C(x) = 1$ is a necessary one—an inductive proof of the following:

Theorem 6.8. *Let $f(z) = A(x)z^2 + B(x)z + 1$, where $A(x), B(x) \in \mathbb{Z}[x]$. Then the $O(n)$ Householder scheme (6.2) has the property that, for $n \geq 0$,*

$$\mathcal{H}_n(f)|_{z(x)=\alpha_k(x)/\alpha_{k+1}(x)} = \frac{\alpha_{(n+2)(k+1)+n}(x)}{\alpha_{(n+2)(k+1)+n+1}(x)}.$$

Using previous work on the Catalan sequence to verify Theorem 6.8, it is easiest to set $n = p$, $k = n - 1$, whence, with $A(x) = x$, $B(x) = -1$, it reads, for $p \geq 0$,

$$\mathcal{H}_p(f)|_{z(x)=P_{n-1}(x)/P_n(x)} = \frac{P_{(p+2)n+p}(x)}{P_{(p+2)n+p+1}(x)}, \tag{6.35}$$

recovering the respective $p = 0, 1, 2$ results

$$\begin{aligned}
\mathcal{H}_0(f)|_{z(x)=P_{n-1}(x)/P_n(x)} &= \frac{P_{2n}(x)}{P_{2n+1}(x)}, \\
\mathcal{H}_1(f)|_{z(x)=P_{n-1}(x)/P_n(x)} &= \frac{P_{3n+1}(x)}{P_{3n+2}(x)}, \\
\mathcal{H}_2(f)|_{z(x)=P_{n-1}(x)/P_n(x)} &= \frac{P_{4n+2}(x)}{P_{4n+3}(x)}, \qquad (6.36)
\end{aligned}$$

as seen (explicitly or otherwise) in (5.12), (5.22) and (5.27); the further instances $p = 3, 4$ are covered by (5.28),(5.29). It is also evident that Theorem 6.8, in generating the specific ratios seen above, supports Conjecture 5.5.

6.3 Schröder and Motzkin Householder Schemes

As a further illustration of the theory described in the previous section, we turn our attention to the Schröder and Motzkin sequences, using the matrix formula (6.15) to generate a series of polynomials for each which are analogous to the Catalan polynomials. Consequently, we are in a position to construct specialised Householder schemes which accommodate the quadratic equation governing the o.g.f. of each of these sequences in the expectation that similar non-linear polynomial identities can be formulated.

6.3.1 Schröder and Motzkin Polynomials

The quadratic equation governing the o.g.f.

$$S(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x} = \sum_{i=0}^{\infty} s_i x^i \tag{6.37}$$

for the Schröder sequence is

$$0 = 1 + (x-1)S(x) + xS^2(x), \tag{6.38}$$

with which, from (6.15), are associated Schröder polynomials[1]

$$\begin{aligned}
S_0(x) &= 1, \\
S_1(x) &= 1 - x, \\
S_2(x) &= 1 - 3x + x^2, \\
S_3(x) &= 1 - 5x + 5x^2 - x^3, \\
S_4(x) &= 1 - 7x + 13x^2 - 7x^3 + x^4, \\
S_5(x) &= 1 - 9x + 25x^2 - 25x^3 + 9x^4 - x^5, \\
S_6(x) &= 1 - 11x + 41x^2 - 63x^3 + 41x^4 - 11x^5 + x^6, \\
S_7(x) &= 1 - 13x + 61x^2 - 129x^3 + 129x^4 - 61x^5 + 13x^6 - x^7,
\end{aligned} \tag{6.39}$$

etc., of general form

$$S_n(x) = \alpha_n(x, x-1, 1) = (1, 0)\begin{pmatrix} 1-x & x \\ -1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Similarly, we see that there exists a set of corresponding Motzkin polynomials

$$\begin{aligned}
M_0(x) &= 1, \\
M_1(x) &= 1 - x, \\
M_2(x) &= 1 - 2x, \\
M_3(x) &= 1 - 3x + x^2 + x^3, \\
M_4(x) &= 1 - 4x + 3x^2 + 2x^3 - x^4, \\
M_5(x) &= 1 - 5x + 6x^2 + 2x^3 - 4x^4, \\
M_6(x) &= 1 - 6x + 10x^2 - 9x^4 + 2x^5 + x^6, \\
M_7(x) &= 1 - 7x + 15x^2 - 5x^3 - 15x^4 + 9x^5 + 3x^6 - x^7,
\end{aligned} \tag{6.40}$$

etc., expressed generally as

$$M_n(x) = \alpha_n(x^2, x-1, 1) = (1, 0)\begin{pmatrix} 1-x & x^2 \\ -1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

[1] Although the notation used here coincides with that used in Chapter 1 to illustrate the mechanism of the Schröder and Motzkin recurrence schemes ((1.16) and (1.20), respectively), it must be emphasised that these sets of polynomials are entirely distinct.

and arising from the equation

$$0 = 1 + (x-1)M(x) + x^2M^2(x) \tag{6.41}$$

for the o.g.f.

$$M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} = \sum_{i=0}^{\infty} m_i x^i. \tag{6.42}$$

The o.g.fs for the polynomials themselves are known to be

$$\frac{1}{1 + (x-1)y + xy^2} = \sum_{n=0}^{\infty} S_n(x) y^n \tag{6.43}$$

and

$$\frac{1}{1 + (x-1)y + x^2y^2} = \sum_{n=0}^{\infty} M_n(x) y^n, \tag{6.44}$$

which we will see later are merely special cases of a more general result (Lemma 6.12).

From (6.39), it is apparent that, similarly to binomial coefficients comprising the non-signed coefficients of the Catalan polynomials, the non-signed coefficients of the Schröder polynomials are instances of the Delannoy numbers (sequence no. A008288 in the O.E.I.S.), a prominent two-dimensional sequence in the area of path-counting problems. The general Schröder polynomial is found to be expressible by

$$S_n(x) = \sum_{i=0}^{n} d_{i,n-i}(-x)^i, \qquad n \geq 0, \tag{6.45}$$

where, for $i, j \geq 0$, the Delannoy numbers $d_{i,j}$ comprise the symmetric array

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 3 & 5 & 7 & 9 & \cdots \\ 1 & 5 & 13 & 25 & 41 & \cdots \\ 1 & 7 & 25 & 63 & 129 & \cdots \\ 1 & 9 & 41 & 129 & 321 & \cdots \\ \vdots & & & & & \ddots \end{matrix} = \begin{matrix} d_{0,0} & d_{0,1} & d_{0,2} & d_{0,3} & d_{0,4} & \cdots \\ d_{1,0} & d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & \cdots \\ d_{2,0} & d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & \cdots \\ d_{3,0} & d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & \cdots \\ d_{4,0} & d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & \cdots \\ \vdots & & & & & \ddots \end{matrix}, \tag{6.46}$$

the ordered non-signed coefficients of the Schröder polynomials representing the anti-diagonals of the above array.

By contrast, there currently exists no O.E.I.S. sequence corresponding to the coefficients (either signed or unsigned) of the Motzkin polynomials.

6.3.2 Polynomial Identities

The phenomenon observed in Section 6.2.3 (whereby input ratios of generalised polynomials to the Householder schemes were seen to form non-linear identities in conjunction with the polynomial ratios output by the schemes) has been shown to recover results seen in Chapter 5. It is found to be repeated in the computations made when Householder algorithms are applied in the context of the Schröder and Motzkin sequences, both of which have $C(x) = 1$ in their governing o.g.f. quadratics (6.38),(6.41). The resulting identities from the $p = 0, 1$ Householder schemes are given below by way of illustration.

Schröder Identities

Householder $p = 0$ (Newton-Raphson) Scheme

$$\begin{aligned} S_{2n}(x) &= S_n^2(x) - xS_{n-1}^2(x), \\ S_{2n+1}(x) &= S_n(x)[(1-x)S_n(x) - 2xS_{n-1}(x)]. \end{aligned} \tag{6.47}$$

Householder $p = 1$ (Halley) Scheme

$$\begin{aligned} S_{3n+1}(x) &= (1-x)S_n^3(x) - 3xS_{n-1}(x)S_n^2(x) + x^2S_{n-1}^3(x), \\ S_{3n+2}(x) &= S_n(x)[(1-3x+x^2)S_n^2(x) - 3x(1-x)S_{n-1}(x)S_n(x) + 3x^2S_{n-1}^2(x)]. \end{aligned} \tag{6.48}$$

Motzkin Identities

Householder $p = 0$ (Newton-Raphson) Scheme

$$\begin{aligned} M_{2n}(x) &= M_n^2(x) - x^2M_{n-1}^2(x), \\ M_{2n+1}(x) &= M_n(x)[(1-x)M_n(x) - 2x^2M_{n-1}(x)]. \end{aligned} \tag{6.49}$$

Householder $p = 1$ (Halley) Scheme

$$\begin{aligned} M_{3n+1}(x) &= (1-x)M_n^3(x) - 3x^2M_{n-1}(x)M_n^2(x) + x^4M_{n-1}^3(x), \\ M_{3n+2}(x) &= M_n(x)[(1-2x)M_n^2(x) - 3x^2(1-x)M_{n-1}(x)M_n(x) + 3x^4M_{n-1}^2(x)]. \end{aligned} \tag{6.50}$$

Remark 6.9. The Schröder and Motzkin specialisations of Lemmas 6.3, 6.4 are

$$\begin{aligned} 0 &= xS_n(x) + (x-1)S_{n+1}(x) + S_{n+2}(x), \\ 0 &= xS_{p-1}(x)S_{q-1}(x) - S_p(x)S_q(x) + S_{p+q}(x), \end{aligned} \tag{6.51}$$

and

$$\begin{aligned} 0 &= x^2M_n(x) + (x-1)M_{n+1}(x) + M_{n+2}(x), \\ 0 &= x^2M_{p-1}(x)M_{q-1}(x) - M_p(x)M_q(x) + M_{p+q}(x). \end{aligned} \tag{6.52}$$

Setting $p = q = n$ in the second identities of (6.51),(6.52) reproduces the first identity in each of (6.47),(6.49).

