Concise Computer Mathematics: Tutorials on Theory and Problems

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Problem booklet

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1 Numbers and Sets

1.1 Brief theoretical background

This section briefly presents the theoretical aspects covered in the tutorial. For more details please check the lecture notes.

Number systems

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\} \quad \text{natural numbers (including zero)} \tag{1.1}$$
$$\mathbb{N}^+ = \{1, 2, 3, \ldots\} \quad \text{natural numbers excluding zero} \tag{1.2}$$
$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, \ldots\} \quad \text{integers (positive and negative)} \tag{1.3}$$
$$\mathbb{Q} = \{q = \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}^+\} \quad \text{rational numbers (fractions)} \tag{1.4}$$
$$\mathbb{R} = \{a.b \quad : a, b \in \mathbb{Z}\} \quad \text{reals - infinite decimals} \tag{1.5}$$
$$\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\} \tag{1.6}$$
$$\mathbb{C} = \{a + bi \quad : a, b \in \mathbb{R}\}, \quad i = \sqrt{-1} \quad \text{complex numbers.} \tag{1.7}$$

Motivation of number sets Number sets appeared from the need to solve various equations, and to have well defined operations. Polynomials that generate solutions in the various number sets are

1. $P(x) = x - 1 \Rightarrow x = 1 \in \mathbb{N}$
2. $P(x) = x + 1 \Rightarrow x = -1 \notin \mathbb{N}$, $\quad x \in \mathbb{Z}$
3. $P(x) = 4x - 1 \Rightarrow x = 1/4 \notin \mathbb{Z}$, $\quad x \in \mathbb{Q}$
4. $P(x) = 2x^2 - 1 \Rightarrow x = \sqrt{2} \notin \mathbb{Q}$, $\quad x \in \mathbb{R}$
5. $P(x) = x^2 + 1 \Rightarrow x = \sqrt{-1} = 1 \notin \mathbb{R}$, $\quad x \in \mathbb{C}$.

Fundamental theorem of algebra (Argand in 1806, Gauss 1816). Every non-constant single-variable polynomial with complex coefficients has at least one complex root (real coefficients and roots being within the definition of complex numbers).

Definitions and notations

- set - collection of similar objects (numbers, symbols, etc)
- $x \in A$ - element x is a member of set A
- U *universe* - contains all the elements of a kind
- $\emptyset$ - set containing no element
- $B \subset A \Leftrightarrow x \in B \Rightarrow x \in A$

- $A = B \Leftrightarrow B \subset A$ and $A \subset B$
- $\mathcal{P}(A)$ - power set - set of all subsets of A
- $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$ cartesian product

Ways to define sets

$$A = \{0, 1, 2, 3\} \qquad \text{(enumeration)}$$
$$B = \{n \in \mathbb{N} : 3 \leq n \leq 100\} \qquad \text{(ellipsis - just when obvious!!!)}$$
$$C = \{x = n^2 : n \in \mathbb{N}\} \qquad \text{(formula)}$$

Other set properties

- finite or infinite
- ordered (any two elements are comparable) vs not ordered

Operations. For sets A and B taken from the universe U we define

- $A \cap B = \{x \in U : x \in A \text{ and } x \in B\}$ (intersection)
- $A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$ (reunion)
- $A \setminus B = \{x \in U : x \in A \text{ and } x \notin B\}$ (difference)
- $A \Delta B = (A \setminus B) \bigcup (B \setminus A)$ (symmetric difference)
- $A^c = \{x \in U : x \notin A\}$ (complement)
- Alternative notations for set complement: $(A)^c = \sim (A) = \complement(A) = U \setminus A$

1.2 Compulsory problems

These are problems anyone should know how to solve.

C 1.1. The set of all subsets of a given set A is denoted by $\mathcal{P}(A)$.

$$A = \{a, b, c\}, \quad \mathcal{P}(A) = ?.$$

C 1.2. (Cartesian product problem) Find the cartesian product $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$ of sets $A = \{1, 2, 3, 6\}$ and $B = \{a, b, c\}$.

C 1.3. Let

$$\begin{aligned} A &= \{\text{'}r\text{'}, \text{'}e\text{'}, \text{'}s\text{'}, \text{'}t\text{'}\} \\ B &= \{\text{'}r\text{'}, \text{'}e\text{'}, \text{'}l\text{'}, \text{'}a\text{'}, \text{'}x\text{'}\} \\ C &= \{\text{'}s\text{'}, \text{'}l\text{'}, \text{'}u\text{'}, \text{'}m\text{'}, \text{'}b\text{'}, \text{'}e\text{'}, \text{'}r\text{'}\} \end{aligned}$$

$$\begin{aligned} \text{Find } & A \cup B \quad A \cup C \quad B \cup C \\ & A \cap B \quad A \cap C \quad B \cap C \\ & A \setminus (A \cap C) \quad B \setminus (A \cup C). \end{aligned}$$

C 1.4. Let the Universe $U = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and let

$$D = \{0, 1, 2\}, \quad E = \{2, 4, 6\}, \quad F = \{1, 3, 5, 7\}.$$

Find $D \setminus F$, $E \cap F$, $E \cup F$ and $(F \cup D)^c$.

C 1.5. $a)$ List the elements of the following sets (enumerate or give a formula).

$$\begin{aligned} &i) \quad \{x \in \mathbb{N}^+ : x \text{ is a multiple of } 4\} \\ &ii) \quad \{x \in \mathbb{Z} : -2 \leq x \leq 3\} \end{aligned}$$

b) List the elements of the following sets (enumerate or give a formula).

$$\begin{aligned} A &= \{x \in \mathbb{N}^+ : x \text{ is prime}\} \\ B &= \{x \in \mathbb{N}^+ : x \text{ is even}\} \\ C &= \{x \in \mathbb{N}^+ : x \text{ is odd}\} \\ D &= \{x \in \mathbb{N}^+ : x \text{ is a multiple of } 6\} \end{aligned}$$

Are any of the above sets A, B, C or D subsets of one of the other sets? Write down your answer using the notation $\subset$.

c) Determine which of the following sets are equal

$$\begin{aligned} E &= \{0, 1, 2, 3\} \\ F &= \{x \in \mathbb{N} : x^2 < 13\} \\ G &= \{x \in \mathbb{N} : (x^2 + 3) < 2\} \end{aligned}$$

C 1.6. i) If $A \subseteq C$ and $C \subseteq B$, does it follow that $A \subseteq B$?
ii) If $A \subseteq C$ and $B \subseteq C$, what can you say about A and B.

C 1.7. If A and B are subsets of the universe, show by constructing examples that each of the following is true.

$$
\begin{aligned}
&i) \quad (A \Delta B)^c = (A \cap B) \cup (A \cup B)^c \\
&ii) \quad (A \setminus B)^c = A^c \cup B
\end{aligned}
$$

C 1.8. Use Venn diagrams to show that the following pairs are logically equivalent.

$$
\begin{aligned}
&i) \quad (A \cap B) \cup (A \cap B)^c = U(\text{universal set}) \\
&ii) \quad (A^c \cap B) \cup A = A \cup B
\end{aligned}
$$

1.3 Supplementary problems

These are problems are not part of the assessment.
S 1.1. (Cartesian product)
(a) Find $A \times B \times C$ for the sets $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ and $C = \{a, b, c\}$.
(b) If the sets A, B and C have m, n and p ($m, n, p \in \mathbb{N}^+$) elements respectively, what is the number of elements of $A \times B \times C$?

S 1.2. Prove that if A, B, C are sets from the Universe U, then:

$$
\begin{aligned}
&i) \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \text{(distributivity of } \cap) \\
&ii) \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{(distributivity of } \cup) \\
&iii) \quad (A \cap B)^c = A^c \cup B^c \quad \text{(de Morgan law for } \cap) \\
&iv) \quad (A \cup B)^c = A^c \cap B^c \quad \text{(de Morgan law for } \cup).
\end{aligned}
$$

S 1.3. Find the sets A, B with the properties

$$
\begin{aligned}
&i) \quad A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\
&ii) \quad A \cap B = \{3, 4, 5\} \\
&iii) \quad A \setminus B = \{2, 6\}
\end{aligned}
$$

S 1.4. Write the following numbers as fractions

$$
\begin{aligned}
&i) \quad x = 0.333 \cdots = 0.(3) \\
&ii) \quad x = 0.23232323 \cdots = 0.(23) \\
&iii) \quad x = 0.1129292929 \cdots = 0.11(29) \\
&iv) \quad x = \sqrt{3}.
\end{aligned}
$$

1.4 Answers for Tutorial 1

Compulsory problems

C 1.1. $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

C 1.2. $A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c), (6,a), (6,b), (6,c)\}$.

C 1.3. $A \cup B = \{‘r‘, ‘e‘, ‘s‘, ‘t‘, ‘l‘, ‘a‘, ‘x‘\}$; $\quad A \cup C = \{‘r‘, ‘e‘, ‘s‘, ‘t‘, ‘l‘, ‘u‘, ‘m‘, ‘b‘\}$;
$B \cup C = \{‘r‘, ‘e‘, ‘l‘, ‘a‘, ‘x‘, ‘s‘, ‘u‘, ‘m‘, ‘b‘\}$; $\quad A \cap B = \{‘r‘, ‘e‘\}$; $\quad A \cap C = \{‘r‘, ‘e‘, ‘s‘\}$;
$B \cap C = \{‘r‘, ‘e‘, ‘l‘\}$; $\quad A \setminus (A \cap C) = \{‘s‘, ‘t‘\}$; $\quad B \setminus (A \cup C) = \{‘a‘, ‘x‘\}$.

C 1.4. $D \setminus F = \{0, 2\}$; $\quad E \cap F = \emptyset$; $E \cup F\{2, 4, 6, 1, 3, 5, 7\}$; $\quad (F \cup D)^c = \{4, 6\}$.

C 1.5. a) i) $\{4, 8, 12, \dots\} = \{4n : n \in \mathbb{N}^+\}$; ii)$\{-2, -1, 0, 1, 2, 3\}$.

$$\begin{aligned}
b)\quad & A = \{2, 3, 5, 7, 11, \dots\} (\text{infinite set}) \\
& B = \{2, 4, 6, 8, \dots\} = \{2n : n \in \mathbb{N}^+\} \\
& C = \{1, 3, 5, 7, \dots\} = \{2n - 1 : n \in \mathbb{N}^+\} = \{2n + 1 : n \in \mathbb{N}\} \\
& D = \{6, 12, 18, \dots \mathbb{N}^+\} = \{6n : n \in \mathbb{N}^+\} \subset B \\
c)\quad & E = F = \{0, 1, 2, 3\}.
\end{aligned}$$

C 1.6. i) yes; ii) nothing
If $A = \{2n : n \in \mathbb{N}^+\} \subset C = \mathbb{N}$, $B = \{2n + 1 : n \in \mathbb{N}^+\} \subset C = \mathbb{N}$) then $A \cap B = \emptyset$.
If $A = \{2n : n \in \mathbb{N}^+\} \subset C = \mathbb{N}$, $B = \{4n : n \in \mathbb{N}^+\} \subset C = \mathbb{N}$) then $B \subset A$.

C 1.7. Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let $A = \{0, 1, 2, 3, 4, 5\}$, $\quad B = \{4, 5, 6, 7\}$.

$$\begin{aligned}
i)\quad & (A \Delta B) = \{0, 1, 2, 3, 6, 7\}, (A \cap B) = \{4, 5\}, (A \cup B)^c = \{8, 9\} \\
& (A \Delta B)^c = (A \cap B) \cup (A \cup B)^c = \{4, 5, 8, 9\} \\
ii)\quad & (A \setminus B)^c = A^c \cup B = \{4, 5, 6, 7, 8, 9\}
\end{aligned}$$

C 1.8. Use Venn diagrams to show that the following pairs are logically equivalent.

$$\begin{aligned}
i)\quad & (A \cap B) \cup (A \cap B)^c = U (\text{universal set}) \\
ii)\quad & (A^c \cap B) \cup A = A \cup B
\end{aligned}$$

Supplementary problems

S 1.1. (b) mnp.

$$(a) A \times B \times C = \{(1,a,a),(1,a,b),(1,a,c),(1,b,a),(1,b,b),(1,b,c),(1,c,a),(1,c,b),(1,c,c),$$
$$(2,a,a),(2,a,b),(2,a,c),(2,b,a),(2,b,b),(2,b,c),(2,c,a),(2,c,b),(2,c,c),$$
$$(3,a,a),(3,a,b),(3,a,c),(3,b,a),(3,b,b),(3,b,c),(3,c,a),(3,c,b),(3,c,c)\};$$

S 1.2. You may use the Venn diagrams where to show that both sides determine the same region. Otherwise, we can use the following argument.
(i) Let $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in (B \cup C)$. This means that $x \in A$ and $x \in B$ or $x \in C$. This means that $x \in (A \cap B)$ or $x \in (A \cap C)$.
For the proof to be complete one also need to start with a x from the right hand side, and show that it belongs to the set at the left.