Remark 6.10. We remarked in Section 4.2.2 that $\{P_n(\pm 1)\}_0^\infty$ each correspond to a known O.E.I.S. sequence. It is also found that sequences A000129 (Pell sequence) and A056594 (a period 4 sequence whose o.g.f. is the inverse of the 4th cyclotomic polynomial) are $\{S_n(-1)\}_0^\infty = \{1, 2, 5, 12, 29, 70, 169, \ldots\}$ and $\{S_n(1)\}_0^\infty = \{1, 0, -1, 0, 1, 0, -1, \ldots\}$. Likewise, $\{M_n(-1)\}_0^\infty = \{1, 2, 3, 4, 5, 6, \ldots\}$ is the sequence of natural numbers (A000027), whilst $\{M_n(1)\}_0^\infty = \{S_n(1)\}_0^\infty$.

6.4 Further Theory and Results

With $\alpha_n(x) = \alpha_n(A(x), B(x), C(x))$ defined as in (6.15), we move on by offering a result in which this general polynomial is expressed directly in terms of a general Catalan polynomial.

Theorem 6.11. *Let $A(x), B(x), C(x) \in \mathbb{Z}[x]$ satisfy the quadratic $0 = A(x)y^2 + B(x)y + C(x)$. For $n \geq 0$,*

$$\alpha_n(x) = \alpha_n(A(x), B(x), C(x)) = [-B(x)]^n P_n\left(\frac{A(x)C(x)}{B^2(x)}\right).$$

We set out three quite different proofs of Theorem 6.11, the third being a succinct proof which overlaps in nature with one in Appendix C for Theorem 6.1.

Proof I. Proof I utilises a result which identifies a closed form for the o.g.f. of the polynomials $\alpha_0(x), \alpha_1(x), \alpha_2(x)$, *etc.* (when specialised as appropriate, Lemma 6.12 is seen to recover (4.9), (6.43) and (6.44) given earlier):

Lemma 6.12. [2] *Let $F(x,y) = \sum_{n=0}^{\infty} \alpha_n(x)y^n$ be the o.g.f. for the polynomials $\alpha_0(x), \alpha_1(x), \alpha_2(x), \ldots$. Then*

$$F(x,y) = \frac{1}{A(x)C(x)y^2 + B(x)y + 1}.$$

[2] A matrix-based proof of Lemma 6.12 is given in Appendix D.

Proof of Lemma 6.12. From Remark 6.5 we note once more that $\alpha_0(x) = 1$, $\alpha_1(x) = -B(x)$. Consider

$$F(x,y) = \sum_{n=0}^{\infty} \alpha_n(x) y^n = \alpha_0(x) + \alpha_1(x)y + \alpha_2(x)y^2 + \cdots , \tag{6.53}$$

and let series $s_1(x,y), s_2(x,y)$ be

$$\begin{aligned} s_1(x,y) &= \sum_{n=0}^{\infty} \alpha_{n+1}(x) y^n = \alpha_1(x) + \alpha_2(x)y + \alpha_3(x)y^2 + \cdots , \\ s_2(x,y) &= \sum_{n=0}^{\infty} \alpha_{n+2}(x) y^n = \alpha_2(x) + \alpha_3(x)y + \alpha_4(x)y^2 + \cdots , \end{aligned} \tag{6.54}$$

so that

$$\begin{aligned} s_1(x,y) &= \frac{F(x,y) - \alpha_0(x)}{y} = \frac{F(x,y) - 1}{y}, \\ s_2(x,y) &= \frac{F(x,y) - \alpha_0(x) - \alpha_1(x)y}{y^2} = \frac{F(x,y) - 1 + B(x)y}{y^2}. \end{aligned} \tag{6.55}$$

Directly from Lemma 6.3 we can now write

$$\begin{aligned} 0 &= A(x)C(x)\alpha_n(x) + B(x)\alpha_{n+1}(x) + \alpha_{n+2}(x) \\ &= \sum_{n=0}^{\infty} [A(x)C(x)\alpha_n(x) + B(x)\alpha_{n+1}(x) + \alpha_{n+2}(x)] y^n \\ &= A(x)C(x) \sum_{n=0}^{\infty} \alpha_n(x) y^n + B(x) \sum_{n=0}^{\infty} \alpha_{n+1}(x) y^n + \sum_{n=0}^{\infty} \alpha_{n+2}(x) y^n \\ &= A(x)C(x)F(x,y) + B(x)s_1(x,y) + s_2(x,y) \\ &= A(x)C(x)F(x,y) + B(x)\left(\frac{F(x,y) - 1}{y}\right) + \frac{F(x,y) - 1 + B(x)y}{y^2} \\ &= \frac{[A(x)C(x)y^2 + B(x)y + 1]F(x,y) - 1}{y^2}, \end{aligned} \tag{6.56}$$

and the result follows. □

Proof I of Theorem 6.11 concludes as follows. Let $G(x,y) = A(x)C(x)y^2 + B(x)y + 1$. Then Lemma 6.12 gives

$$\frac{\partial^n}{\partial y^n}\{F(x,y)\} = \frac{\partial^n}{\partial y^n}\left\{\frac{1}{G(x,y)}\right\} \tag{6.57}$$

which, employing Theorem 6.1,

$$\begin{aligned} &= (-1)^n n! \frac{(\frac{\partial G}{\partial y})^n}{G^{n+1}(x,y)} P_n\left(\frac{A(x)C(x)G(x,y)}{(\frac{\partial G}{\partial y})^2}\right) \\ &= (-1)^n n! \frac{[2A(x)C(x)y + B(x)]^n}{[A(x)C(x)y^2 + B(x)y + 1]^{n+1}} P_n\left(\frac{A(x)C(x)[A(x)C(x)y^2 + B(x)y + 1]}{[2A(x)C(x)y + B(x)]^2}\right). \end{aligned} \tag{6.58}$$

Since, however, we can write down that

$$\alpha_n(x) = \frac{1}{n!}\frac{\partial^n}{\partial y^n}\{F(x,y)\}\Bigg|_{y=0} \tag{6.59}$$

then by (6.57),(6.58) the r.h.s. of (6.59) duly simplifies to give

$$\alpha_n(x) = [-B(x)]^n P_n\left(\frac{A(x)C(x)}{B^2(x)}\right). \square \tag{6.60}$$

Proof II. We require the following subsidiary result, from which Theorem 6.11 is immediate:

Lemma 6.13. *Noting that $P_{-1}(x) = 0$, then for $n \geq 0$,*

$$\begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} P_n(-xy) \\ yP_{n-1}(-xy) \end{pmatrix}.$$

Proof of Lemma 6.13. By induction. The result holds for $n = 0$, both sides being $(1,0)^T$. Suppose, therefore, that it is true for some $n = k \geq 0$, and consider

$$\begin{aligned}
\begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}^{k+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}^{k} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix} \begin{pmatrix} P_k(-xy) \\ yP_{k-1}(-xy) \end{pmatrix} \qquad \text{(by assumption)} \\
&= \begin{pmatrix} P_k(-xy) + xyP_{k-1}(-xy) \\ yP_k(-xy) \end{pmatrix} \\
&= \begin{pmatrix} P_{k+1}(-xy) \\ yP_k(-xy) \end{pmatrix}
\end{aligned} \tag{6.61}$$

as required, since the linear recurrence (4.5) gives, with x, n replaced with $-xy, k-1$, resp., $P_k(-xy) + xyP_{k-1}(-xy) = P_{k+1}(-xy)$. $\square$

Proof II of Theorem 6.11 now follows easily, for Lemma 6.13 gives

$$\begin{pmatrix} 1 & -A(x)/B(x) \\ C(x)/B(x) & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} P_n(A(x)C(x)/B^2(x)) \\ \frac{C(x)}{B(x)}P_{n-1}(A(x)C(x)/B^2(x)) \end{pmatrix}, \tag{6.62}$$

whence, starting from the definition of $\alpha_n(x)$ (6.15),

$$\begin{aligned}
\alpha_n(x) &= (1,0)\begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= [-B(x)]^n(1,0)\begin{pmatrix} 1 & -A(x)/B(x) \\ C(x)/B(x) & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= [-B(x)]^n(1,0)\begin{pmatrix} P_n(A(x)C(x)/B^2(x)) \\ \frac{C(x)}{B(x)}P_{n-1}(A(x)C(x)/B^2(x)) \end{pmatrix} \\
&= [-B(x)]^n P_n\left(\frac{A(x)C(x)}{B^2(x)}\right). \square
\end{aligned} \tag{6.63}$$

Finally, for this short third proof, we utilise the same recurrence for $Q_n(u,v) = (-v)^n P_n(u)$ as given in Appendix C in the form of (C1),

$$0 = uv^2 Q_n(u,v) + vQ_{n+1}(u,v) + Q_{n+2}(u,v). \tag{6.64}$$

Proof III. Setting

$$u(x) = \frac{A(x)C(x)}{B^2(x)}, \qquad v(x) = B(x), \tag{6.65}$$

(6.64) now reads, in terms of $Q_n(x) = Q_n(u(x), v(x))$,

$$0 = A(x)C(x)Q_n(x) + B(x)Q_{n+1}(x) + Q_{n+2}(x). \tag{6.66}$$

Comparing with Lemma 6.3, and noting that $Q_0(x) = P_0(u(x)) = 1 = \alpha_0(x)$, $Q_1(x) = -v(x)P_1(u(x)) = -v(x) = -B(x) = \alpha_1(x)$, we have shown that, $\forall n \geq 0$, $\alpha_n(x) = Q_n(u(x), v(x)) = [-v(x)]^n P_n(u(x))$, which is Theorem 6.11. □