(ii) $x \in A \cup (B \cap C) \Leftrightarrow x \in A$ or $x \in (B \cap C)$. If $x \in A$, it clearly belongs to $(A \cup B)$ and $(A \cup C)$, therefore to their intersection. If $x \in (B \cap C)$, it belongs to both B and C, therefore to $(A \cup B)$ and $(A \cup C)$ and $(A \cup B) \cap (A \cup C)$.

(iii) and (iv) - check with Venn diagrams.

S 1.3. One can use a Venn diagram to show that

$$A = \{2,3,4,5,6\}$$
$$B = \{1,3,4,5,7,8,9,10\}$$

S 1.4. i) $x = 0.333\cdots = 0.(3)$.
Multiplying by 10 we obtain $10x = 3 + x$, which shows that $x = \frac{1}{3}$.

ii) $x = 0.23232323\cdots = 0.(23)$.
Multiplying by 100 we obtain $100x = 23 + x$, which shows that $x = \frac{23}{99}$.

iii) $x = 0.1129292929\cdots = 0.11(29)$.
Multiplying by 100 the fraction becomes $100x = 11 + 0.(29) = 11 + \frac{29}{99}$.
In the end, we obtain $x = \frac{1129}{9900}$.

iv) $\quad x = \sqrt{3}$.
This is not a rational number. To prove this we assume there are two natural numbers p, q s.t they have no common divisors and $\frac{p}{q} = \sqrt{3}$. This is equivalent to $p^2 = 3q^2$ and because p and q are natural numbers, p^2 is a multiple of 3. This can only happen when p itself is a multiple of 3, therefore $p = 3p_1$. In this case, $(3p_1)^2 = 3q^2$, which gives $3p_1^2 = q^2$. Using the same argument as before, $q = 3q_1$. However, this would mean that 3 is a common divisor of p and q, which is a contradiction. This ends the proof.

2 Relations

2.1 Brief theoretical background

Notations and definitions

- A binary relation R is a triplet (X, Y, G), where G is a subset of $X \times Y$,
- X - domain, Y - co-domain
- G - graph (we usually identify the relation R with its graph G!!!)
- If the pair (x, y) is in G (or R) we write xRy (x is in relation with y) or $x \sim y$

Operations on relations Let X, Y be sets, $A \subset X$, $B \subset Y$ be subsets, and let $R, S \subset X \times Y$ be two relations. The following operations are defined:

- $R^{-1} = \{(y, x) \in Y \times X : (x, y) \in R\} \subset Y \times X$ (inverse)
- $A \lhd R = \{(x, y) \in X \times Y : (x, y) \in R \text{ and } x \in A\}$ (restriction)
- $R \rhd B = \{(x, y) \in X \times Y : (x, y) \in R \text{ and } y \in B\}$ (co-restriction)
- $R(A) = \{(y \in Y : \text{ there is a pair } (a, y) \in R \text{ with } a \in A\}$ (image - this is a SET!)
- $R \cup S = \{(x, y) : (x, y) \in R \text{ or } (x, y) \in S\} \subset X \times Y$ (union)
- $R \cap S = \{(x, y) : (x, y) \in R \text{ and } (x, y) \in S\} \subset X \times Y$ (intersection)

Composition of binary relations Let $R \subset X \times Y$ and $S \subset Y \times Z$ be two relations.

$$S \circ R = \{(x, z) \in X \times Z : \text{ there exists } y \in Y \text{ with } (x, y) \in R \text{ and } (y, z) \in S\}.$$

Special relations over a set

- reflexive: for all $x \in X$ it holds that xRx
- symmetric: For all $x, y \in X$ it holds that if xRy then yRx
- antisymmetric: For all distinct $x, y \in X$, if xRy and yRx then $x = y$
- transitive: For all $x, y, z \in X$, if xRy and yRz then xRz
- total: For all $x, y \in X$, it holds that xRy or yRx
- equivalence: a relation that is reflexive, symmetric and transitive.

N-ary relations Let $n \geq 2$ and $A_1, \ldots, A_n$ sets. A n-ary relation is a set $R \subset A_1 \times \cdots \times A_n$ of n-tuples $(a_1, \ldots, a_n)$ s.t. $a_j \in A_j$, $j = 1, \ldots, n$.

2.2 Compulsory problems

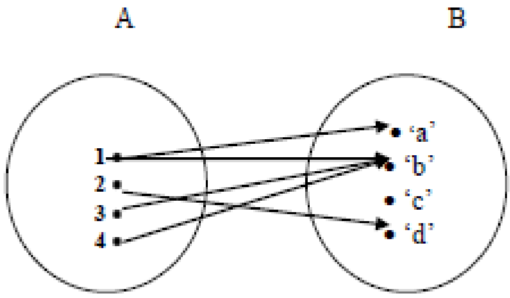
C 2.1. Let C be the set of characters and $Y = \{"apple", "pie"\}$. Find the define the graph of the relation

$$APPEARS \subset C \times Y$$

where $(c, s) \in APPEARS$ if the character c appears in the string s.

C 2.2. A relation $R \subset A \times B$, where $A = \{1, 2, 3, 4\}$ and $B = \{'a', 'b', 'c', 'd'\}$ is shown in the figure below. (i) Give R as a table; (ii) Give R^{-1} as a set of pairs.

Relation types



C 2.3. Let $X = \{1, 2, 3, 4\}$, $Y = \{A, B, C, D, E, F\}$ and the relations

$$R \subset X \times Y = \{(1, A), (1, C), (2, A), (3, D), (4, D)\}$$
$$S \subset Y \times Y = \{(E, A), (E, B), (B, D), (C, F), (A, F)\}.$$

Calculate each of the following sets or relations:

$(i)\{1, 2\} \lhd R$	$(ii) S \rhd \{A, C, D\}$	$(iii) R(\{1, 2\})$	$(iv) S \circ R$
$(v) S \circ S$	$(vi) S \circ S \circ S$	$(vii) R \circ R^{-1}$	

C 2.4. Suppose that T is a set of students, S is a set of sports, and $LIKES \subset T \times S$ is a relation, as defined below

$$T = \{Anne, Bill, Carol, Dave, Emma, Frankie\}$$
$$S = \{hockey, netball, tennis\}$$
$$LIKES = \{(Anne, tennis), (Anne, hockey), (Bill, tennis), (Carol, netball), (Dave, hockey)\}.$$

If $A = \{Bill, Dave, Frankie\} \subset T$ and $B = \{hockey, tennis\} \subset S$ then find $A \lhd LIKES$, $LIKES \rhd B$ and $LIKES(A)$.

C 2.5. Let $A \subset X$, $B \subset Y$ be subsets, and $R, S \subset X \times Y$ be two relations. Prove that

$$(A \lhd R) \rhd B = A \lhd (R \rhd B). \tag{2.1}$$

C 2.6. Let $R \subset X \times Y$, $S \subset Y \times Z$ and $T \subset Z \times U$ be relations. Show that

$$(i) \quad (S \circ R)^{-1} = R^{-1} \circ S^{-1}$$
$$(ii) \quad (T \circ S) \circ R = T \circ (S \circ R)$$

C 2.7. For each of the relations given below, decide whether or not it is:
(a) many-one; (b) one-many; (c) one-one; (d) onto;
The relations are
(i) $R_1 = \{(s,t) \in S \times S : s \neq "" \text{ and } t \text{ is the string } s \text{ with its first character deleted}\}$
(ii) $R_2 = \{(c,s) \in C \times S : LEN(s) \geq 3 \text{ and } c \text{ is the third character of string } s\}$
(iii) $R_3 = \{(s,n) \in S \times N : s \neq "" \text{ and } n \text{ is the ASCII code of the first character in } s\}$.

C 2.8. Check whether the following relations $R \subset X \times X$ are equivalences
(i) $X = \mathbb{N}$, $(x,y) \in R$ if $x = y$.
(ii) $X = \mathbb{R}$, $(x,y) \in R$ if $x \geq y$.
(iii) $X = \mathbb{Z}$, $(x,y) \in R$ if x is divisible by y.

2.3 Supplementary problems

S 2.1. Let X and Y be sets of cardinalities m and n respectively. What is the minimum possible cardinality of relation $R \subset X \times Y$? What is the maximal cardinality of R ?

S 2.2. Let X, Y be sets, $A_1, A_2 \subset X$, $B_1, B_2 \subset Y$ be subsets, and let $R, S \subset X \times Y$ be two relations. Prove that

$$(A_1 \cap A_2) \lhd R = (A_1 \lhd R) \cap (A_2 \lhd R)$$
$$(A_2 \cup A_2) \lhd R = (A_1 \lhd R) \cup (A_2 \lhd R)$$
$$R \rhd (B_1 \cap B_2) = (R \rhd B_1) \cap (R \rhd B_2)$$
$$R \rhd (B_1 \cup B_2) = (R \rhd B_1) \cup (R \rhd B_2)$$
$$R(A_1 \cap A_2) = R(A_1) \cap R(A_2)$$
$$R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$$

S 2.3. Let $STUDENTS$ be the set of students in a college, $COURSES$ be the set of courses the college runs, and $SESSIONS$ be the set of sessions (such as: June 21, a.m.) at which exam papers are taken. (Students may take more than one course, and a course may have more than one exam paper.) Relations $TAKING$ and $EXAM$ are defined as:

$$TAKING = \{(s,c) \subset STUDENTS \times COURSES : \text{ student } s \text{ is taking course } c\}$$
$$EXAM = \{(c,d) \subset COURSES \times SESSIONS : \text{ there is a paper for course } c \text{ in session } d\}$$

Let F be the set of female students, and C the set of computing courses.
(i) Express in English each of the following:

$(a) C \lhd EXAM$ $\quad$ $(b) EXAM(C)$ $\quad$ $(c) EXAM \circ TAKING$
$(d) EXAM^{-1} \circ EXAM$ $\quad$ $(e) EXAM \circ (TAKING \rhd C)$

(ii) Using F, C, $TAKING$ and $EXAM$ and operations on binary relations, express
  (a) The relation (s, c) where s is a female student taking the computing course c.
  (b) The set of courses with no female students.
  (c) The set of exam sessions when the student '$A107X$' has a paper to sit.

S 2.4. HOBBIES is a relation which is defined below:

$$\{(Sarah, painting), (Tom, painting), (Joe, football), (Janet, football), (Susie, hockey), (Sarah, sketching), (Craig, sketching)\}$$

and also, the following subsets are considered

$$\begin{aligned} Professional &= \{Joe, Sarah\} \\ Amateur &= \{Susie, Tom, Janet, Craig\} \\ Arty &= \{sketching, painting\} \\ Games &= \{football, hockey\} \end{aligned}$$

Write down the following:

$$\begin{aligned} i) &\quad HOBBIES \rhd Arty \\ ii) &\quad Professional \lhd HOBBIES \\ iii) &\quad Amateur \lhd HOBBIES \rhd Games \end{aligned}$$

S 2.5. Let $R \subset X \times Y$ be a binary relation. Express in English each of the propositions:

$$\begin{aligned} (i) &\quad \forall x \in X[\exists y \in Y[(x, y) \in R]] \\ (ii) &\quad \forall y \in Y[\exists x \in X[(x, y) \in R]] \end{aligned}$$

S 2.6. Prove whether the following binary relations are equivalences. If the relation is not an equivalence relation, state why it fails to be one.

(i) $X = \mathbb{Z} : a \sim b$ if $ab > 0$.
(ii) $X = \mathbb{R} : a \sim b$ if $a - b$ is rational .
(iii) $X = \mathbb{R} : a \sim b$ if $a \geq b$.
(iv) $X = \mathbb{C} : a \sim b$ if $|a| = |b|$.

2.4 Answers for Tutorial 2

Compulsory problems

C 2.1. $APPEARS =$

C	Y
$'a'$	$''apple''$
$'p'$	$''apple''$
$'l'$	$''apple''$
$'e'$	$''apple''$
$'p'$	$''pie''$
$'i'$	$''pie''$
$'e'$	$''pie''$

C 2.2. (i)

1	$'a'$
1	$'b'$
2	$'d'$
3	$'b'$
4	$'b'$

(ii) $R^{-1} = \{('a', 1), ('b', 1), ('b', 3), ('b', 4), ('d', 1)\}$.