As a result of Theorem 6.11, closed forms are available for the Schröder and Motzkin polynomials as corollaries to it. It is straightforward to show, using Theorem 6.11 in conjunction with the closed form (4.6) for $P_n(x)$ and denoting a discriminant function $\sqrt{B^2(x) - 4A(x)C(x)}$ by $\Delta(x) = \Delta(A(x), B(x), C(x))$, that for $n \geq 0$,

$$\alpha_n(x) = \frac{1}{2^{n+1}} \frac{[-B(x) + \Delta(x)]^{n+1} - [-B(x) - \Delta(x)]^{n+1}}{\Delta(x)}, \tag{6.67}$$

or

$$\alpha_n(x) = A^n(x) \frac{R_+^{n+1}(x) - R_-^{n+1}(x)}{R_+(x) - R_-(x)} \tag{6.68}$$

where

$$\begin{aligned} R_+(x) &= \frac{-B(x) + \Delta(x)}{2A(x)}, \\ R_-(x) &= \frac{-B(x) - \Delta(x)}{2A(x)}. \end{aligned} \tag{6.69}$$

The Schröder and Motzkin polynomials are now delivered as follows:

Corollary 6.14. *With $\Delta_S(x) = \Delta(x, x-1, 1) = \sqrt{1 - 6x + x^2}$, then for $n \geq 0$,*

$$\begin{aligned} S_n(x) &= (1-x)^n P_n\left(\frac{x}{(1-x)^2}\right) \\ &= \frac{1}{2^{n+1}} \frac{[1 - x + \Delta_S(x)]^{n+1} - [1 - x - \Delta_S(x)]^{n+1}}{\Delta_S(x)}. \end{aligned}$$

Corollary 6.15. *With $\Delta_M(x) = \Delta(x^2, x-1, 1) = \sqrt{1 - 2x - 3x^2}$, then for $n \geq 0$,*

$$\begin{aligned} M_n(x) &= (1-x)^n P_n\left(\frac{x^2}{(1-x)^2}\right) \\ &= \frac{1}{2^{n+1}} \frac{[1 - x + \Delta_M(x)]^{n+1} - [1 - x - \Delta_M(x)]^{n+1}}{\Delta_M(x)}. \end{aligned}$$

Remark 6.16. We note that (6.67) can be formulated from consideration of the characteristic equation $0 = \lambda^2 + B(x)\lambda + A(x)C(x)$ associated with Lemma 6.3 (this gives the aforementioned discriminant function $\Delta(x)$). With roots $\lambda_{1,2}(x) = \frac{1}{2}(-B(x) \pm \Delta(x))$ and general solution $\alpha_n(x) = A_1(x)\lambda_1^n(x) + A_2(x)\lambda_2^n(x)$ (for $\lambda_1 \neq \lambda_2$), the result follows from applying the initial conditions $\alpha_0(x) = 1$, $\alpha_1(x) = -B(x)$ to solve for $A_1(x), A_2(x)$. Such a route to it is alluded to when stating (4.6), which is recovered by (6.67) with $\Delta(x) = \Delta(A(x), B(x), C(x)) = \Delta(x, -1, 1)$.

To conclude, we state one more result—and specialisations thereof which in turn lead to consideration of Padé approximation in the context discussed previously—before summarising the work presented in this chapter and indicating future possible topics for study.

Lemma 6.17. *Let $N(x)$ be an o.g.f. satisfying the quadratic $0 = A(x)N^2(x) + B(x)N(x) + C(x)$—where $A(x), B(x), C(x) \in \mathbb{Z}[x]$—from which are defined general polynomials $\alpha_n(x), \beta_n(x)$ as in (6.16) and (6.17). Then for $n \geq 1$,*

$$N^n(x) = \frac{\alpha_{n-1}(x)N(x) + \beta_{n-1}(x)}{A^{n-1}(x)}.$$

Proof. Noting that the result holds for $n = 1$ (both sides are $N(x)$), we assume it holds for some $n = k \geq 1$ and argue inductively. Consider

$$\begin{aligned}
N^{k+1}(x) &= N(x)N^k(x) \\
&= N(x)\left(\frac{\alpha_{k-1}(x)N(x) + \beta_{k-1}(x)}{A^{k-1}(x)}\right) \qquad \text{(by assumption)} \\
&= \frac{\alpha_{k-1}(x)N^2(x) + \beta_{k-1}(x)N(x)}{A^{k-1}(x)} \\
&= \frac{1}{A^{k-1}(x)}\left(\alpha_{k-1}(x)\frac{[-B(x)N(x) - C(x)]}{A(x)} + \beta_{k-1}(x)N(x)\right) \\
&= \frac{1}{A^k(x)}\{[A(x)\beta_{k-1}(x) - B(x)\alpha_{k-1}(x)]N(x) - C(x)\alpha_{k-1}(x)\} \\
&= \frac{1}{A^k(x)}[\alpha_k(x)N(x) + \beta_k(x)] \qquad (6.70)
\end{aligned}$$

on employing equations (6.19) with $n = k - 1$. The inductive step is upheld, and the proof completed. □

If we re-write Lemma 6.17 as, using (6.19),

$$N^{n+2}(x) = \frac{N(x)\alpha_{n+1}(x) - C(x)\alpha_n(x)}{A^{n+1}(x)}, \qquad n \geq 0, \tag{6.71}$$

then the Catalan specialisation $A(x) = x$, $B(x) = -1$ (not actually used here) and $C(x) = 1$ of (6.71) (with $N(x) = G(x)$, $\alpha_n(x) = P_n(x)$) recovers the result seen in Lemma 4.3 for powers of the Catalan sequence o.g.f. $G(x)$, whilst those versions for Schröder and Motzkin o.g.f. powers are immediate:

Corollary 6.18. *For $n \geq 0$,*

$$S^{n+2}(x) = \frac{S(x)S_{n+1}(x) - S_n(x)}{x^{n+1}}.$$

Corollary 6.19. *For $n \geq 0$,*

$$M^{n+2}(x) = \frac{M(x)M_{n+1}(x) - M_n(x)}{x^{2(n+1)}}.$$

6.5 Padé Approximation by Polynomial Ratio

It was previously shown that $\forall n \geq 0$, the ratio $P_n(x)/P_{n+1}(x)$ of Catalan polynomials is an order $(\lfloor \frac{1}{2}n \rfloor, \lfloor \frac{1}{2}(n+1) \rfloor)$ Padé approximant of the Catalan sequence o.g.f. $G(x)$ (refer to Theorem 5.6 and its associated proof); we are now able to make corresponding statements in relation to Schröder and Motzkin polynomials, and their respective o.g.fs.

Recall that given a function $f(x)$ and integers $m, p \geq 0$, the order (m, p) Padé approximant of $f(x)$ is the rational function

$$\frac{u(x)}{v(x)} = \frac{u_0 + u_1 x + u_2 x^2 + \cdots + u_m x^m}{1 + v_1 x + v_2 x^2 + \cdots + v_p x^p} \tag{6.72}$$

which, when expanded as a Maclaurin series, has cancellation strictly in its first $m + p + 1$ terms with the corresponding series form of $f(x)$. In other words, for some $\Theta(x) \in \mathbb{Z}[[x]]$ with non-zero lead term,

$$u(x) - f(x)v(x) = x^{m+p+1}\Theta(x). \tag{6.73}$$

There always exists a reduced order (m, p) approximant to the series form of $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$ (for which $u(x)$ and $v(x)$ are relatively prime, with $u(0) = f_0$, $v(0) = 1$).

6.5.1 Approximation of the Schröder Polynomials

Consider first the role of the Schröder polynomials in potentially forming Padé approximants in a similar fashion by successive ratio. If the general ratio $S_n(x)/S_{n+1}(x)$ is to be such an approximant then, since by Corollary 6.18 $S_n(x) - S(x)S_{n+1}(x) = x^{n+1}\Theta_n^S(x)$ (where $\Theta_n^S(x) = -S^{n+2}(x) \in \mathbb{Z}[[x]]$), we must have that $\deg\{S_n(x)\} + \deg\{S_{n+1}(x)\} = n$. This is never true, however, for $\deg\{S_n(x)\} = n$, and so we conclude that *for $n \geq 0$ $S_n(x)/S_{n+1}(x)$ is not a Padé approximant of $S(x)$.*

6.5.2 Approximation of the Motzkin Polynomials

Corollary 6.19, noting that $\deg\{M_n(x)\} = \lfloor\lfloor\frac{1}{3}n\rfloor + \frac{1}{3}(2n+1)\rfloor$, leads with a little work to the following conclusion: *for $n \geq 0$ the ratio $M_n(x)/M_{n+1}(x)$ of Motzkin polynomials is not a Padé approximant of $M(x)$ excepting those values $n = 0, 3, 6, 9, \ldots,$ when it is an approximant of order $(n, n+1)$.* In this latter instance, setting $n = 3k$ $(k = 0, 1, 2, \ldots,)$ it is elementary to show that $\deg\{M_n(x)\} + \deg\{M_{n+1}(x)\} = n + (n+1) = 2n+1$, which is the required condition.[3] Noting that $M_0(x) = 1$ is degree zero, the sequence of Motzkin polynomial degrees $\{\deg\{M_n(x)\}\}_1^\infty = \{1, 1, 3, 4, 4, 6, 7, 7, 9, \ldots\}$ is listed as O.E.I.S. sequence no. A117571, being those coefficients of terms in the expansion of $(1+2x^2)/[(1-x)(1-x^3)]$.