C 2.3.

$$\begin{aligned}
&(i) \quad \{1,2\} \lhd R = \{(1, A), (1, C), (2, A)\}, \\
&(ii) \quad S \rhd \{A, C, D\} = \{(E, A), (B, D)\}, \\
&(iii) \quad R(\{1,2\}) = \{A, C\}, \\
&(iv) \quad S \circ R = \{(1, F), (2, F)\}, \\
&(v) \quad S \circ S = \{(E, F), (E, D)\}, \\
&(vi) \quad S \circ S \circ S = \emptyset, \\
&(vii) \quad R \circ R^{-1} = \{(A, A), (A, C), (C, A), (C, C), (D, D)\}.
\end{aligned}$$

C 2.4.

$$\begin{aligned}
A \lhd LIKES &= \{(Bill, tennis), (Dave, hockey)\} \\
LIKES \rhd B &= \{(Anne, tennis), (Anne, hockey), (Bill, tennis), (Dave, hockey)\} \\
LIKES(A) &= \{tennis, hockey\}.
\end{aligned}$$

C 2.5. Both sides represent the set

$$(A \lhd R) \rhd B = A \lhd (R \rhd B) = \{(a, b) \in R : \ a \in A, b \in B\}. \tag{2.2}$$

C 2.6. Let $R \subset X \times Y$, $S \subset Y \times Z$ and $T \subset Z \times U$ be relations. Show that

$$\begin{aligned}
(i) \quad (z,x) \in (S \circ R)^{-1} &\Leftrightarrow (x,z) \in (S \circ R) \\
&\Leftrightarrow \exists y \in Y \text{ s.t } (x,y) \in R, (y,z) \in S \\
&\Leftrightarrow \exists y \in Y \text{ s.t } (y,x) \in R^{-1}, (z,y) \in S^{-1} \\
&\Leftrightarrow (z,x) \in R^{-1} \circ S^{-1} \\
(ii) \quad (x,t) \in (T \circ S) \circ R &\Leftrightarrow \exists y \in Y \text{ s.t } (x,y) \in R, (y,t) \in T \circ S \\
&\Leftrightarrow \exists y \in Y, z \in Z \text{ s.t } (x,y) \in R, (y,z) \in S, (z,t) \in T, \\
&\Leftrightarrow \exists z \in Z \text{ s.t } (x,z) \in S \circ R, (z,t) \in T, \\
&\Leftrightarrow (x,t) \in T \circ (S \circ R).
\end{aligned}$$

C 2.7. (i) many-one; (ii) one-many; (iii) many-one.

C 2.8. (i) $X = \mathbb{N}$, $(x,y) \in R$ if $x = y$. Yes, because it is (a) reflexive: $x = x$; (b) symmetric: $x = y$ implies $y = x$ (c) transitive: $x = y, y = z$ implies $x = z$.

(ii) $X = \mathbb{R}$, $(x,y) \in R$ if $x \geq y$. Not an equivalence, because it is
(a) reflexive: $x \geq x$; (b) **not** symmetric (actually antisymmetric): $x \geq y$ does not imply $y \geq x$ (ex. $x = 3, y = 2$) (c) transitive: $x \geq y, y \geq z$ implies $x \geq z$.

(iii) $X = \mathbb{Z}$, $(x,y) \in R$ if x is divisible by y. As above, it is not symmetric.

Supplementary problems

S 2.1. $\min = 0; \max = mn$.

S 2.2. One just needs to use the definitions of $\cap, \cup, \lhd, \rhd$ and of the image of a subset, and the double inclusion to prove the pairs of sets are equal.

S 2.3. (i) Express in English each of the following:

$(a) C \lhd EXAM =$ (c,d): there is a paper in computing c in the exam session d
$(b) EXAM(C) =$ the sessions d having computing exams
$(c) EXAM \circ TAKING =$ (s,d) the students s taking an exam in session d
$(d) EXAM^{-1} \circ EXAM =$ (c,e) the pairs of courses c,e having a common session
$(e) EXAM \circ (TAKING \rhd C) =$ (s,d) the students s taking computing exam in session d

(ii) (a) $F \lhd TAKING \rhd C$; (b) $TAKING(STUDENTS \setminus F)$;
(c) EXAM(TAKING($'A107X'$)).

S 2.4. The answer is

$$i) HOBBIES \rhd Arty = \{(Sarah, painting), (Tom, painting), (Sarah, sketching), (Craig, sketching)\}$$
$$ii) Professional \lhd HOBBIES = \{(Sarah, painting), (Joe, football), (Sarah, sketching)\}$$
$$iii) Amateur \lhd HOBBIES \rhd Games = \{(Janet, football), (Susie, hockey)\}.$$

S 2.5. (i) for all x in X, there is y in Y s.t. (x,y) is in R;
(ii) for all y in Y, there is x in X s.t. (x,y) is in R.

S 2.6. (i) $X = \mathbb{Z} : a \sim b$ if $ab > 0$. **Yes:** reflexive, symmetric and transitive.

(ii) $X = \mathbb{R} : a \sim b$ if $a - b$ is rational. **Yes:** reflexive, symmetric and transitive.

(iii)$X = \mathbb{R} : a \sim b$ if $a \geq b$. **No:** reflexive, **antisymmetric** and transitive.

(iv) $X = \mathbb{C} : a \sim b$ if $|a| = |b|$. **Yes:** reflexive, symmetric and transitive.

3 Relation schemes and Functions

3.1 Brief theoretical background

Notations and definitions

- Binary relation: subset $R \subset X \times Y$.
- Inverse of relation R: $R^{-1} \subset Y \times X$ containing reversed pairs in R.
- Relation scheme: table with its column headings (called attributes). The entries in the table are called "tuples".
- Key: Set K of attributes for relation R s.t.
  1) no relation in R can have two tuples with the same values in K, but different on some other attribute
  2) no subset of K has a determining property
- Function between X and Y:
  Definition 1): Relation between a set of inputs and a set of permissible outputs s.t. each input is related to exactly one output.
  Definition 2): relation $R \subset X \times Y$ s.t. for all $x \in X$, there is a single $y \in Y$ s.t. xRy.
- Functions: X-domain, Y-codomain, $y = R(x) \in Y$ - image of x.
- Types of functions: linear, quadratic, continuous, trigonometric, odd, even, etc.
- Composition of functions (similar to relations): in general $g(f(x)) \neq f(g(x))$!!!

Examples of functions

- $Card : \mathcal{P}(\mathcal{X}) \rightarrow \mathbb{N}, \quad Card(\mathcal{A}) =$ nr of elements in subset $\mathcal{A}$
- $d : X \times Y \rightarrow \mathbb{R}^+, \quad d(x, y) =$ distance between x and y
- $Mark\_CM$: Students in this room $\rightarrow [0\%, 100\%]$ (marks at Comp Maths)
- Functions defined on $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$: well defined $(+, \times)$ and ill defined $(-, /)$.

General properties of functions $f : X \rightarrow Y$

- injective: different elements have different images
  $$x_1, x_2 \in X : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$
- surjective: all elements in co-domain are image of elements from the domain.
  $$y \in Y : \text{ there is } x \in X s.t. f(x) = y$$
- bijective (1-on-1): a function that is both injective and surjective
- a function f only has an inverse f^{-1} if it is bijective!

Applications

- Theory of cardinals:
  1) Two sets have same cardinal $X \sim Y$ if there is a 1-1 function between them
  2) Finite sets: same cardinal means they have the same number of elements
  3) Infinite sets: $\mathbb{N} \sim \mathbb{Z} \sim \mathbb{Q}$ - countable sets
  4) $\mathbb{R}$ - not countable

- Datatypes (definition): A set, together with the basic defining functions.

- Datatype string:
  1) Sets: C set of all characters, S set of all (ordered) sequences of characters
  2) Notation: 'a' - character, "string" - string
  3) Functions defining the datatype string (all but LEN):

  - $LEN : S \to \mathbb{N}$ - number of characters in string s

  - $FIRST : S \to C$ - selects the first character in string s
    The domain of FIRST is $S \setminus \{"\ "\}$ (i.e. s is not an empty string).

  - $REST : S \to S$ - deletes the first character from string s
    The domain of REST is $S \setminus \{"\ "\}$ (i.e. s is not an empty string)

  - $ISEMPTY : S \to \{true, false\}$ - checks whether a string is empty

  - $ADDFIRST : C \times S \to S$ adds character c at the front of string s

  - $STR : C \to S$ - makes the string $"c"$ out of the character $'c'$.

- Functions ASC and CHR (for characters):

  - ASCII code (Definition): a number between 0 and 255 representing a keyboard character on one byte (8 bit).
  - $ASC : C \to \{0, \ldots, 255\}$ converts a character to its ASCII code
  - $CHR : \{0, \ldots, 255\} \to C$ converts an ASCII code to the appropriate character (inverse of ASC).
  - Example: $ASC('a') = 97, CHR(98) = 'b'$.

- Special notations and operations

  - $+_{\mathbb{R}}, =_{\mathbb{N}}, <_C, =_s$ - operations defined on sets: $\mathbb{R}$(real), $\mathbb{N}$(natural), C(char), S(string)
  - Example: $=_C$ is well defined when **both** inputs are in set C.
  - $>_c$: orders two characters by comparing their ASCII codes.
  - $+_s$ concatenates two strings $("a" +_s "xe" = "axe")$

3.2 Compulsory problems

When solving the problems one has to notice that the operations with index ($" +_s "," =_c "," =_{\mathbb{N}} "," +_{\mathbb{R}} "$) refer to operations on the sets: S (strings), C (characters), $\mathbb{N}$ (natural numbers) and $\mathbb{R}$ real numbers.

Make sure you understand the theory, before solving the problems!

C 3.1. Calculate $ASC(x) +_{\mathbb{N}} 32$ for $x = `A`$ and $x = `p`$.
Note: the apostrophe notation $`h`$ denotes the character h, $"h"$ denotes the string h.

C 3.2. Which of the following are true ? If false or invalid explain why.

$$(a)\, `a` <_c `A`; \quad (b)\, `2` <_c `Q`; \quad (c)\, 3 =_c `3`.$$

C 3.3. Which of the following are true, false (or invalid). Explain your decision.

$(a)\, STR(`c`) =_s "c"; \quad (b)\, "d" =_c `d`; \quad (c)\, ASC(`c`) \in C; \quad (d)\, STR(ASC(`b`)) =_s "98"$

C 3.4. Given $s = "Hal"$, $t = "PRINCE"$ write down the following where possible:
$(a)\, LEN(s); \quad (b)\, LEN(t); \quad (c)\, s +_{\mathbb{R}} t; \quad (d)\, FIRST(s);$
$(e)\, REST(t); \quad (f) t +_s s; \quad (g)\, t +_s `s`; \quad (h)\, REST(s) +_s "ice".$

C 3.5. Find $FIRST("12345")$; $LEN("takeaway")$.

C 3.6. Evaluate the following where possible, or say if the expression is undefined.
(a) $ASC(`D`) +_{\mathbb{N}} ASC(`d`)$;
(b) $CHR(6.5)$;
(c) $STR(`H`)$;
(d) $STR(ASC(`A`))$;
(e) $REST("Louise")$;
(f) $FIRST("THELMA")$;
(g) $LEN("time" +_s "out")$;
(h) $REST(REST("frog"))$;
(i) $FIRST(REST("toad"))$;
(j) $FIRST(REST("K"))$;
(k) $STR(FIRST("frog")) +_s REST(REST("Xi"))$;

C 3.7. Each of the following statements defines an element IMPLICITLY. Say which element IS defined
or whether there is no such element and explain why.
or that there are many possible elements and describe them.
a) $c \in S$ such that $c +_s "ed" =_s "pushed"$;
b) $t \in S$ such that $LEN(t) = 6$;
c) $r \in S$ such that $LEN(r) = 0$;
d) $v \in C$ such that $ASC(v) = 109$;
e) $w \in N$ such that $LEN(w) = 1$.

3.3 Supplementary problems

S 3.1. Find the domain and codomain of the following functions $f : X \to Y$:
(a) $f(x) = x^2$
(b) $f(x) = \sqrt(x)$
(c) $X = [2,4]$ and $f(x) = x + 2$

S 3.2. Define the function $CAPITAL\_CONVERT : S \to S$ that replaces a string by the same string, with the first letter replaced by the upper case symbol for that letter. i.e. $fish \to Fish$, $ada \to Ada$. Hint: Check the notations booklet to find the ASCII codes of lower and upper letters, and for the functions of datatype string.

S 3.3. State which of the following functions is injective, surjective or bijective:
a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x + 3;$
b) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 5;$
c) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2 - 2x + 3;$
d) $f : \mathbb{R} \to \mathbb{R}^+, \quad f(x) = x^2 - 2x + 1;$
e) $f : \mathbb{R}^+ \to \mathbb{R}^+, \quad f(x) = x^2 - 2x + 1;$
f) $f : \mathbb{N} \to \mathbb{N}, \quad f(n) = 2n;$
g) $f : \mathbb{N} \to 3\mathbb{N}, \quad f(n) = 3n;$ What can you say about the cardinals of $\mathbb{N}$ and $3\mathbb{N}$?
h) $f : \mathbb{N} \to \mathbb{Z}, \quad f(n) = n/2$ (n even), and $f(n) = -(n+1)/2$ (n odd).

3.4 Answers for Tutorial 3

Compulsory problems

C 3.1. $ASC('A') = 65$, $ASC('p') = 112$, therefore we have 97 and 144.

C 3.2. (a) false; (b) true; (c) invalid.

C 3.3. (a) true; (b) invalid; (c) false; (d) invalid.

C 3.4. (a) 3; (b) 6; (c) invalid; (d) 'H'; (e) "RINCE";
(f) "PRINCEHal"; (g) invalid; (h) "alice".