Remark 6.20. Those Motzkin polynomial ratios $M_{3n}(x)/M_{3n+1}(x)$, $n = 0, 1, 2,$ *etc.*, shown to be Padé approximants of $M(x)$, satisfy (6.72) in that the lead term of $\alpha_n(x)$ generally, and so $M_n(x)$ in particular, have $\forall n \geq 1$ a (constant) lead term of unity.

Remark 6.21. We emphasise, for clarity, that the definition of a Padé approximant used here seems to be a standard one, with the non-zero lead term of $\Theta(x)$ referring implicitly to the *constant* term therein. Clearly, since the difference $S_n(x) - S(x)S_{n+1}(x)$ is a power series in terms of only $O(x^{n+1})$ and higher, the ratio $S_n(x)/S_{n+1}(x)$—when written in the form of (6.72)—can never provide an approximant to $S(x)$ without $\Theta(x)$ containing inverse powers of x which violates the definition of an approximant (we would have to write $S_n(x) - S(x)S_{n+1}(x) = x^{2(n+1)}\Theta(x)$, where $\Theta(x) = -S^{n+2}(x)/x^{n+1}$). We clarify the conclusion made for ratios of successive Motzkin polynomials as approximants to $M(x)$ by noting that the difference $M_n(x) - M(x)M_{n+1}(x)$—being a series comprising terms in $x^{2(n+1)}$ and higher order—offers an extra power of x to $\Theta(x)$ when n is not a multiple of 3, in which cases the lead term of $\Theta(x)$ would fail to possess a standalone constant; by the definition we are using this is not a Padé approximant, although it could possibly be regarded as an approximant of different type (actually offering an extra single term of agreement between $M_n(x)/M_{n+1}(x)$ and $M(x)$).

Directly from Lemma 6.17 it is possible to deduce the following statement:

Corollary 6.22. *Let $N(x)$ be an o.g.f. satisfying the quadratic* $0 = A(x)N^2(x) + B(x)N(x) + C(x)$*—where $A(x), B(x), C(x) \in \mathbb{Z}[x]$—from which is defined the general polynomial* $\alpha_n(x) =$

[3] Corollary 6.19 gives, writing $\Theta_n^M(x) = -M^{n+2}(x) \in \mathbf{Z}[[x]]$, that the difference $M_n(x) - M(x)M_{n+1}(x)$ has the form $x^{2(n+1)}\Theta_n^M(x)$.

$(1,0)\mathbf{M}(x)\binom{1}{0}$, *with* $\mathbf{M}(x)$ *as in (6.16). Then, if* $A(x), C(x)$ *are of particular form* $A(x) = x^a$, $C(x) = c$, *for integers* $a \geq 1$, $c \neq 0$, *the ratio* $\alpha_n(x)/\alpha_{n+1}(x)$ *is, for* $n \geq 0$, *a Padé approximant of* $\frac{1}{c}N(x)$ *of order* $(\deg\{\alpha_n(x)\}, \deg\{\alpha_{n+1}(x)\})$ *if* $\deg\{\alpha_n(x)\} + \deg\{\alpha_{n+1}(x)\} = a(n+1) - 1$.

Corollary 6.22 recovers those conditions pertaining to the Catalan, Schröder and Motzkin polynomials as forming ratio approximants to their respective o.g.fs $G(x)$, $S(x)$ and $M(x)$ (for which $c = 1$ in each case and values of a are, in order, $a = 1, 1, 2$).

6.6 Summary

This chapter marks the conclusion of our study into the development of generalised algebraic Householder schemes for integer sequences, and as such, a few final observations will now be made regarding the results presented here, along with some speculative remarks regarding topics appropriate for future investigation.

Given a sequence o.g.f. $N(x)$ which satisfies the general quadratic $0 = A(x)N^2(x) + B(x)N(x) + C(x)$ $(A(x), B(x), C(x) \in \mathbb{Z}[x])$, we have developed here some theory which can be applied to any such sequence (of which there are many in existence), showing that associated polynomial families exist and how—in the case when $C(x) = 1$—non-linear identities are generated naturally from so-called Householder schemes when applied algebraically. Instances relating to the Schröder and Motzkin sequences have been presented in full, taking place within the framework of results given in previous work on Catalan polynomials to which reference has been made throughout. It remains to be seen

- if sequences whose o.g.fs satisfy a cubic equation with functional coefficients, or one of higher order, lend themselves to the type of treatment set out here through any associated polynomial families;
- if polynomial families such as those seen here have any application in the context of producing iterated generating functions as described in Chapter 3 (for the Catalan sequence).

Chapter 7

Further Identities for Polynomial Families

7.1 Introduction

In this chapter, we utilise some of the recurrence properties known to hold for the Catalan polynomials in the construction of an additional set of identities involving both the polynomials and their derivatives. Following the work presented in Chapter 6, we are also in a position to derive new identities for generalised polynomial families, the Schöder and Motzkin polynomials being two specific instances which we use as examples.

One identity involving derivatives of the Catalan polynomials has already arisen, this being (4.25) (originally from Lidl *et al.* (1993)):

$$0 = n(n-1)P_n(x) + [n + 2(3-2n)x]\frac{dP_n(x)}{dx} + x(4x-1)\frac{d^2P_n(x)}{dx^2}. \tag{7.1}$$

7.2 Identities for Catalan Polynomials

7.2.1 Summary of Properties

As we have seen in previous chapters, various forms exist for the general $(n+1)$th Catalan polynomial. Summarising those discovered so far, for $n \geq 0$ we have the binomial sum

$$P_n(x) = \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i} (-x)^i, \tag{7.2}$$

the hypergeometric series form

$$P_n(x) = {}_2F_1\left(\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}(n-1) \\ -n \end{matrix} \,\middle|\, 4x\right), \tag{7.3}$$

the two matrix forms

$$P_n(x) = \left(\sqrt{x}\right)^n \left(1, 1/\sqrt{x}\right) \begin{pmatrix} 0 & -1 \\ 1 & \frac{1}{\sqrt{x}} \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.4}$$

and

$$P_n(x) = (1,0) \begin{pmatrix} 1 & x \\ -1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{7.5}$$

and closed form

$$P_n(x) = \frac{1}{2^{n+1}} \frac{\left(1+\sqrt{1-4x}\right)^{n+1} - \left(1-\sqrt{1-4x}\right)^{n+1}}{\sqrt{1-4x}}, \tag{7.6}$$

this last being derived from the basic linear recurrence

$$0 = xP_n(x) - P_{n+1}(x) + P_{n+2}(x); \qquad P_0(x) = P_1(x) = 1. \tag{7.7}$$

The bi-variate function

$$\frac{1}{1-t+xt^2} = \sum_{n=0}^{\infty} P_n(x) t^n \tag{7.8}$$

has also been found to act as an o.g.f. for the Catalan polynomials.

7.2.2 Identities

Identity I

$$\frac{d}{dx}[P_{n+1}(x) - xP_{n-1}(x)] = -(n+1)P_{n-1}(x), \qquad n \geq 1.$$

Proof. Consider, from (7.2),

$$\begin{aligned} P_{n+1}(x) - xP_{n-1}(x) &= \sum_{i=0}^{\lfloor \frac{1}{2}(n+1) \rfloor} \binom{n+1-i}{i} (-x)^i - x \sum_{i=0}^{\lfloor \frac{1}{2}(n-1) \rfloor} \binom{n-1-i}{i} (-x)^i \\ &= \sum_{i=0}^{\lfloor \frac{1}{2}(n+1) \rfloor} \binom{n+1-i}{i} (-x)^i + \sum_{i=0}^{\lfloor \frac{1}{2}(n-1) \rfloor} \binom{n-1-i}{i} (-x)^{i+1} \\ &= \sum_{i=0}^{\lfloor \frac{1}{2}(n+1) \rfloor} \binom{n+1-i}{i} (-x)^i + \sum_{i=1}^{\lfloor \frac{1}{2}(n+1) \rfloor} \binom{n-i}{i-1} (-x)^i \\ &= 1 + \sum_{i=1}^{\lfloor \frac{1}{2}(n+1) \rfloor} \left[\binom{n+1-i}{i} + \binom{n-i}{i-1}\right] (-x)^i \\ &= 1 + (n+1) \sum_{i=1}^{\lfloor \frac{1}{2}(n+1) \rfloor} \frac{1}{i} \binom{n-i}{i-1} (-x)^i. \end{aligned} \tag{7.9}$$

Thus,

$$\begin{aligned}\frac{d}{dx}[P_{n+1}(x) - xP_{n-1}(x)] &= -(n+1) \sum_{i=1}^{\lfloor \frac{1}{2}(n+1) \rfloor} \binom{n-i}{i-1} (-x)^{i-1} \\ &= -(n+1) \sum_{i=0}^{\lfloor \frac{1}{2}(n-1) \rfloor} \binom{n-1-i}{i} (-x)^{i} \\ &= -(n+1)P_{n-1}(x). \ \square \qquad (7.10)\end{aligned}$$

Identity II

$$\frac{dP_n(x)}{dx} = -\frac{1}{2^{n-1}} \sum_{i=0}^{n-2} (i+2) 2^i P_i(x), \qquad n \geq 2.$$