C 3.5. '1', 9.

C 3.6. (a) 68+100=168; (b) invalid; (c) "H"; (d) STR(65) invalid;
(e) "ouise"; (f) "T"; (g) 7; (h) "og" ; (i) "o"; (j) FIRST(" ") - invalid - the function is only defined for non-empty strings; (k) $"f"+_s" " = "f"$.

C 3.7. (a) "push";
(b) any 6 character string;
(c) " ";
(d) $ASC('m') = 109$, so $v = 'm'$;
(e) $w \in \mathbb{N}$ is not a string, therefore $LEN(w)$ is invalid.

Supplementary problems

S 3.1. (a) $f(x) = x^2$: If $X = \mathbb{R}$, then $R^+ \subset Y$;
(b) $f(x) = \sqrt(x)$: $X \subset R^+$;
(c) $X = [2,4]$ and $f(x) = x + 2$: $[4,6] \subset Y$.

S 3.2. The solution is: domain $s \in S \setminus \{" "\}$ and $97 \leq_{\mathbb{N}} ASC(FIRST(s)) \leq_{\mathbb{N}} 122$, codomain: $t = ADDFIRST(CHR(ASC(FIRST(s)) -_{\mathbb{N}} 32), REST(s))$.

Remark: five other functions $ASC, CHR, FIRST, REST, ADDFIRST$ have all been used to make the function $CAPITAL\_CONVERT$.

S 3.3. (a) bijective; (b) neither; (c) neither; (d) surjective; (e) bijective; (f) bijective; (g) bijective; (h) bijective.

4 Boolean types, logic and quantifiers

4.1 Brief theoretical background

Boolean values: $\mathbb{B} = \{true, false\}$

Boolean operations:

- NOT: $\neg : \mathbb{B} \to \mathbb{B}$ (standard notation, some books may use $'' \sim ''$)
- OR: $\vee : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$
- AND: $\wedge : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$
- IMPLIES: $\Rightarrow : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$
- EQUIVALENT: $\Leftrightarrow : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$

p	q	¬p	p∨q	p∧q	p⇒q	p⇔q
T	T	F	T	T	T	T
T	F	F	T	F	F	F
F	T	T	T	F	T	F
F	F	T	F	F	T	T

Equivalence between operations defined on boolean variables and set operations
Let A, B be two subsets of universe U and p and q be defined as

$$p(x) \text{ is true if and only if } x \in A$$
$$q(x) \text{ is true if and only if } x \in B$$

- $\neg p(x)$ is true if and only if $x \in A^c = \complement A = \sim A = U \setminus A$.
- $p \vee q(x)$ is true if and only if $x \in A \cup B$.
- $p \wedge q(x)$ is true if and only if $x \in A \cap B$.

Definitions:

- Sentence: statement that takes a value true or false.
- Proposition: sentence with no variables.
- Predicate: sentence which contains variables.
- Special predicates:
  - Tautology: always true
  - Contradiction: always false

De Morgan rules

$$(i) \quad \neg(p \wedge q) = \neg p \vee \neg q$$
$$(ii) \quad \neg(p \vee q) = \neg p \wedge \neg q$$

Disjunctive normal form (d.n.f)

Definition: Equivalent way of writing the truth table, using the variables i.e. $p, q, \neg p, etc....$ first using $\wedge$, then using $\vee$.

p	q	r	H
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

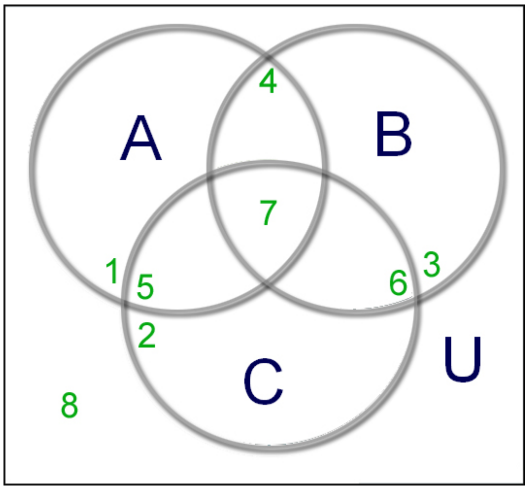


Figure 1: D.n.f using Truth tables and Venn diagrams

The d.n.f of H is

$$d.n.f(H) = (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r).$$

The corresponding regions in the Venn diagram: $7, 1, 2$.

Quantifiers

- universal quantifier: $\forall$ for all
- existential quantifier: $\exists$ there exists

4.2 Compulsory problems

C 4.1. Write the negations for each of the following statements
a) John is six feet tall and he weighs at least 200 pounds.
b) The bus was late or Tom's watch was slow.

C 4.2. Construct a truth table for the expressions
(i) ¬(¬p∧q).
(ii) (p∨¬q)∧r
(iii) Construct a truth table for (p∧q)∨r and for (p∨r)∧(q∨r) hence show that these two statements are logically equivalent.
(iv) Check whether (p∨q)∧¬r and (p∧¬r)∨(q∧¬r) are equivalent.

C 4.3. For $x \in \mathbb{R}$, suppose that p, q, and r are the propositions given below:

$$\begin{aligned} p: &\quad x = -4 \\ q: &\quad x = +4 \\ r: &\quad x^2 = 16 \end{aligned}$$

Which of the following are true ?

$$a)\ p \Rightarrow r; \quad b)\ r \Rightarrow p; \quad c)\ q \Rightarrow r; \quad d)\ r \Rightarrow q; \quad e)\ p \Leftrightarrow r;$$
$$f)\ q \Leftrightarrow r; \quad g)\ (p \vee q) \Rightarrow r; \quad h)\ r \Rightarrow (p \vee q); \quad i)\ (p \vee q) \Leftrightarrow r;$$

C 4.4. Decide which of the sentences below, are universally true, which are universally false, and which are neither. Write the predicates in symbolic form.
i) $(x+3)^2 > 8$, $x \in \mathbb{N}$
ii) $2x = 9$, $x \in \mathbb{N}$.

C 4.5. Find the d.n.f. of the following formulas:
i) $\neg(\neg p \wedge q)$
ii) $\neg(\neg p \vee q)$
iii) $p \wedge (q \vee r)$;

C 4.6. Use quantifiers to express formally the following propositions:
a. There is a natural number whose square is greater than 7
b. Whatever real number x is, $x^2 = -(x^2+1)$ is false.

C 4.7. Decide, by constructing truth tables, whether the following sets of statements are logically equivalent

(i) $(a \wedge b) \vee b$ and $a \wedge b$
(ii) $(a \wedge b) \vee a$ and a
(iii) $(\neg a \wedge b \wedge c) \vee ((a \wedge b) \vee c)$ and $b \wedge c$
(iv) $a \Rightarrow b$ and $\neg b \Rightarrow \neg a$.

C 4.8. Below, y is a variable of data type **Nat**, c is of data type **Char** and s and t are of data type **Str**. Represent each of the following conditions formally, using only standard functions of the data types **Nat**, **Char**, **Str** and $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow$.
i) y is either less than three, or greater than six but less than 10. However y is not seven.
ii) The first character of s is 'b' and the second character of s is 'a'.
iii) If s starts with 'c' then t starts with 'c'.
iv) s starts with the same character as t, but their second characters are not the same.
v) c is an upper case vowel, and the first two characters of t are the same if and only if the first two characters of s are the same.
vi) s is the string "fred" or y has the value 16, but not both.

4.3 Supplementary problems

S 4.1. Decide whether each of the following propositions is true or false:
i) $\sqrt{27} \geq 5$;
ii) $3^2 + 2^3 < \sqrt{250}$;
iii) $3^4 < 4^3$.

S 4.2 Repeat question **C 4.4.** for $x \in \mathbb{R}$.

S 4.3 Decide whether each of the following predicates is true or false.
i) $\forall x \in \mathbb{R}[x^2 >_{\mathbb{R}} 0]$;
ii) $\exists t \in \mathbb{N}[t^2 =_{\mathbb{N}} 7]$;
iii) $\exists s \in \mathbb{N}[s^2 =_{\mathbb{N}} 9]$.

S 4.4 Decide which of the sentences below are universally true in $\mathbb{R}$, which are universally false and which are neither.
i) $(x+3)^2 > 6$; ii) $x = x + 1$; iii)$2x = 7$; iv) $(x-1)^2 = x^2 - 2x + 1$.

S 4.5 Let R be the proposition "Roses are red" and B be the proposition "Violets are blue". Express each of the following propositions as logical expressions:
a) If roses are not red, then violets are not blue;
b) Roses are red or violets are not blue;
c) Either roses are red or violets are blue (but not both);
Use a truth table to show that (a) and (b) are equivalent.

S 4.6 Find the d.n.f. of the following formulas and the regions in the Venn diagram:
(i) $\neg p \wedge (q \Rightarrow p)$
(ii) $\neg p \wedge (\neg q \vee r)$;
(iii) $\neg p \wedge (q \vee r)$;

4.4 Answers for Tutorial 4

Compulsory problems

C 4.1. John is either not six feet tall, or he weighs less than 200 pounds.

C 4.2. In each case, one needs to build the table for 2 and 3 variables.
For simplicity, the final expressions are called H or G (to avoid too wide tables).
(i) Let H= $\neg(\neg p \wedge q)$. The corresponding truth table is

p	¬p	q	¬p∧q	H
1	0	1	0	1
1	0	0	0	1
0	1	1	1	0
0	1	0	0	1

(ii) Let H=$(p \vee \neg q) \wedge r$. The corresponding truth table is

p	q	r	¬q	p∨¬q	H
1	1	1	0	1	1
1	1	0	0	1	0
1	0	1	1	1	1
1	0	0	1	1	0
0	1	1	0	0	0
0	1	0	0	0	0
0	0	1	1	1	1
0	0	0	1	1	0

(iii) Let G=$(p \wedge q) \vee r$ and H=$(p \vee r) \wedge (q \vee r)$. The combined truth table for G and H is

p	q	r	p∧q	p∨r	q∨r	G	H
1	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	0	0	0
0	1	1	0	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	1	1	1	1
0	0	0	0	0	0	0	0

hence the two statements are logically equivalent.
(iv)Let G=$(p \vee q) \wedge \neg r$ and H=$(p \wedge \neg r) \vee (q \wedge \neg r)$.
The combined truth table for G and H is

p	q	r	¬r	p∨q	p∧¬r	q∧¬r	G	H
1	1	1	0	1	0	0	0	0
1	1	0	1	1	1	1	1	1
1	0	1	0	1	0	0	0	0
1	0	0	1	1	1	0	1	1
0	1	1	0	1	0	0	0	0
0	1	0	1	1	0	1	1	1
0	0	1	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0

hence the two statements are logically equivalent.

C 4.3. *a) T; b) F; c) T; d) F; e) F; f) F; g) T; h) T; i) T;*

C 4.4. (i) UT; (ii) UF.

C 4.5. We first find the truth table for each expression.
Note: The truth tables for (i) and (ii) were solved together to save time and space.
Let G= $\neg(\neg p \wedge q)$ and H= $\neg(\neg p \vee q)$. The combined truth table for G and H is

p	¬p	q	¬q	¬p∧q	¬p∨q	G	H
1	0	1	0	0	1	1	0
1	0	0	1	0	0	1	1
0	1	1	0	1	1	0	0
0	1	0	1	0	1	1	0

One obtains (the 3 variables method is covered in the theoretical background)
i) $(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$;
ii) $p \wedge \neg q$;
iii) $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge \neg r)$;

C 4.6. i) $\exists x \in \mathbb{N}$ s.t. $x^2 > 7$; ii) $\forall x \in \mathbb{R}, x^2 \neq -(x^2+1)$.

C 4.7. (i) yes; (ii) yes; (iii) no; (iv) yes.

C 4.8. (i) $((y < 3) \vee ((y < 10) \wedge (y > 6))) \wedge \neg(y = 6)$;
(ii) $FIRST(s) = \text{‘}b\text{‘} \wedge FIRST(FIRST(s)) = \text{‘}a\text{‘}$;
(iii) $(FIRST(s) = \text{‘}c\text{‘}) \Rightarrow FIRST(t) = \text{‘}c\text{‘}$;
(iv) $FIRST(s) = FIRST(t) \wedge \neq (FIRST(FIRST(s)) = FIRST(FIRST(t))$;
(v) $\Big(65 \leq ASC(c) \leq 90 \wedge FIRST(t) = FIRST(FIRST(t))\Big) \Leftrightarrow FIRST(s) = FIRST(FIRST(s))$;
(vi) $(s = \text{"}fred\text{"} \vee y = 16) \wedge \neg(s = \text{"}fred\text{"} \wedge y = 16)$.

Supplementary problems

S 4.1. (i) T; (ii) F; (iii) F.

S 4.2. (i) UT; (ii) neither.

S 4.3. (i) F; (ii) F; (iii) T.

S 4.4. i) UT; ii) UF; iii) neither; iv) UT.

S 4.5. a) $\neg R \Rightarrow \neg B$; b) $R \vee \neg B$; c) $(R \vee B) \wedge \neg (R \wedge B)$

S 4.6. One needs to construct the truth table for each expression, then write the brackets corresponding to the 1 (or T) values and finally connect them using $\vee$ (disjunction).

5 C.n.f, negation of quantifiers, proof and argument

5.1 Brief theoretical background

In general, one may use the set $\{0, 1\}$ instead of the boolean values $\mathbb{B} = \{True, False\}$.

Disjunctive normal form

Disjunctive normal form (d.n.f.) combines the basic literals, first using $\wedge$, then using $\vee$.

Exercise. Find the d.n.f of $\neg(\neg p \wedge q)$.

Step 1: Truth table

p	¬p	q	¬q	¬p∧q	¬(¬p∧q)
1	0	1	0	0	1
1	0	0	1	0	1
0	1	1	0	1	0
0	1	0	1	0	1

Step 2: Identify 1 entries in the last column

Step 3: For each 1 such entry , choose the 1 entries in the first 4 columns their $\wedge$ is 1.

Solution: $d.n.f(\neg(\neg p \wedge q)) = (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$

Useful formulas for logical operations

- **De Morgan's Laws**

$$\neg(p \wedge q) = \neg p \vee \neg q$$
$$\neg(p \vee q) = \neg p \wedge \neg q$$

- **Distributive Law**

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

- **Associative Law**

$$p \wedge (q \wedge r) = (p \wedge q) \wedge (p \wedge r)$$
$$p \vee (q \vee r) = (p \vee q) \vee (p \vee r)$$

Conjunctive normal form (c.n.f)

Conjunctive normal form (c.n.f.) combines the basic literals, first using $\vee$, then using $\wedge$.

Algorithm for finding the c.n.f of a formula H:

Step 1: Find the d.n.f of ¬H
Step 2: Find the negation of this d.n.f (basically ¬(¬ H))

Observation: Applying ¬ has the following effect:

- p becomes ¬p (and of course, ¬p becomes p)
- $\wedge$ becomes $\vee$
- $\vee$ becomes $\wedge$

Negation of quantifiers

- $\forall$ becomes $\exists$
- $\exists$ becomes $\forall$

p	q	p⇒q	¬q	¬p
1	1	1	0	0
1	0	0	1	0
0	1	1	0	1
0	0	1	0	1

Deduction rules

Arguments

Valid	**Fallacious**
Modus Ponens	Converse error
$\underline{p \Rightarrow q; p}$	$\underline{p \Rightarrow q; q}$
q	p
Modus Tollens	Inverse error
$\underline{p \Rightarrow q; \neg q}$	$\underline{p \Rightarrow q; \neg p}$
$\neg p$	$\neg q$

5.2 Compulsory questions

C 5.1. Find the d.n.f and c.n.f of the following statements
(i) $p \vee (q \wedge r)$;
(ii) $\neg((p \wedge q) \vee r)$;
(iii) $p \Leftrightarrow (q \wedge r)$;

C 5.2. Given the following truth table for a formula H, find the disjunctive normal form for H and hence find the conjunctive normal form of H.

p	q	H
1	1	1
1	0	0
0	1	0
0	0	1

C 5.3. Given the following truth table for a formula G, find the d.n.f for G and hence find the c.n.f of G. Also identify the corresponding regions in the Venn diagram.

p	q	r	G
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	1

C 5.4. Write the English statements that are the negations of each of the sentences
(i) All dogs bark;
(ii) Some birds fly;
(iii) No cat likes to swim;
(iv) No computer science student does not know mathematics.

C 5.5. Find the negation of the following statements and evaluate their truth values:
(i) $\forall$ x $\in \mathbb{R}$ [$x^2 - 6x + 10 \geq 0$]; (ii) $\forall$ x $\in \mathbb{R}$ [$x^2 \geq x$];
(iii) $\exists$ x $\in \mathbb{N}^+$ [$x^2 + 2x - 2 \leq 0$]; (iv) $\exists$ x $\in \mathbb{Q}$ [$x^2 = 3$].

C 5.6. Prove Modus Ponens, Modus Tollens, Inverse Error and Converse Error using the truth table from the previous page.

5.3 Supplementary questions

S 5.1. Find the d.n.f and c.n.f of the following expressions:
(i) $\neg(p \Rightarrow q)$; (ii) $p \Rightarrow (q \wedge r)$; (iii) $\neg(p \Leftrightarrow q)$.

S 5.2. Find the c.n.f of

$$\begin{array}{ll} (i) & \neg(p \Rightarrow \neg q) \\ (ii) & \neg p \wedge (q \Rightarrow p) \\ (iii) & \neg((p \wedge q) \vee r) \\ (iv) & \neg((p \vee q) \wedge r) \\ (v) & ((p \vee q) \wedge \neg r) \end{array}$$

S 5.3. Interpret the following quantified propositions in English and state which are true and which are false:
(i) $\forall$ s $\in S$ [($\neg$(s = "")) $\Leftrightarrow$ (LEN(s) $\geq$ 1)];
(ii) $\forall$ s $\in S$ [$\exists$ c $\in C$ [c = FIRST(s)]];
(iii) $\forall$ s $\in S$ [$\exists$ c $\in C$ [(s = "") $\vee$ c = FIRST(s))]].

S 5.4. Use Deduction Rules to prove which of the following arguments are valid and which are fallacious, you may only use a deduction rule that you have already proved via truth tables. No credit will be given for decisions that are not justified.

(i) If the apple is ripe, then it will be sweet;
The apple is sweet.
Therefore the apple is ripe.

(ii) If I go to the pub, I won't finish my revision;
If I don't finish my revision, I won't do well in the exam tomorrow.
Thus if I go to the pub, I won't do well in the exam tomorrow.

(iii) Gill is playing rugby if Tom is not in class;
If Gill is not playing rugby then Tom will be in class
Therefore Tom is not in class or Gill is not playing rugby.

(iv) Sam is studying maths or Sam is studying economics;
Sam is required to take logic if he is studying maths.
So Sam is an economics student or he is not required to take logic.

(v) English men wear bowler hats;
The man is wearing a bowler hat.
Therefore he must be English.

S 5.5. Comment on the following argument.
Storing on floppy disk is better than nothing. Nothing is better than a hard disk drive. Therefore, storing on floppy disk is better than a hard disk drive.

5.4 Answers for Tutorial 5

Compulsory problems

C 5.1. We first need to find the truth table for the expressions
A= $p \vee (q \wedge r)$, B= $\neg((p \wedge q) \vee r)$ and C= $p \Leftrightarrow (q \wedge r)$ and their negations.

p	q	r	q∧r	p∧q	(p∧q)∨r	A	B	C	¬A	¬B	¬C
1	1	1	1	1	1	1	0	1	0	1	0
1	1	0	0	1	1	1	0	0	0	1	1
1	0	1	0	0	1	1	0	0	0	1	1
1	0	0	0	0	0	1	1	0	0	0	1
0	1	1	1	0	1	1	0	0	0	1	1
0	1	0	0	0	0	0	1	1	1	0	0
0	0	1	0	0	1	0	0	1	1	1	0
0	0	0	0	0	0	0	1	1	1	0	0

(i) Using the columns corresponding to A and ¬A we obtain
d.n.f(A)= $(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r)$.
c.n.f(A)= ¬d.n.f(¬A)= $(p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r) \wedge (p \vee q \vee r)$.
(ii) Using the columns corresponding to B and ¬B we obtain
d.n.f(B) = $(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$.
c.n.f(B)= $(\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r) \wedge (p \vee q \vee \neg r)$.
(iii) Using the columns corresponding to C and ¬C we obtain
d.n.f(C) = $(p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$.
c.n.f(C)= ¬d.n.f(¬C)= $(\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r)$.

C 5.2. As seen above, we add the column corresponding to ¬H and obtain
d.n.f(H)= $(p \wedge q) \vee (\neg p \wedge \neg q)$; c.n.f(H)= ¬d.n.f(¬H)= $(\neg p \vee q) \wedge (p \vee \neg q)$.

C 5.3. As seen above, we add the column corresponding to ¬G and obtain
d.n.f(G) = $(p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$.
c.n.f(G)= ¬d.n.f(¬G)= $(\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r)$.

C 5.4. (i) There is at least one dog which can't bark;
(ii) There is at least one bird which can't fly;
(iii) There is at least one cat which likes to swim;
(i) There is at least one computer science student who knows mathematics.

C 5.5. (i) $\exists$ x $\in \mathbb{R}$ [$x^2 - 6x + 10 < 0$] F; (ii) $\exists$ x $\in \mathbb{R}$ [$x^2 < x$] T;
(iii) $\forall$ x $\in \mathbb{N}^+$ [$x^2 + 2x - 2 > 0$] T; (iv) $\forall$ x $\in \mathbb{Q}$ [$x^2 \neq 3$] T.

C 5.6. The detailed proof is available in the lecture slides (Lecture 5).

6 Vectors and complex numbers

6.1 Brief theoretical background

Vector: A quantity having **direction** as well as **magnitude**.

Notations: A vector is an ordered collection of n elements (components).

$\overrightarrow{a}$, $\underline{\mathbf{a}}$, $\overrightarrow{AB}$, $\mathbf{a}, \hat{\mathbf{a}}$

$\mathbf{a} = (a_1, a_2, a_3), \quad \mathbf{a} = (a_x, a_y, a_z), \quad \mathbf{a} = [a_x, a_y, a_z]$

$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \ (column)$

$A(a_1, a_2, a_3), B(b_1, b_2, b_3) \Rightarrow \overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3).$

Properties and operations:

- **Equality:** Two vectors are equal if they have the same modulus and direction.
- **Addition:**
  - $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (triangle rule for geometric vectors)
  - Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors. Their sum is
  $$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$
- **Multiplication by a scalar:** Let $\mathbf{a} = (a_1, a_2, a_3)$ be a vector and $\lambda \in \mathbb{R}$. Then
$$\lambda\mathbf{a} = \lambda(a_1, a_2, a_3) = (\lambda a_1, \lambda a_2, \lambda a_3).$$
- **Substraction:** $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}$.

Special vectors:

- zero vector: $\mathbf{0} = (0, 0, 0) \in \mathbb{R}^3$ (3D) or $\mathbf{0} = (0, 0) \in \mathbb{R}^2$ (2D)
- Unit vectors: $\|\mathbf{a}\| = 1$. Sometimes denoted by $\hat{\mathbf{a}}$.

Centre of mass: The system of points $A_1, \ldots, A_n$ has the centre of mass M s.t.

$$\overrightarrow{OM} = \frac{\overrightarrow{OA_1} + \overrightarrow{OA_2} + \cdots + \overrightarrow{OA_n}}{n}.$$

Centre of mass (with weights): The system of points $A_1, \ldots, A_n$ with masses $m_1, \ldots, m_n$ has the centre of mass M s.t.

$$\overrightarrow{OM} = \frac{m_1\overrightarrow{OA_1} + m_2\overrightarrow{OA_2} + \cdots + m_n\overrightarrow{OA_n}}{m_1 + m_2 + \cdots + m_n}.$$

Polygon addition rule: Let A, B, C, D be points in plane (space). Then

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \mathbf{0}. \quad \text{(same is true for a polygon } A_1, \ldots, A_n\text{)}$$

Dot (scalar) product: Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors.