Proof. Identity I can be re-written trivially using (7.7) in the form

$$(n+1)P_{n-1}(x) = \frac{d}{dx}[P_n(x) - 2P_{n+1}(x)]. \qquad (7.11)$$

Denoting by $'$ a derivative with respect to x, this gives, employing it for values of $n \geq 1$,

$$\begin{aligned} 2^0 \cdot 2P_0(x) &= 2^0 P_1'(x) - 2^1 P_2'(x), \\ 2^1 \cdot 3P_1(x) &= 2^1 P_2'(x) - 2^2 P_3'(x), \\ 2^2 \cdot 4P_2(x) &= 2^2 P_3'(x) - 2^3 P_4'(x), \\ &\vdots \\ 2^{n-3} \cdot (n-1)P_{n-3}(x) &= 2^{n-3} P_{n-2}'(x) - 2^{n-2} P_{n-1}'(x), \\ 2^{n-2} \cdot nP_{n-2}(x) &= 2^{n-2} P_{n-1}'(x) - 2^{n-1} P_n'(x), \qquad (7.12) \end{aligned}$$

and adding the l.h.s. terms creates a "telescoping" effect in the r.h.s., so that

$$\sum_{i=0}^{n-2} 2^i \cdot (i+2)P_i(x) = 2^0 P_1'(x) - 2^{n-1} P_n'(x) = -2^{n-1} P_n'(x), \qquad (7.13)$$

whence the result. $\square$

Remark 7.1. We can evaluate Identity II at $x = 0$, which, noting that for $n \geq 2$, $P_n(0) = 1$ and $P_n'(0) = -(n-1)$, then contracts to $\sum_{i=0}^{n} (i+2)2^i = (n+1)2^{n+1}$ or, rearranging (noting that the sum $\sum_{i=0}^{n} 2^i$ is but a geometric series with closed form $2^{n+1} - 1$), the simpler result $\sum_{i=0}^{n} i2^i = 2[1 + (n-1)2^n]$.

Identity III

$$(1-4x)\frac{dP_n(x)}{dx} = (n+1)P_{n-1}(x) - 2nP_n(x), \qquad n \geq 1.$$

Proof. We first establish a couple of subsidiary identities to which we make appeal as part of the proof. Let $\mathbf{N}$ be a $k \times k$ matrix such that $|\mathbf{N} - \mathbf{I}_k| \neq 0$, where $\mathbf{I}_k$ is the $k \times k$ identity matrix. Then

$$\sum_{i=0}^{n} \mathbf{N}^i = (\mathbf{N}^{n+1} - \mathbf{I}_k)(\mathbf{N} - \mathbf{I}_k)^{-1} \tag{7.14}$$

and

$$\sum_{i=0}^{n} i\mathbf{N}^i = (n\mathbf{N}^{n+2} - (n+1)\mathbf{N}^{n+1} + \mathbf{N})(\mathbf{N} - \mathbf{I}_k)^{-2}. \tag{7.15}$$

Utilising the fact that $P_n(x)$ has the matrix form (7.5), we set for convenience $\mathbf{R}$ to be the row vector $(1, 0)$, $\mathbf{M} = \mathbf{M}(x)$ to be the 2×2 matrix $(1, x; -1, 0)$ and $\mathbf{C}$ to be the column vector $(1, 0)^T$, so that $P_n(x) = \mathbf{R}\mathbf{M}^n\mathbf{C}$.

This allows us to write, from Identity II,

$$\begin{aligned} -2^{n-1}P_n'(x) &= \sum_{i=0}^{n-2} (i+2)2^i P_i(x) \\ &= \mathbf{R}\left[\sum_{i=0}^{n-2} (i+2)2^i \mathbf{M}^i\right]\mathbf{C} \\ &= \mathbf{R}f(\mathbf{M}, n)\mathbf{C}, \end{aligned} \tag{7.16}$$

where $f(\mathbf{M}, n) = f(\mathbf{M}(x), n) = \sum_{i=0}^{n-2}(i+2)2^i\mathbf{M}^i$. We first split $f(\mathbf{M}, n)$ into a sum of terms

$$f(\mathbf{M}, n) = \sum_{i=0}^{n-2} i(2\mathbf{M})^i + 2\sum_{i=0}^{n-2}(2\mathbf{M})^i. \tag{7.17}$$

Directly from (7.15) we have

$$\sum_{i=0}^{n-2} i(2\mathbf{M})^i = [(n-2)(2\mathbf{M})^n - (n-1)(2\mathbf{M})^{n-1} + 2\mathbf{M}](2\mathbf{M} - \mathbf{I}_2)^{-2}, \tag{7.18}$$

and likewise (7.14) yields

$$\begin{aligned} \sum_{i=0}^{n-2}(2\mathbf{M})^i &= [(2\mathbf{M})^{n-1} - \mathbf{I}_2](2\mathbf{M} - \mathbf{I}_2)^{-1} \\ &= [(2\mathbf{M})^{n-1} - \mathbf{I}_2](2\mathbf{M} - \mathbf{I}_2)(2\mathbf{M} - \mathbf{I}_2)^{-2}, \end{aligned} \tag{7.19}$$

together giving

$$\begin{aligned} f(\mathbf{M}, n) &= [n(2\mathbf{M})^n - (n+1)(2\mathbf{M})^{n-1} - 2\mathbf{M} + 2\mathbf{I}_2](2\mathbf{M} - \mathbf{I}_2)^{-2} \\ &= \frac{n(2\mathbf{M})^n - (n+1)(2\mathbf{M})^{n-1} - 2\mathbf{M} + 2\mathbf{I}_2}{1 - 4x}, \end{aligned} \tag{7.20}$$

since

$$(2\mathbf{M}-\mathbf{I}_2)^{-2} = \begin{pmatrix} 1 & 2x \\ -2 & -1 \end{pmatrix}^{-2} = \frac{1}{1-4x}\mathbf{I}_2. \tag{7.21}$$

Finally, we back-substitute (7.20) in (7.16) to complete the proof:

$$\begin{aligned}
-2^{n-1}(1-4x)P_n'(x) &= \mathbf{R}[n(2\mathbf{M})^n - (n+1)(2\mathbf{M})^{n-1} - 2\mathbf{M} + 2\mathbf{I}_2]\mathbf{C} \\
&= -\mathbf{R}[(n+1)(2\mathbf{M})^{n-1} + 2\mathbf{M} - n(2\mathbf{M})^n - 2\mathbf{I}_2]\mathbf{C} \\
&= -[(n+1)2^{n-1}\mathbf{R}\mathbf{M}^{n-1}\mathbf{C} + 2\mathbf{R}\mathbf{M}\mathbf{C} - n2^n\mathbf{R}\mathbf{M}^n\mathbf{C} - 2\mathbf{R}\mathbf{I}_2\mathbf{C}] \\
&= -[(n+1)2^{n-1}P_{n-1}(x) + 2P_1(x) - n2^nP_n(x) - 2P_0(x)] \\
&= -2^{n-1}[(n+1)P_{n-1}(x) - 2nP_n(x)]. \ \square
\end{aligned} \tag{7.22}$$

Identity IV

$$(n+2)x\frac{dP_n(x)}{dx} = (n+1)\frac{dP_{n+1}(x)}{dx} - n\frac{dP_{n+2}(x)}{dx}, \qquad n \geq 0.$$

Proof. Recalling (7.11),

$$(n+1)P_{n-1}(x) = P_n'(x) - 2P_{n+1}'(x), \tag{7.23}$$

or, equivalently,

$$(n+2)P_n(x) = P_{n+1}'(x) - 2P_{n+2}'(x), \tag{7.24}$$

we can substitute both of these into Identity III to read

$$(1-4x)P_n'(x) = P_n'(x) - 2P_{n+1}'(x) - 2n\frac{1}{n+2}[P_{n+1}'(x) - 2P_{n+2}'(x)], \tag{7.25}$$

which gives the result after a little rearrangement. $\square$

7.3 Identities for Generalised Polynomials

7.3.1 Summary of Properties

Any sequence of integers whose o.g.f. $N(x)$ satisfies the quadratic

$$0 = A(x)N^2(x) + B(x)N(x) + C(x), \tag{7.26}$$

with functional coefficients $A(x), B(x), C(x) \in \mathbb{Z}[x]$, can be considered to give rise naturally to a family of associated polynomials

$$\begin{aligned}
\alpha_n(x) &= \alpha_n(A(x), B(x), C(x)) \\
&= (1,0)\begin{pmatrix} -B(x) & A(x) \\ -C(x) & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\end{aligned} \tag{7.27}$$

the first few of which have explicit general form

$$
\begin{aligned}
\alpha_0(x) &= 1,\\
\alpha_1(x) &= -B(x),\\
\alpha_2(x) &= B^2(x) - A(x)C(x),\\
\alpha_3(x) &= 2A(x)B(x)C(x) - B^3(x),\\
\alpha_4(x) &= B^4(x) - 3A(x)B^2(x)C(x) + A^2(x)C^2(x),\\
\alpha_5(x) &= 4A(x)B^3(x)C(x) - 3A^2(x)B(x)C^2(x) - B^5(x),
\end{aligned} \tag{7.28}
$$

etc.; for the specialisation $A(x) = x$, $B(x) = -1$, $C(x) = 1$ then (7.26) becomes the familiar equation $0 = xN^2(x) - N(x) + 1$ for the Catalan sequence o.g.f. and (7.27) contracts to (7.5) ($\alpha_n(x, -1, 1) = P_n(x)$, with (7.28) recovering the Catalan polynomials $P_0(x) = P_1(x) = 1$, $P_2(x) = 1 - x$, $P_3(x) = 1 - 2x$, $P_4(x) = 1 - 3x + x^2$, $P_5(x) = 1 - 4x + 3x^2, \ldots$).