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3 \\ &= \|a\| \|b\| \cos\theta\end{aligned}$$

Applications of the scalar product:

- The **modulus** of vector $\mathbf{a} = (a_1, a_2, a_3)$ is obtained from the formula

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

- The **angle** θ between vectors **a** and **b** is

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|a\| \|b\|}$$

- Vectors **a** and **b** are **perpendicular** if and only if

$$\theta = \pi/2 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$$

- **Unit vectors** for **a** (same direction, modulus one):

$$\mathbf{a} = \|\mathbf{a}\| \hat{\mathbf{a}}$$

Cross product (Vector product) Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors. The cross (vector) product is equal to

$$\mathbf{a} \times \mathbf{b} = \|a\| \|b\| \sin\theta \, \hat{\mathbf{n}}.$$

Standard base: $\mathbf{i} = (1, 0, 0)$; $\mathbf{j} = (0, 1, 0)$; $\mathbf{k} = (0, 0, 1)$.
The vector product for the standard base:

$$\begin{aligned}\mathbf{i} \times \mathbf{j} = \mathbf{k}; &\quad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} = \mathbf{i}; &\quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j}; &\quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\ \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} &= \mathbf{k} \times \mathbf{k} = \mathbf{0}\end{aligned}$$

Vector product in coordinate notation: For vectors

$$\begin{aligned}\mathbf{a} &= (a_1, a_2, a_3) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}. \\ \mathbf{b} &= (b_1, b_2, b_3) = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}.\end{aligned}$$

the vector product is defined as

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\end{aligned}$$

Applications of the cross (vector) product:

- The **area** of a triangle between vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is obtained from the formula

$$S = \|\mathbf{a} \times \mathbf{b}\|$$

- The **angle** θ between vectors **a** and **b** is

$$\sin\theta = \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|a\|\|b\|}$$

- Vectors **a** and **b** are **parallel** if and only if

$$\theta = 0 \Leftrightarrow \sin\theta = 0 \Leftrightarrow \mathbf{a} \times \mathbf{b} = 0$$

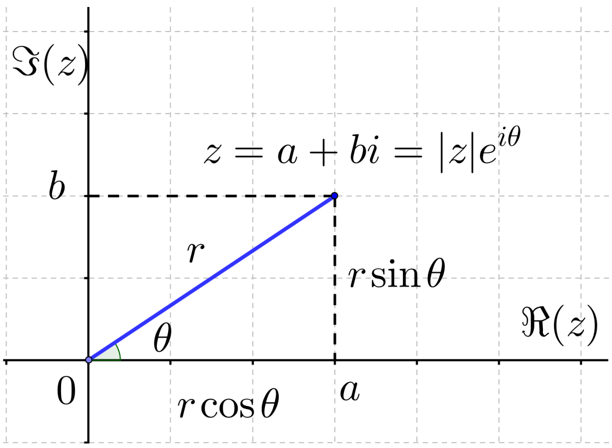
- Find a vector perpendicular to a plane. Take points $A, B, C \in \mathbb{R}^3$:

$$\overrightarrow{AB} \times \overrightarrow{AC} \quad \perp \quad \overrightarrow{AB}$$
$$\overrightarrow{AB} \times \overrightarrow{AC} \quad \perp \quad \overrightarrow{AC}$$

Complex numbers: The imaginary number: $i = \sqrt{-1}$ solves the equation $x^2 + 1 = 0$.

Notation:

$$z = a + bi, \quad a, b \in \mathbb{R}$$
$$a = \Re(z) = r\cos\theta \text{ (real part)}$$
$$b = \Im(z) = r\sin\theta, \text{ (imaginary part)}$$
$$r = |z| = \sqrt{a^2 + b^2}, \text{ (modulus)}$$
$$\theta \in [0, 2\pi) \text{ (argument)}$$



The complex conjugate The conjugate of $z = a + bi$ is the number $\overline{z} = a - bi$.

$$\begin{aligned}
\overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2} \\
\overline{z_1 z_2} &= \overline{z_1}\,\overline{z_2} \\
\overline{z_1 / z_2} &= \overline{z_1} / \overline{z_2} \\
\Re(z) &= \frac{1}{2}(z + \overline{z}), \\
\Im(z) &= \frac{1}{2i}(z - \overline{z})
\end{aligned}$$

Modulus property:

$$|z|^2 = z\overline{z} = (a + bi)(a - bi) = a^2 + b^2.$$

Basic operations with complex numbers: Let $z_1 = a + bi$ and $z_2 = c + di$. The following operations are defined:

- Addition:

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$$

- Substraction:

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$$

- Multiplication (remember that $i^2 = -1$)

$$\begin{aligned}
z_1 z_2 &= (a + bi)(c + di) \\
&= ac + adi + bci + bdi^2 \\
&= (ac - bd) + (ad + bc)i
\end{aligned}$$

- Division (remember that $i^2 = -1$ and $z\overline{z} = |z|^2$)

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\
&= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i
\end{aligned}$$

6.2 Compulsory questions

C 6.1. Decide which of the following are vector quantities: Velocity, mass, acceleration, weight, area, temperature, force, potential energy, volume.

C 6.2. In a system of vectors , a force of 10 newtons is represented by a line 5cm long. Find the magnitude of the force vector represented in the system by a line 2cm long.

C 6.3. i) Find the magnitude of the following vectors:

$$\mathbf{a} = -2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}; \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}; \quad \mathbf{c} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}; \quad \mathbf{d} = 2\mathbf{j} + 3\mathbf{k}.$$

ii) Find unit vectors parallel to the following vectors

$$\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}; \quad \mathbf{v} = 0.5\mathbf{i} + 2\mathbf{j} - \mathbf{k}; \quad \mathbf{w} = \mathbf{i} - 2\mathbf{j};$$

iii) Find $x \in \mathbb{R}$ such that the vector $\mathbf{c} = (x, 1, 1)$ is perpendicular on $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

C 6.4. i) Evaluate the dot products of the following pairs of vectors:

$$\begin{array}{ll} a) & \mathbf{p} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}, \quad \mathbf{q} = 3\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} \\ b) & \mathbf{m} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}, \quad \mathbf{l} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} \\ c) & \mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, \quad \mathbf{u} = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \end{array}$$

ii) Use the definition of the dot product and your answers to part i) to determine the angles between the pairs of vectors in part i).

iii) For each pair of vectors in i), find a vector that is perpendicular on both vectors.

C 6.5. Evaluate the cross products of the following vectors:

$$\begin{array}{ll} a) & \mathbf{p} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \mathbf{q} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k} \\ b) & \mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \quad \mathbf{b} = -2\mathbf{i} + 4\mathbf{j} + \mathbf{k} \\ c) & \mathbf{r} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \quad \mathbf{s} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}. \end{array}$$

C 6.6. Let $z_1 = 2 + 3i$ and $z_2 = 3 - 4i$. Compute the following

$$\begin{array}{ll} (1) & z_1 + z_2; \\ (2) & z_1 - z_2; \\ (3) & z_1 z_2; \\ (4) & |z_1|; \\ (5) & z_1/z_2; \\ (6) & z_1 - \overline{z}_2; \\ (7) & z_1\overline{z}_2; \\ (8) & z_2\overline{z}_2; \\ (9) & z_1^3; \\ (10) & z_1^4; \end{array}$$

6.3 Supplementary questions

S 6.1. Let ABCD be a quadrilateral. Find each of the single vectors equivalent to
(i) $\overrightarrow{AB} + \overrightarrow{BC}$;
(ii) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$;
(iii) $\overrightarrow{AD} + \overrightarrow{DC}$;
(iv) $\overrightarrow{BC} + \overrightarrow{CD}$;
(v) $\overrightarrow{AB} + \overrightarrow{DA}$;

S 6.2. O, A, B, C, D are five points such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{AB} = \mathbf{b}$, $\overrightarrow{AB} = \mathbf{a} + 2\mathbf{b}$ and $\overrightarrow{OD} = 2\mathbf{a} - \mathbf{b}$. Express $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of **a** and **b**.
S 6.3. Let ABCD be a quadrilateral, whose vertices have coordinates A(1,1), B(7,3), C(10,12), D(8,2). Find the vectors
(i) $\overrightarrow{AB}$; (ii) $\overrightarrow{AC}$; (iii) $\overrightarrow{AD}$; (iv) $\overrightarrow{BD}$;

S 6.4. For the quadrilateral ABCD in **S 6.3.** , check whether the diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular. If not, what is the angle between them ?

S 6.5. Let ABCD be a quadrilateral, whose vertices are A(1,1), B(7,3), C(10,12), D(8,2).
(i) Find the centre of mass G for the quadrilateral ABCD.
(ii) Find the centres of mass G_1, G_2, G_3, G_4 corresponding to each of the triangles ABC, BCD, CDA, DAB.
(iii) Find the centre of mass G_0 for the quadrilateral $G_1G_2G_3G_4$.
What is the relation between G_0 and G?

S 6.6. Answer the same questions as in problem **S 6.5.** , when the masses 1kg, 2kg, 3kg and 4kg are attached to A, B, C and D, respectively.

S 6.7. Compute the following expressions:
(i) $(1 + i) + (3 - 4i)$
(ii) $\frac{1+i}{1-i}$
(iii) i^{2012}

S 6.8. Solve in $\mathbb{C}$ the following equations:
(i) $z^2 + 4z + 2 = 0$;
(ii) $z^2 + 4z + 4 = 0$;
(iii) $z^2 + 4z + 5 = 0$.
(iv) $ix^2 + (1 - 4i)x + 4i - 1 = 0$.

S 6.9. Let $z = 2 + 3i$. Find the complex number $u = a + bi$ s.t. $u^2 = z$.

S 6.10. Given that z is not real and $|z| = 1$, prove that the number $w = \frac{z-1}{z+1}$ is a pure imaginary number.

6.4 Answers for Tutorial 6

Compulsory problems

C 6.1. Vectors: velocity, acceleration, force.

C 6.2. 4 newtons.

C 6.3. (i) $\|\mathbf{a}\| = \sqrt{29}$; $\|\mathbf{b}\| = \sqrt{21}$; $\|\mathbf{c}\| = \sqrt{18}$; $\|\mathbf{d}\| = \sqrt{13}$;

(ii) $\hat{\mathbf{u}} = \frac{1}{\sqrt{21}}(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$; $\hat{\mathbf{v}} = \frac{2}{\sqrt{21}}(0.5\mathbf{i} + 2\mathbf{j} - \mathbf{k})$; $\hat{\mathbf{w}} = \frac{1}{\sqrt{5}}(\mathbf{i} - 2\mathbf{j})$;

(iii) $\mathbf{c} \perp \mathbf{u}$: x=-2; $\mathbf{c} \perp \mathbf{v}$: x=-2; $\mathbf{c} \perp \mathbf{w}$: x=2.

C 6.4. (i) a) $\mathbf{p} \cdot \mathbf{q} = -9$; b) $\mathbf{m} \cdot \mathbf{l} = -10$; c) $\mathbf{v} \cdot \mathbf{u} = 0$.

(ii) a) $\cos\theta_{\mathbf{p},\mathbf{q}} = \frac{-9}{\sqrt{14}\sqrt{34}}$; b) $\cos\theta_{\mathbf{m},\mathbf{l}} = \frac{-10}{\sqrt{38}\sqrt{17}}$; c) $\cos\theta_{\mathbf{v},\mathbf{u}} = 0$.

(iii) In each case we have to find a vector $\mathbf{a} = (x, y, z)$ s.t.
a) $\mathbf{a} \cdot \mathbf{p} = \mathbf{a} \cdot \mathbf{q} = 0$. Solution: $\mathbf{a} = (t, 9/5t, 17/10t)$, $t \in \mathbb{R}$.
b) $\mathbf{a} \cdot \mathbf{m} = \mathbf{a} \cdot \mathbf{l} = 0$. Solution: $\mathbf{a} = (t, -7/2t, -16/13t)$, $t \in \mathbb{R}$.
c) $\mathbf{a} \cdot \mathbf{v} = \mathbf{a} \cdot \mathbf{u} = 0$. Solution: $\mathbf{a} = (t, -t, 1/2t)$, $t \in \mathbb{R}$.

C 6.5. a) $\mathbf{p} \times \mathbf{q} = -3\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}$; b) $\mathbf{a} \times \mathbf{b} = 14\mathbf{i} + 5\mathbf{j} + 8\mathbf{k}$; c) $\mathbf{r} \times \mathbf{s} = 9\mathbf{i} - 9\mathbf{j} - 3\mathbf{k}$.

C 6.6. (1) $5 - i$; (2) $-1 + 7i$; (3) $18 + i$; (4) $\sqrt{13}$; (5) $(-6 + 17i)/25$; (6) $-1 - i$; (7) $-6 + 17i$; (8) 25; (9) $-46 + 9i$; (10) $119 - 12i$.

Supplementary problems

S 6.1. (i) $\overrightarrow{AC}$; (ii) $\overrightarrow{AD}$; (iii) $\overrightarrow{AC}$; (iv) $\overrightarrow{BD}$; (v) $\overrightarrow{DB}$;
S 6.2.
S 6.3.
S 6.4.
S 6.5.
S 6.6.
S 6.7.
S 6.8.
S 6.9.
S 6.10.

7 Matrices

7.1 Brief theoretical background

Matrix: array of numbers called **elements**. Horizontal components are called **rows** (or lines) while the vertical components are called **columns**. A matrix with m rows and n columns is of **dimension** (size, order) $m \times n$.

Example: The following matrix has **2 rows** and **4 columns**

$$\begin{pmatrix} 2 & 0 & 1 & 3 \\ 1 & 3 & 1 & 2 \end{pmatrix}$$

Serious Definition: A matrix of order $m \times n$ in X is a **function**

$$\mathbf{A} : \{1, \ldots, m\} \times \{1, \ldots, n\} \to X, \quad \mathbf{A}(i, j) = a_{ij}.$$

Special times of matrices (orderwise):

- row matrix

$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$$

- column matrix

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \text{ or } \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

- square matrix - same number of rows as columns (i.e. m=n)

$$\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \text{ or } \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

Matrix notation

- Matrices are usually denoted by capital letters.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$$

- a matrix of order $m \times n$ is denoted by

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

- a_{ij} denotes the element in row i and column j.

Special matrices

- **The zero matrix:** all its elements are zero. The zero 2×2 matrix is

$$\underline{\mathbf{0}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- **Diagonal matrix:** Square, elements outside the main diagonal are zero.

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \ldots$$

- **The identity matrix:** Diagonal matrix with 1 on the main diagonal.