The Schröder and Motzkin polynomials are recoverable via the specialisations $A(x) = x$, $B(x) = x - 1$, $C(x) = 1$ and $A(x) = x^2$, $B(x) = x - 1$, $C(x) = 1$, respectively, or in other words, $S_n(x) = \alpha_n(x, x-1, 1)$ and $M_n(x) = \alpha_n(x^2, x-1, 1)$.

It has also previously been shown that the generalised polynomials can be expressed directly in terms of the Catalan polynomials according to

$$\alpha_n(x) = [-B(x)]^n P_n\left(\frac{A(x)C(x)}{B^2(x)}\right). \tag{7.29}$$

7.3.2 Identities

Identity V

For $n \geq 1$,

$$\frac{d}{dx}\{\alpha_n(x)\} = [(n+1)\omega_1(x)]\alpha_{n-1}(x) + [n\omega_2(x)]\alpha_n(x),$$

where

$$
\begin{aligned}
\omega_1(x) &= \frac{2A(x)C(x)\frac{d}{dx}\{B(x)\} - B(x)[A(x)\frac{d}{dx}\{C(x)\} + C(x)\frac{d}{dx}\{A(x)\}]}{B^2(x) - 4A(x)C(x)},\\
\omega_2(x) &= \frac{B(x)\frac{d}{dx}\{B(x)\} - 2[A(x)\frac{d}{dx}\{C(x)\} + C(x)\frac{d}{dx}\{A(x)\}]}{B^2(x) - 4A(x)C(x)}.
\end{aligned}
$$

Proof. Our starting point is (7.29). Differentiating both sides w.r.t. x, and denoting by $P_n'(A(x)C(x)/B^2(x))$ the derivative of P_n w.r.t. its argument, we obtain

$$\begin{aligned}\frac{d}{dx}\{\alpha_n(x)\} &= n[-B(x)]^{n-1}\frac{d}{dx}\{-B(x)\}P_n\left(\frac{A(x)C(x)}{B^2(x)}\right)\\ &\qquad + [-B(x)]^n P_n'\left(\frac{A(x)C(x)}{B^2(x)}\right)\frac{d}{dx}\left\{\frac{A(x)C(x)}{B^2(x)}\right\}\\ &= n\frac{1}{B(x)}\frac{d}{dx}\{B(x)\}\alpha_n(x)\\ &\qquad + [-B(x)]^n P_n'\left(\frac{A(x)C(x)}{B^2(x)}\right)\frac{d}{dx}\left\{\frac{A(x)C(x)}{B^2(x)}\right\}\end{aligned} \tag{7.30}$$

using (7.29), which is rearranged to

$$P_n'\left(\frac{A(x)C(x)}{B^2(x)}\right) = \frac{\frac{d}{dx}\{\alpha_n(x)\} - n\frac{1}{B(x)}\frac{d}{dx}\{B(x)\}\alpha_n(x)}{[-B(x)]^n\frac{d}{dx}\left\{\frac{A(x)C(x)}{B^2(x)}\right\}}, \tag{7.31}$$

holding for $n \geq 0$. Now, from Identity III,

$$\frac{d}{dx}\{P_n(x)\} = \frac{(n+1)P_{n-1}(x) - 2nP_n(x)}{1-4x}, \qquad n \geq 1, \tag{7.32}$$

which as $x \to A(x)C(x)/B^2(x)$ reads

$$P_n'\left(\frac{A(x)C(x)}{B^2(x)}\right) = \frac{B^2(x)[(n+1)P_{n-1}(A(x)C(x)/B^2(x)) - 2nP_n(A(x)C(x)/B^2(x))]}{B^2(x)-4A(x)C(x)}, \tag{7.33}$$

whence, equating (7.31),(7.33) and employing (7.29) once more,

$$\begin{aligned}&\frac{d}{dx}\{\alpha_n(x)\} - n\frac{1}{B(x)}\frac{d}{dx}\{B(x)\}\alpha_n(x)\\ &\qquad = -B^2(x)\frac{d}{dx}\left\{\frac{A(x)C(x)}{B^2(x)}\right\}\frac{(n+1)B(x)\alpha_{n-1}(x) + 2n\alpha_n(x)}{B^2(x)-4A(x)C(x)}\end{aligned} \tag{7.34}$$

after a little work; finally, noting that

$$\frac{d}{dx}\left\{\frac{A(x)C(x)}{B^2(x)}\right\} = \frac{B(x)[A(x)\frac{d}{dx}\{C(x)\} + C(x)\frac{d}{dx}\{A(x)\}] - 2A(x)C(x)\frac{d}{dx}\{B(x)\}}{B^3(x)}, \tag{7.35}$$

(7.34) leads to Identity V after further algebraic manipulation. □

Instances of Identity V

The Catalan specialisation of Identity V is merely (7.32), whilst versions for the aforementioned Schröder and Motzkin polynomials are seen to be, for $n \geq 1$,

$$(1-6x+x^2)\frac{d}{dx}\{S_n(x)\} = [(n+1)(1+x)]S_{n-1}(x) - [n(3-x)]S_n(x), \tag{7.36}$$

and

$$(1-2x-3x^2)\frac{d}{dx}\{M_n(x)\} = [2(n+1)x]M_{n-1}(x) - [n(1+3x)]M_n(x), \tag{7.37}$$

both of which have been verified computationally using the algebraic closed forms for $S_n(x)$, $M_n(x)$ delivered by (7.27); many other examples could, of course, have been given here.

Identity VI

For $n \geq 1$,

$$-(n+1)\frac{\omega_1^n(x)}{\omega_2(x)}\sum_{i=1}^{n}\left(-\frac{\omega_2(x)}{\omega_1(x)}\right)^i \frac{1}{i(i+1)}\frac{d}{dx}\{\alpha_i(x)\} = (n+1)\omega_1^n(x) + [(-1)^{n+1}\omega_2^n(x)]\alpha_n(x).$$

Proof. It can be shown inductively (though the details are omitted here for brevity) that an arbitrary recurrence

$$y_n = \delta_n z_{n-1} + \varepsilon_n z_n, \qquad n \geq 1, \tag{7.38}$$

leads to the following identity:

$$\sum_{i=1}^{n}\left[\prod_{j=i+1}^{n}\delta_j \prod_{l=1}^{i-1}\varepsilon_l\right](-1)^{i+1}y_i = z_0\prod_{j=1}^{n}\delta_j + (-1)^{n+1}z_n\prod_{l=1}^{n}\varepsilon_l. \tag{7.39}$$

Setting $y_n = y_n(x) = \frac{d}{dx}\{\alpha_n(x)\}$, $z_n = z_n(x) = \alpha_n(x)$, $\delta_j = \delta_j(x) = (j+1)\omega_1(x)$ and $\varepsilon_l = \varepsilon_l(x) = l\omega_2(x)$ (so that (7.38) reads as Identity V), and noting that by definition (7.27) $\alpha_0(x) = 1 = z_0$, Identity VI is yielded directly from (7.39) after elementary simplification of both sides. □

Instances of Identity VI

The Catalan, Schröder and Motzkin polynomial instances of the result are, for $n \geq 1$,

$$(n+1)(1-4x)\sum_{i=1}^{n}\frac{2^{i-1}}{i(i+1)}\frac{d}{dx}\{P_i(x)\} = n+1-2^nP_n(x), \tag{7.40}$$

$$(n+1)(1-6x+x^2)(1+x)^n\sum_{i=1}^{n}\left(\frac{3-x}{1+x}\right)^i\frac{1}{i(i+1)}\frac{d}{dx}\{S_i(x)\}$$
$$= (3-x)\{(n+1)(1+x)^n - (3-x)^nS_n(x)\}, \tag{7.41}$$

$$(n+1)(1-2x-3x^2)(2x)^n\sum_{i=1}^{n}\left(\frac{1+3x}{2x}\right)^i\frac{1}{i(i+1)}\frac{d}{dx}\{M_i(x)\}$$
$$= (1+3x)\{(n+1)(2x)^n - (1+3x)^nM_n(x)\}. \tag{7.42}$$

As before, combining (7.40), (7.41) and (7.42) with the closed forms for $P_n(x)$, $S_n(x)$ and $M_n(x)$ respectively allows for a general computational verification of each of the above, supporting the proof.

7.4 Summary

In this chapter, development of previous work on Catalan polynomials has led to the formulation of six new identities; four for the Catalan polynomials themselves, and two additional identities for a general polynomial family, in which the Catalan, Schröder and Motzkin polynomials feature as special cases. All identities involve polynomial derivatives, with some displaying a considerable level of complexity. Note that the derivative element $\frac{d}{dx}\{\alpha_i(x)\}$ within the sum of Identity VI can be replaced with a corresponding expression in $\alpha_{i-1}(x), \alpha_i(x)$ directly from Identity V to create an additional identity in which derivatives are absent; in some sense the result is one in which a "telescoping" effect is evident.

Chapter 8

Summary and Conclusions

In this thesis, an investigation into the analysis of so-called iterated generating functions, and the schemes that produce them, has been made. The presentation has been set out in an orderly way which reflects the natural chronology of the work. The primary results obtained in each chapter can be summarised thus:

Starting with the notion of an iterated generating function, the basic iterative method demonstrated by Larcombe and Fennessey (1999) has been shown to have an application in generating integer sequences other than the Catalan sequence.

In Chapter 2, arbitrary finite sequences have been briefly examined in context, and a method devised for reproducing such sequences by means of first-order recurrence schemes.