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots$$

Equality of matrices Two matrices are **equal** if they have

- same dimension
- equal corresponding elements

Matrix operations

- **Transpose of a matrix:** The transpose of a matrix changes columns with rows. First column becomes the first row, second column becomes the second row, etc.

$$\text{If } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

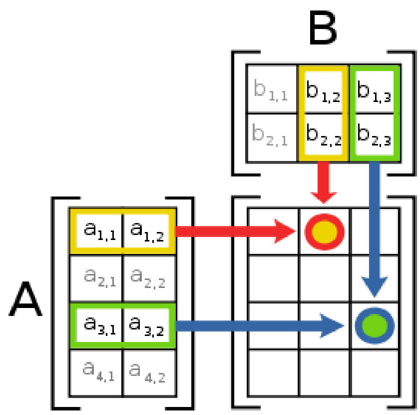
  Remark: The transpose of an $m \times n$ matrix is an $n \times m$ matrix.

- **Addition and substraction:**
  - element by element operation
  - matrices need to be of the same order (dimension, size)
  - addition and substraction are done element by element

- **Multiplication by a scalar:**
  Let $\mathbf{A}$ be a matrix and $\lambda \in \mathbb{R}$ a scalar (number).
  The matrix $\lambda\mathbf{A}$ is obtained by multiplying each element of $\mathbf{A}$ by λ.

$$\lambda\mathbf{A} = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$$

- **Multiplication of two matrices:** the rule of **row-column multiplication**. Let **A**, **B** be matrices of sizes $m \times n$ and $n \times p$ respectively.
  The element c_{ij} in $\mathbf{C} = \mathbf{AB}$ is obtained by multiplying
  - the i-th row of **A** by
  - the j-th column of **B**



- determinant of 2×2 matrix

$$|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \det A$$

- inverse of 2×2 matrix
  Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix. The inverse matrix is

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Solution of a linear system of equations
Consider the system of linear equations

$$\begin{aligned} 2x + 3y &= 13 \\ 3x + 4y &= 18 \end{aligned}$$

This can be written in matrix form as

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underline{x} = \mathbf{A}^{-1}\underline{b} = (-1)\begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix} \begin{bmatrix} 13 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

7.2 Compulsory questions

C 7.1. State the dimension of each of the following matrices:

$$(a) \quad \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{bmatrix}, \quad (b) \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 2 & 3 \\ 3 & 1 \end{bmatrix}, \quad (c) \quad \mathbf{C} = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad (d) \quad \mathbf{D} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

C 7.2. Indicate whether the following statements are true or false, for

$$(a) \quad \mathbf{A} = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \quad (b) \quad \mathbf{B} = \begin{bmatrix} 5 & 3 & 1 \end{bmatrix},$$

$$(c) \quad \mathbf{C} = \begin{bmatrix} 10/2 & 6/2 & 1 \end{bmatrix}, \quad (d) \quad \mathbf{D} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad (e) \quad \mathbf{E} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$

(i) $\mathbf{D} = \mathbf{A}$; (ii) $\mathbf{D} = \mathbf{E}$; (iii) $\mathbf{B} = \mathbf{C}$; (iv) $\mathbf{A} = \mathbf{B}$; (v) $\mathbf{D} - \mathbf{C} = \underline{0}$.

C 7.3. Let **A**, **B** and **C** be the matrices defined below

$$\mathbf{A} = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix},$$

Compute each of the following
(i) $\mathbf{A} + \mathbf{C}$; (ii) $\mathbf{A} - \mathbf{B}$; (iii) $2\mathbf{A}$; (iv) $-3\mathbf{C}$; (v) $\mathbf{A} - 2\mathbf{B} + 4\mathbf{C}$; (vi) $\mathbf{B}^T$; (vii) $\mathbf{B} + \mathbf{C}^T$.

C 7.4. Form the product of the following matrices, stating whether the result is a column or row vector:

$$(i) \quad \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \quad (ii) \quad \begin{bmatrix} 5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix},$$

$$(iii) \quad \begin{bmatrix} 5 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad (iv) \quad \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \end{bmatrix},$$

C 7.5. Find the determinant of the matrix $\begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}$, and find its inverse.
Use the result to solve the pair of simultaneous equations

$$5x + 3y = 19$$
$$4x + 2y = 14$$

C 7.6.

$$\text{Let } \mathbf{A} = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & 2 \\ 5 & 2 & 1 \end{bmatrix},$$

Compute (i) **AB**; (ii) **AC**; (iii) **CA**.
Show that $\mathbf{AC} \neq \mathbf{CA}$, therefore the matrix multiplication is generally not true.

7.3 Supplementary questions

S 7.1. Compute (i) $\mathbf{A}+\mathbf{B}$; (ii) $\mathbf{A}^T+\mathbf{B}^T$; (iii) $(\mathbf{A}+\mathbf{B})^T$ for the matrices below.

$$\text{Let } \mathbf{A} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}, \text{ and } \quad \mathbf{B} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 2 & 3 \end{bmatrix}$$

What is the connection between the results for parts (i) and (ii) above?

S 7.2. Compute (i) $\mathbf{C}+\mathbf{D}$; (ii) $\mathbf{C}-\mathbf{D}$; (iii) $\mathbf{C}+\mathbf{C}^T$; (iv) $(\mathbf{C}-\mathbf{D})^T$; (v) $(\mathbf{C}^T)^T$ for

$$\mathbf{C} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}, \text{ and } \quad \mathbf{D} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 2 & 3 \end{bmatrix}$$

S 7.3. Compute **AI** and **IA** for the matrices

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}, \text{ and } \quad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

What does this suggest about the multiplication by the identity matrix ?

S 7.4. Find (i) $\mathbf{A}^T$; (ii) $\mathbf{A}\mathbf{A}^T$ and (iii) $\mathbf{A}^T\mathbf{A}$ for the matrix $\mathbf{A} = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$.

S 7.5. Given the following matrices

$$\mathbf{A} = \begin{bmatrix} 5 & 3 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2 & 3 & 2 \\ 5 & 2 & 1 \end{bmatrix},$$

State the size of the following products and compute the ones that exist.
(i) **AB**; (ii) **BA**; (iii) **CD**; (iv) **DC**; (v) **DB**; (vi) **BD**; (vii) $\mathbf{A}^2$. (viii) $\mathbf{C}^2$.

S 7.6. Evaluate the determinants of the following matrices:

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 2 & 3 & 2 \\ 5 & 2 & 1 \\ 1 & 4 & 3 \end{bmatrix}.$$

Find the inverses of the 2×2 matrices **B**, **C** and **D**.

S 7.7. Solve the following systems of linear equations using the inverse matrix method.

$$\begin{array}{llll} 3x-y=4; & 2x-y=-11; & 4x-3y=6; & 3x-4y=-9 \\ x+2y=13; & x+2y=2; & -2x+y=-4; & x+2y=2 \end{array}$$

S 7.8. State the reason why the inverse matrix method may not be applied to the system of equations $4x+6y=12; 1+3y=14$.

7.4 Answers for Tutorial 7

Compulsory problems

C 7.1. (a) 2×3; (b) 3×2; (c) 1×2; (d) 3×1.

C 7.2. (a) F; (b) F; (c) T; (d) F; (e) F.

C 7.3. (i) $\mathbf{A}+\mathbf{C} = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 4 & 5 \end{bmatrix}$; (ii) $\mathbf{A}-\mathbf{B} = \begin{bmatrix} 4 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$; (iii) $2\mathbf{A} = \begin{bmatrix} 10 & 6 & 2 \\ 4 & 2 & 6 \end{bmatrix}$;
(iv) -3$\mathbf{C} = \begin{bmatrix} 0 & -3 & -3 \\ -6 & -9 & -6 \end{bmatrix}$; (v) $\mathbf{A}-2\mathbf{B}+4\mathbf{C} = \begin{bmatrix} 3 & 3 & -1 \\ 2 & 7 & 7 \end{bmatrix}$; (vi) $\mathbf{B}^T = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 3 & 2 \end{bmatrix}$;
(vii) operation not defined.

C 7.4. (i) row matrix: $\begin{bmatrix} 13 & 21 \end{bmatrix}$; (ii) row matrix: $\begin{bmatrix} 16 & 22 & 19 \end{bmatrix}$;
(iii) column matrix: $\begin{bmatrix} 16 \\ 13 \end{bmatrix}$; (iv) square matrix: $\begin{bmatrix} 5 & 10 & 20 \\ 3 & 6 & 12 \\ 1 & 2 & 4 \end{bmatrix}$;

C 7.5. $\det \mathbf{A} = -2$; $\mathbf{A}^{-1} = \begin{bmatrix} -1 & 1.5 \\ 2 & -2.5 \end{bmatrix}$; $x = 2; y = 3$.

C 7.6. (i) $\mathbf{AB} = \begin{bmatrix} 19 & 20 & 24 \\ 12 & 10 & 17 \\ 11 & 11 & 17 \end{bmatrix}$; (ii) $\mathbf{AC} = \begin{bmatrix} 11 & 16 & 12 \\ 17 & 11 & 7 \\ 12 & 10 & 7 \end{bmatrix}$; (iii) $\mathbf{CA} = \begin{bmatrix} 5 & 2 & 5 \\ 22 & 11 & 15 \\ 32 & 18 & 13 \end{bmatrix}$.

Supplementary problems

S 7.1.
S 7.2.
S 7.3.
S 7.4.
S 7.5.
S 7.6.
S 7.7.
S 7.8.

8 Computer graphics in 2D: Matrix transformations

8.1 Brief theoretical background

Matrix representation: Matrices are useful for representing

- **Points:** represented by a column matrix (position vector)

$$X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

- **Lines:** defined by two points and represented by a 2×2 matrix.
  The matrix $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$ represents a line through points $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

- **Polygons:** Represented by a $2 \times n$ matrix.
  The matrix $\begin{bmatrix} 3 & -1 & 1 \\ 2 & 4 & 1 \end{bmatrix}$ represents the triangle $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Matrix transformations: Consider the point X and a transformation matrix T given by

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Matrix T pre-multiplies point X, generating another point X^*

$$TX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = X^*$$

Example: The point $X = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is transformed by the matrix $T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$X^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = TX = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(-1) + 2(2) \\ 3(-1) + 4(2) \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Transformation of lines:

- Take two points X, Y and a transformation T
- Apply the transformation T to both X, Y (they determine a line XY)
- The transformed points X^*, Y^* determine a **new line** X^*Y^*

Remark: Linear transformations change lines into lines.

8.2 Linear transformations in 2D

Consider the point X, vector V and a transformation matrix T given by

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad V = \begin{bmatrix} u \\ v \end{bmatrix}, \quad T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The following linear transformations can be defined.

Translation: The translation of point X by vector V is defined by

$$X^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = X + V = \begin{bmatrix} x + u \\ y + v \end{bmatrix}.$$

Scaling: Scales the horizontal and vertical coordinates. The transformation matrix is

$$T = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}; \quad X^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix} = TX = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \beta y \end{bmatrix}.$$

Rotation: The line from (0,0) to the point (x, y), is rotated by an angle θ.

By convention **anticlockwise movement is considered positive**.
We say that (x, y) has rotated about the origin to (x^*, y^*). The transformation matrix is:

$$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Reflections:

- in the X axis: **changes the sign of the y co-ordinate**

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \quad X^* = TX = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

- in the Y axis: **changes the sign of the x co-ordinate**

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; \quad X^* = TX = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

- in the $y = x$ line: **interchanges the x and y co-ordinates**.

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad X^* = TX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

- in the $y = -x$ line: **interchanges x and y co-ordinates and changes their signs**.

$$T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}; \quad X^* = TX = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$$

Shearing (optional): A shear is a **distortion**. The transformation matrix

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The 'off-diagonal' elements b and c determine the kind of shear produced.

- b produces x-direction shear,
- c produces y-direction shear.

Combination of transformations: If the point X is subject to two transformations T_1 and T_2, then it is subject to the composed transformation $T = T_2T_1$.

$$X^* = T_2(T_1X) = (T_2T_1)X = TX.$$

8.3 Compulsory questions

C 8.1. A straight line joins the points $(3, 2)$ and $(4, 5)$. Describe the transformed line produced by the matrix $T = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$.

C 8.2. Write down a transformation matrix which
a) scales vertical lines by a factor of 3 and leaves horizontal lines unchanged.
b) scales horizontal lines by a factor of 2 and vertical lines by a factor of 5.

C 8.3. The point (3,1) is rotated anticlockwise about the origin through an angle of 45°. Calculate the position of the new point.

C 8.4. Write down the following matrices.
a) Rotation about the origin, anti- clockwise through $\pi/6$ radians.
b) Rotation about the origin, clockwise through $\pi/6$.
c) Use the appropriate matrix to find the new position of the point $(2, 3)$ after it has been rotated anti-clockwise through $\pi/6$ about the origin.
d) Calculate the product of the two matrices found in (a) and (b). Explain your answer.