The focus switches in Chapter 3 to infinite sequences, where an algorithm is developed for recovering the recurrence scheme of a suitable sequence. This leads to the emergence of the Catalan polynomials, these being of particular interest not simply due to their role in creating Catalan sequence iterated generating function schemes, but also due to their properties which have been noted (using existing results from the literature) and further developed in Chapter 4.

In Chapters 5 and 6, the creation of non-linear Catalan polynomial identity pairs through the algebraic execution of the suite of (numeric) Householder root algorithms has been presented, and the Catalan polynomials themselves have been shown to have applications in providing Padé approximants to the Catalan sequence o.g.f. Moreover, this phenomenon is sometimes repeated for other polynomials associated with their namesake sequences.

To conclude, a number of identities linking the Catalan (and generalised) polynomials with their derivatives have been presented in Chapter 7, and verified computationally for the Catalan, Schröder and Motzkin instances.

As a general remark, it is difficult to overstate the usefulness of computer algebra systems in this type of work. Without the availability of Maple, the CAS package used here, or a similar alternative, it is reasonable to assume that the majority of the computations presented in this thesis could not otherwise have been performed, the complexity of the algorithms through which they are generated precluding the possibility of their derivation by hand. Indeed, many of the theorems discussed in this work were initially formulated using empirical evidence gained through computational experimentation.

It is clear that where areas suitable for future study are concerned, a number of opportunities exist. Apart from the specifics noted at the end of each chapter, one obvious general possibility would be to re-examine all of the topics covered in this thesis in relation to sequences whose governing o.g.fs are cubic or higher degree, but some of the concepts considered here seem non-trivial to extend in this way. However, even though we have restricted our attention to iterated generating functions schemes for sequences with a quadratic o.g.f. equation, we have nonetheless found many interesting problems to examine, with features which will be of interest within the general area of discrete mathematics.

In addition, two areas of immediate further interest have arisen as a direct consequence of this study (although neither involves iterated generating functions in anything more than a peripheral role): firstly, ongoing work concerning the Catalan polynomials, where particular topics of interest are their factorisation/divisibility properties, their generalisation as polynomials, and their role in the construction and analysis of periodic sequences; and secondly, the development of closed forms (based on matrices and binomial sums) for the Householder-derived identities presented in Chapter 6, which have since been published as Clapperton *et al.* (2012).

Finally, it is worth noting that not all sequences can be produced by means of iterated generating functions since there are only a countable number of schemes and an uncountable number of sequences. This means that there must exist “impossible” sequences with no generating schemes of the type considered here; a class of such sequences is established in Clapperton *et al.* (2011b).

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Appendices

Appendix A

In this exercise we apply (3.21), matching coefficients of terms of the polynomials $G_1(x), G_2(x)$, $G_3(x), \ldots,$ to Catalan sequence terms as before, but this time truncating the polynomials *en route*. First, we clearly have $[x^0]\{G_1(x)\} = [x^0]\{\alpha + \beta x\} = \alpha = c_0 = 1$ once more. This time, however, we use $G_1(x) = 1$ (truncated) in the next iteration, which is

$$\begin{aligned} G_2(x) &= 1 + \beta x + (\gamma + \delta x)G_1(x) + (\varepsilon + \zeta x)G_1^2(x) \\ &= 1 + \gamma + \varepsilon + (\beta + \delta + \zeta)x, \end{aligned} \tag{A1}$$

giving $[x^0]\{G_2(x)\} = c_0 = 1 = 1 + \gamma + \varepsilon$, *i.e.*,

$$0 = \gamma + \varepsilon, \tag{A2}$$

and

$$[x^1]\{G_2(x)\} = c_1 = 1 = \beta + \delta + \zeta. \tag{A3}$$

With $G_2(x) = 1 + x$ in correct truncated form,

$$\begin{aligned} G_3(x) &= 1 + \beta x + (\gamma + \delta x)G_2(x) + (\varepsilon + \zeta x)G_2^2(x) \\ &= 1 + \gamma + \varepsilon + (\beta + \gamma + \delta + 2\varepsilon + \zeta)x + (\delta + \varepsilon + 2\zeta)x^2 + \zeta x^3 \\ &= 1 + (1 + \gamma + 2\varepsilon)x + (\delta + \varepsilon + 2\zeta)x^2 + \zeta x^3, \end{aligned} \tag{A4}$$

employing (A2),(A3), from which we may write $[x^1]\{G_3(x)\} = c_1 = 1 = 1 + \gamma + 2\varepsilon$, *i.e.*,

$$0 = \gamma + 2\varepsilon, \tag{A5}$$

and

$$[x^2]\{G_3(x)\} = c_2 = 2 = \delta + \varepsilon + 2\zeta. \tag{A6}$$

Now $\gamma = \varepsilon = 0$ by (A2) and (A5), whence (A6) becomes

$$\delta + 2\zeta = 2. \tag{A7}$$

Equations (A3),(A7) are those of (3.27). If we now modify the recurrence accordingly and impose it—with $G_2(x) = 1 + x + 2x^2$ (truncated)—to generate $G_4(x)$, we merely reproduce (3.29) (see (3.28)), and in turn the scheme (3.30) from the repeat β, δ, ζ solution.

Appendix B

In this section, we verify Lemma 4.3, noting that the case $n = 1$ is already accounted for in its proof; those steps in any individual case which draw on a previous one are so obvious as to be left unmarked:

$\underline{n = 2}$:

Rearranging the quadratic (1.9) it reads trivially

$$xC^2(x) = C(x) - 1 = P_1(x)C(x) - P_0(x). \tag{B1}$$

$\underline{n = 3}$:

Again (1.9) gives

$$\begin{aligned} x^2C^3(x) &= xC^2(x) - xC(x) \\ &= C(x) - 1 - xC(x) \\ &= (1-x)C(x) - 1 \\ &= P_2(x)C(x) - P_1(x). \end{aligned} \tag{B2}$$

$\underline{n = 4}$:

Again (1.9) gives

$$\begin{aligned} x^3C^4(x) &= x^2C^3(x) - x^2C^2(x) \\ &= x^2C^3(x) - x[xC^2(x)] \\ &= (1-x)C(x) - 1 - x[C(x) - 1] \\ &= (1-2x)C(x) - (1-x) \\ &= P_3(x)C(x) - P_2(x). \end{aligned} \tag{B3}$$

$\underline{n = 5}$:

Again (1.9) gives

$$\begin{aligned} x^4C^5(x) &= x^3C^4(x) - x^3C^3(x) \\ &= x^3C^4(x) - x[x^2C^3(x)] \\ &= (1-2x)C(x) - (1-x) - x[(1-x)C(x) - 1] \\ &= (1-3x+x^2)C(x) - (1-2x) \\ &= P_4(x)C(x) - P_3(x). \end{aligned} \tag{B4}$$

Instances of Lemma 4.3 for $n > 5$ are dealt with similarly.

Appendix C

For completeness, two additional proofs of Theorem 6.1 are presented here, the first of which utilises analytic function theory and originates from Clapperton *et al.* (2010, Appendix A), its inclusion therein having been suggested by a referee.

Proof I. Consider the recurrence (4.5). Replacing x by u and defining, for $n \geq 0$, $Q_n(u,v) = (-v)^n P_n(u)$ it reads

$$0 = uv^2 Q_n(u,v) + vQ_{n+1}(u,v) + Q_{n+2}(u,v). \tag{C1}$$

Denoting, w.r.t. a general complex function $F(z)$, say, $\mathcal{D}_n(z)$ to be

$$\mathcal{D}_n(z) = \frac{1}{n!}\frac{d^n}{dz^n}\left\{\frac{1}{F(z)}\right\}, \tag{C2}$$

the standard theory of analytic functions gives that

$$\mathcal{D}_n(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{d\zeta}{(\zeta - z)^{n+1}F(\zeta)}, \tag{C3}$$

where $\mathcal{C}$ is a small circle around z (fixed) in the complex plane, and $\mathcal{D}_0, \mathcal{D}_1$, $\mathcal{D}_2$, *etc.*, occur in the (Taylor) expansion

$$\frac{1}{F(\zeta)} = \sum_{n=0}^{\infty} \mathcal{D}_n(z)(\zeta - z)^n. \tag{C4}$$

For $F(z) = f(z) = f(z;x) = A(x)z^2 + B(x)z + C(x)$ quadratic in z, then (denoting the partial derivative $\partial/\partial z$ by $'$, and suppressing the dependency of f on x as in the main inductive proof) expanding $f(\zeta)$ about z we have

$$\begin{aligned} f(\zeta) &= f(z) + f'(z)(\zeta - z) + \frac{f''(z)}{2!}(\zeta - z)^2 \\ &= f(z) + f'(z)(\zeta - z) + A(x)(\zeta - z)^2, \end{aligned} \tag{C5}$$

upon which (C4) reads

$$\begin{aligned} 1 &= f(\zeta)\sum_{n=0}^{\infty}\mathcal{D}_n(z)(\zeta - z)^n \\ &= A(x)\sum_{n=0}^{\infty}\mathcal{D}_n(z)(\zeta - z)^{n+2} + f'(z)\sum_{n=0}^{\infty}\mathcal{D}_n(z)(\zeta - z)^{n+1} + f(z)\sum_{n=0}^{\infty}\mathcal{D}_n(z)(\zeta - z)^n, \end{aligned} \tag{C6}$$

in turn yielding the recurrence

$$0 = A(x)\mathcal{D}_n(z) + f'(z)\mathcal{D}_{n+1}(z) + f(z)\mathcal{D}_{n+2}(z), \qquad n \geq 0, \tag{C7}$$

subject to starting values $\mathcal{D}_0(z) = 1/f(z)$, $\mathcal{D}_1(z) = -f'(z)/f^2(z)$. Now, setting

$$u(z) = u(z;x) = \frac{A(x)f(z)}{f'^2(z)}, \qquad v(z) = \frac{f'(z)}{f(z)}, \tag{C8}$$

it follows from (C1) that $Q_n(z)/f(z)$ (that is to say, $Q_n(u(z),v(z))/f(z)$) also satisfies (C7), and since (checking the first two initial values are correct)

$$\begin{aligned}\frac{Q_0(z)}{f(z)} &= \frac{P_0(u(z))}{f(z)} = \frac{1}{f(z)} = \mathcal{D}_0(z),\\ \frac{Q_1(z)}{f(z)} &= \frac{-v(z)P_1(u(z))}{f(z)} = \frac{-f'(z)/f(z)}{f(z)} = \mathcal{D}_1(z),\end{aligned} \tag{C9}$$

then we have established that

$$\mathcal{D}_n(z) = \frac{Q_n(u(z),v(z))}{f(z)}, \qquad n \geq 0. \tag{C10}$$

Thus, with $u(z), v(z)$ as chosen in (C8), and quadratic $f(z) = A(x)z^2 + B(x)z + C(x)$, then $\frac{d^n}{dz^n}\{1/f(z)\} = n!\mathcal{D}_n(z) = n!Q_n(u(z),v(z))/f(z) = n![-v(z)]^n P_n(u(z))/f(z)$, and Theorem 6.1 is proven. □

Note, as a minor point of interest, that $u(z;x) = A(x)f(z)/f'^2(z)$ appears as $-Q(z;x)$ in the inductive proof of Theorem 6.1 (see (6.10) of Section 6.2.1), and also as $\hat{Q}(z;x)$ in the later proof of Lemma 6.6 (Section 6.2.2); we will also see $u(z), v(z)$ (C8) deployed as variables $\tau(z), \rho(z)$ in Proof II below.

Proof II. This proof is from first principles. The Taylor expansion for a function $t(z)$, say, about a general point z is (for small h)

$$\begin{aligned} t(z+h) &= t(z) + ht'(z) + \frac{h^2}{2!}t''(z) + \frac{h^3}{3!}t'''(z) + \cdots \\ &= \sum_{i=0}^{\infty} \frac{h^i}{i!}\frac{d^i}{dz^i}\{t(z)\}, \end{aligned} \tag{C11}$$

which in the case when $t(z) = f(z) = f(z;x) = A(x)z^2 + B(x)z + C(x)$ reduces to

$$f(z+h) = f(z) + hf'(z) + \frac{h^2}{2}f''(z), \tag{C12}$$

so that as a power series

$$\begin{aligned} \frac{1}{f(z+h)} &= \left\{f(z) + hf'(z) + \frac{h^2}{2}f''(z)\right\}^{-1} \\ &= \frac{1}{f(z)}\left\{1 + h\frac{f'(z)}{f(z)} + \frac{h^2}{2}\frac{f''(z)}{f(z)}\right\}^{-1} \\ &= \frac{1}{f(z)}\left\{1 + h\frac{f'(z)}{f(z)}\left[1 + h\frac{f'(z)}{f(z)}\frac{f(z)f''(z)}{2f'^2(z)}\right]\right\}^{-1} \\ &= \frac{1}{f(z)}\sum_{i=0}^{\infty}(-1)^i\left\{h\frac{f'(z)}{f(z)}\left[1 + h\frac{f'(z)}{f(z)}\frac{f(z)f''(z)}{2f'^2(z)}\right]\right\}^i. \end{aligned} \tag{C13}$$

Now by (C11) it follows that

$$\frac{1}{f(z+h)} = \sum_{i=0}^{\infty} \frac{d^i}{dz^i}\left\{\frac{1}{f(z)}\right\}\frac{h^i}{i!}, \tag{C14}$$

whence, by comparison with (C13) we have directly that, for $n \geq 0$,

$$\frac{d^n}{dz^n}\left\{\frac{1}{f(z)}\right\} = \frac{1}{f(z)}[h^n/n!]\{\Omega(h;z)\}, \tag{C15}$$

where

$$\Omega(h;z) = \sum_{i=0}^{\infty}(-1)^i\left\{h\frac{f'(z)}{f(z)}\left[1+h\frac{f'(z)}{f(z)}\frac{f(z)f''(z)}{2f'^2(z)}\right]\right\}^i. \tag{C16}$$

To pick out the desired coefficient of $h^n/n!$ in $\Omega(h;z)$ we simply write $\rho(z) = f'(z)/f(z)$, $\tau(z) = f(z)f''(z)/2f'^2(z)$, giving first

$$\begin{aligned}
\Omega(h;z) &= \sum_{i=0}^{\infty}(-1)^i h^i \rho^i(z)[1+h\rho(z)\tau(z)]^i \\
&= \sum_{i=0}^{\infty}(-1)^i h^i \rho^i(z)\sum_{j=0}^{i}\binom{i}{j}[h\rho(z)\tau(z)]^j \\
&= \sum_{i=0}^{\infty}(-1)^i h^i \rho^i(z)\sum_{j=0}^{\infty}\binom{i}{j}[h\rho(z)\tau(z)]^j \\
&= \sum_{i,j=0}^{\infty}(-1)^i \rho^{i+j}(z)\tau^j(z)\binom{i}{j}h^{i+j} \\
&= \sum_{s=0}^{\infty}\left\{\sum_{i+j=s}(-1)^i\tau^j(z)\binom{i}{j}\right\}(\rho(z)h)^s.
\end{aligned} \tag{C17}$$

Thus, for $n \geq 0$,

$$\begin{aligned}
[h^n/n!]\{\Omega(h;z)\} &= n!\rho^n(z)\sum_{i+j=n}(-1)^i\tau^j(z)\binom{i}{j} \\
&= n!\rho^n(z)\sum_{j=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^{n-j}\tau^j(z)\binom{n-j}{j} \\
&= [-\rho(z)]^n n!\sum_{j=0}^{\lfloor\frac{1}{2}n\rfloor}\binom{n-j}{j}[-\tau(z)]^j \\
&= [-\rho(z)]^n n! P_n(\tau(z)) \qquad \text{(by definition (4.1))} \\
&= [-f'(z)/f(z)]^n n! P_n(f(z)f''(z)/2f'^2(z)) \\
&= [-f'(z)/f(z)]^n n! P_n(A(x)f(z)/f'^2(z));
\end{aligned} \tag{C18}$$

combined with (C15), Theorem 6.1 is established. □

Appendix D

We establish Lemma 6.12 with a matrix-based argument. With reference to the matrix $\mathbf{M}(x)$ (6.16) and associated functions $\alpha_n(x), \beta_n(x)$ defined in (6.17), let functions $F_\alpha(x,y)$ and $F_\beta(x,y)$ be the respective o.g.fs for these polynomials $\alpha_n(x)$ and $\beta_n(x)$, *i.e.*,

$$\begin{aligned} F_\alpha(x,y) &= \sum_{n=0}^{\infty} \alpha_n(x) y^n, \\ F_\beta(x,y) &= \sum_{n=0}^{\infty} \beta_n(x) y^n, \end{aligned} \tag{D1}$$

and define a vector function

$$\mathbf{F}(x,y) = \begin{pmatrix} F_\alpha(x,y) \\ F_\beta(x,y) \end{pmatrix}. \tag{D2}$$

Then, and employing (6.17),

$$\begin{aligned} \mathbf{F}(x,y) &= \begin{pmatrix} \sum_{n=0}^{\infty} \alpha_n(x) y^n \\ \sum_{n=0}^{\infty} \beta_n(x) y^n \end{pmatrix} \\ &= \sum_{n=0}^{\infty} \begin{pmatrix} \alpha_n(x) \\ \beta_n(x) \end{pmatrix} y^n \\ &= \sum_{n=0}^{\infty} \mathbf{M}^n(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} y^n \\ &= \mathbf{S}(x,y) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned} \tag{D3}$$

say, where, writing $\mathbf{I}_2$ for the 2×2 identity matrix,

$$\begin{aligned} \mathbf{S}(x,y) &= \sum_{n=0}^{\infty} [\mathbf{M}(x) y]^n \\ &= [\mathbf{I}_2 - \mathbf{M}(x) y]^{-1} \\ &= \frac{1}{A(x)C(x)y^2 + B(x)y + 1} \begin{pmatrix} 1 & A(x)y \\ -C(x)y & 1 + B(x)y \end{pmatrix}. \end{aligned} \tag{D4}$$

Substituting $\mathbf{S}(x,y)$ into (D3) gives

$$\mathbf{F}(x,y) = \frac{1}{A(x)C(x)y^2 + B(x)y + 1} \begin{pmatrix} 1 \\ -C(x)y \end{pmatrix}, \tag{D5}$$

with components

$$F_\alpha(x,y) = \frac{1}{A(x)C(x)y^2 + B(x)y + 1}, \tag{D6}$$

which is Lemma 6.12, and

$$F_\beta(x,y) = -\frac{C(x)y}{A(x)C(x)y^2 + B(x)y + 1} \tag{D7}$$

(which latter is also available by writing $F_\beta(x,y) = \sum_{n=0}^{\infty} \beta_n(x)y^n = \beta_0(x) + \sum_{n=1}^{\infty} \beta_n(x)y^n = \sum_{n=1}^{\infty} \beta_n(x)y^n$ (since $\beta_0(x) = 0$) $= y\sum_{n=0}^{\infty} \beta_{n+1}(x)\ y^n = -C(x)y\sum_{n=0}^{\infty} \alpha_n(x)y^n$ (by (6.19)) $= -C(x)yF_\alpha(x,y)$).