C 8.5. Consider the 2D unit square having coordinates (0,0), (0,1), (1,0), (1,1).
Determine the new co-ordinates of the figure after the following transformations:
(1) Translation of vector (1,2); (2) horizontal scaling of factor 2;
(3) vertical scaling of factor 3; (4) mixed scaling of factors 3 and 4;
(5) Rotation of angle $\pi/6$; (6) Rotation of angle 90°; (7) Reflection in the X axis;
(8) Reflection in the Y axis; (9) Reflection in the $y = x$ line;
(10) Reflection in the $y = -x$ line.

C 8.6. Point (2,3) is reflected in Y axis, then reflected in $y = x$, and finally rotated by 90°. Find the co-ordinates of the final point.

8.4 Supplementary questions

S 8.1. Show that for any transformation matrix R, which represents a rotation, that $R^T = R^{-1}$. (i.e. the transpose of a rotation matrix is equal to its inverse).

S 8.2. Let R be the rotation matrix about the origin through angle θ.
a) State the rotation matrix S representing a **clockwise** rotation through an angle θ.
b) Calculate R^{-1} using the result of Question **S 8.1.** above.
c) What can be concluded from parts (a) and (b) above.

S 8.3. Write down transformation matrices which will:
a) scale vertical lines by a factor of 5 and leave horizontal lines unchanged.
b) scale horizontal lines by a factor of 4 and vertical lines by a factor 0.25.
c) Use the above matrices to find the new co-ordinates of the points (2,1) and (-2,-1), performing each transformation separately.

S 8.4. Point A(2,-1) is reflected in X axis, then rotated by 90° and finally reflected in $y = -x$. Find the co-ordinates of the final point.

S 8.5. Find the coordinates of the unit square defined in **C 8.6.**, subject to the shear transformations given by the matrices:
(a) $T = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$; (b) $T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$; (c) $T = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$; (d) $T = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.

8.5 Answers for Tutorial 8

Compulsory problems

C 8.1. The first line is $\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$. The transformed line is

$$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 26 \\ 7 & 7 \end{bmatrix}$$

C 8.2. a) $T = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$; b) $T = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$.

C 8.3. The new point is given by: $T \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$.

C 8.4. Using the formulas for the rotation matrix we obtain

(a) $R = \begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$.

(b) $S = \begin{bmatrix} \cos(-\pi/6) & -\sin(-\pi/6) \\ \sin(-\pi/6) & \cos(-\pi/6) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$.

(c) $R \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.2321 \\ 3.5981 \end{bmatrix}$.

(d) The product is the identity matrix, as the two transformations cancel each other.

C 8.5. First, the unit square is represented by the matrix $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(1) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix}$.

(2) $T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(3) $T = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$; $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix}$.

(4) $T = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$; $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 3 \\ 0 & 0 & 4 & 4 \end{bmatrix}$.

(5) $T = \begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix}$

(6) $T = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

(7) $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$.

(8) $T = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(9) $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

(10) $T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$; $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$.

C 8.6. The following matrix transformations are involved

$$T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The final transformation is

$$T = T_3T_2T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

The co-ordinates of the transformed point are $(-3, -2)$.

Supplementary problems

S 8.1. The transformation matrix for a rotation of angle θ is $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, whose inverse is the rotation of angle $-\theta$, which satisfies

$$R(-\theta) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R(\theta)^T,$$

because $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$.

S 8.2. The answer can easily be obtained from **S 8.3.** and **C 8.4.**

S 8.3. a) $T = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$; b) $T = \begin{bmatrix} 1/4 & 0 \\ 0 & 4 \end{bmatrix}$. (c) (10, 1), (-10, -1); (1/2,8), (-1/2, -8).

S 8.4. The following matrix transformations are involved

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

The final transformation is

$$T = T_3T_2T_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The co-ordinates of the transformed point are $(-2, 1)$.

S 8.5. (a) $T = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

9 Elements of graph theory

9.1 Brief theoretical background

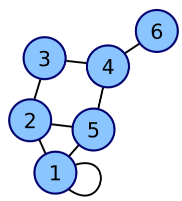
Graph: mathematical structure used to model pairwise relations between objects from a certain collection.

Basic elements of a graph G:

- **vertices:** (nodes) $V(G)$: the set of points
- **edges:** (arcs) $E(G)$: lines connecting two vertices (directed or not)
- **loops:** edge that connects a vertex to itself
- **order** (degree) of a node: number of 'incident' edges.
- **trail:** sequence of arcs, where the end node of one arc is the start of the next.
- **path:** trail in which no node is passed through more than once
- **cycle:** path with an extra arc joining the final node to the initial node.

Example:

$V(G) = \{1, 2, 3, 4, 5, 6\}$.
$E(G) = \{(1, 1), (1, 2), (3, 4), (4, 6), (4, 5), (5, 1), (5, 2)\}$.
$\text{ord}(3) = 2$.
loop: $(1, 1)$
trail: $1 - 2 - 3 - 4 - 5 - 2 - 3$.
path: $1 - 2 - 3 - 4 - 5 - 6$.
cycle: $1 - 2 - 5 - 1$.



Special graphs

- **connected:** It is possible to reach any node from any node
- **complete:** Any two vertices are connected by an arc (Notation: K_3, K_4, etc.)
- **planar:** It can be drawn so that the arcs do not cross
- **simple:** A graph with no loops or multiple arcs
- **tree:** Simple graph with no cycles
- **spanning tree** (for a connected, undirected graph): subgraph that is a tree and connects all the vertices together.
- **network:** A graph with weighted arcs

Adjacency matrix: For a graph G with n vertices this is a $n \times n$ matrix such that

- non-diagonal entry a_{ij}: number of edges from vertex i to vertex j,
- diagonal entry a_{ii}: number of edges from vertex i to itself taken
  - once - directed graphs
  - twice - undirected graphs

Standard operations on binary trees:
The set of all trees with vertices in set X is denoted by $BTREE(X)$.
For a given tree $t \in BTREE(X)$ the following operations are defined

- **ROOT:** $BTREE(X) \to X$ (finds root)
- **LEFT:** $BTREE(X) \to BTREE(X)$ (finds left branch)
- **RIGHT:** $BTREE(X) \to BTREE(X)$ (finds right branch)
- **ISEMPTYTREE:** $BTREE(X) \to B$ (checks if tree is empty)

For t and r trees in $BTREE(X)$ and $s \in X$ one can define the function:

- **MAKE:** $BTREE(X) \times X \times BTREE(X) \to BTREE(X)$
  The output of **MAKE**(t, s, r) is a tree with left branch t, root s and right branch r.

Network problems

- minimum spanning tree (Kruskal, Prim)
- shortest path (Dijkstra)

Kruskal's algorithm (minimum spanning tree):

- List the arcs in order of increasing weight
- Choose the arc with least weight
- Build a tree by working down the list choosing arcs provided they do **not** form a cycle when added to the arcs already chosen
- Stop when no more arcs can be chosen

Prims's algorithm (minimum spanning tree):

- Choose a node
- Build a tree by choosing the minimum weight arc that joins a node that has not yet been chosen to one that has. Add this arc and the node at its end to the tree
- Repeat the tree building process until all the nodes have been chosen

Shortest path problem: Find the minimal cost path between two vertices in a graph.

9.2 Compulsory problems

C 9.1. Prove that the sum of the degrees of the vertices of any finite graph is even.

C 9.2. Draw the graphs represented by the following adjacency matrices:

(i)

	a	b	c	d	e
a	0	1	1	0	1
b	1	0	1	1	1
c	1	1	0	0	0
d	0	1	0	0	1
e	1	1	0	1	0

;

(ii)

	a	b	c	d	e
a	0	1	0	0	1
b	1	0	1	1	0
c	0	1	1	1	0
d	0	1	1	0	1
e	1	0	0	1	1

;

(iii)

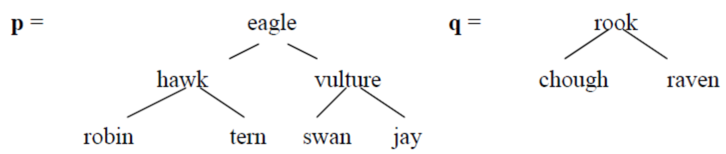
	a	b	c	d	e
a	0	1	0	0	1
b	0	0	0	0	0
c	0	0	0	0	0
d	0	0	0	0	1
e	0	0	0	0	0

C 9.3. a) Write down the information in **C 9.2.** as a relation $R \subset X \times X$ where $X = \{a, b, c, d, e\}$ i.e. as a set of ordered pairs.

b) Which if any of the graphs in **C 9.2.** have the following properties: connected, complete, directed, undirected, contain a cycle, contain a loop, is a tree, contains a vertex which is even, contains a vertex which is odd? Justify your answers.

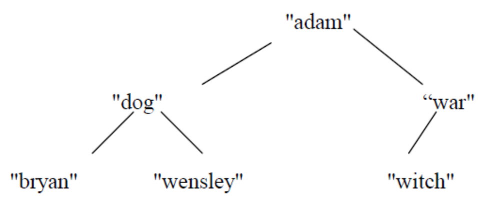
C 9.4. In a group of people John likes Mary, Brian and Emma; Brian likes Mary and Sue; Mary likes John and Sue; Emma likes Mary, John and Brian; Sue likes nobody. Draw a graph showing who likes who.

C 9.5. Given the trees p and q from BTREE(BIRDS) defined below



Write down the following:
i) LEFT(RIGHT(p));
ii) ISEMPTYTREE(RIGHT(RIGHT(q)));
iii) ROOT(LEFT(p));
iv) MAKE(q, gull, q);
v) MAKE (LEFT(p), ROOT(q), RIGHT(p));
vi) MAKE(LEFT(LEFT(q)), ROOT(p), MAKE(q, puffin, RIGHT(p))).

C 9.6. Given the tree u from BTREE(S) defined below:



Evaluate each of the following:
i) LEFT(RIGHT(LEFT(u)));
ii) RIGHT(LEFT(RIGHT(u)));
iii) ROOT(u) $+_S$ ROOT(LEFT(RIGHT(u)).

9.3 Supplementary problems

S 9.1. Show that a tree with n vertices has exactly $n-1$ edges.

S 9.2. Prove that a complete graph with n vertices (K_n) contains $n(n-1)/2$ edges.

S 9.3. Show that every simple graph has two vertices of the same degree.

S 9.4. Use standard functions of Btree(X) to give a formal description in standard form of a function REPL that inputs a binary tree and an item from X, and replaces the root of the left subtree by the input item.

S 9.5. Using standard functions of Btree(X), describe the following functions.
i) A function F that inputs a vertex and a binary tree, and replaces the root of the tree by the input vertex.
ii) A function G that inputs a binary tree and exchanges the left and right subtrees.
Give an example in each case to illustrate what the function does.

S 9.6. Let a network be given by the distance matrix

	A	B	C	D	E	F	G
A	-	10	5	4	3	-	-
B	10	-	8	9	7	-	2
C	5	8	-	-	5	-	-
D	4	9	-	-	10	6	4
E	3	7	5	10	-	8	9
F	-	-	-	6	8	-	11
G	-	2	-	4	9	11	-

(a) Find the minimum spanning tree using Prim's algorithm.
(b) Find the minimum spanning tree using Kruskal's algorithm.
(c) Find the shortest path between A and G.

9.4 Answers for Tutorial 9

Compulsory problems
C 9.1. Each arc contributes with 2 to the sum of degrees, therefore this sum should be twice the number of arcs.

C 9.2. You just have to draw the vertices $\{a, b, c, d, e\}$ and directed segments from x to y if $(x, y) = 1$ in the adjacency matrix.

C 9.3. We write down the directed segments corresponding to each matrix
(i) $G_1 = \{(a,b),(a,c),(a,e),(b,a),(b,c),(b,d),(b,e),(c,a),(c,b),(d,b),(d,e),(e,a),(e,b),(e,d)\}$;
(ii) $G_2 = \{(a,b),(a,e),(b,a),(b,c),(b,d),(c,b),(c,c),(c,d)(d,b),(d,c),(d,e)(e,a),(e,d),(e,e)\}$;
(iii) $G_3 = \{(a,b),(a,e),(d,e)\}$.

C 9.4. The adjacency matrix is

	John	Mary	Brian	Emma	Sue
John	0	1	1	1	0
Mary	1	0	0	0	1
Brian	0	1	0	0	1
Emma	1	1	1	0	0
Sue	0	0	0	0	0

Drawing the graph is straightforward.

C 9.5. i) swan; ii) False; iii) hawk;
iv) Tree whose left branch is q, root is ”gull” and right branch is q;
v) Tree whose left branch is LEFT(p), root is ”rook” and right branch is RIGHT(p);
vi) Tree whose left branch is empty, root is ”eagle” and right branch is another tree; this smaller tree has left branch q, root ”puffin” and right branch RIGHT(p).

C 9.6. i) empty; ii) empty; iii) ”adamwitch”.

Supplementary problems
S 9.1.
S 9.2.
S 9.3.
S 9.4.
S 9.5.
S 9.6.

10 Elements of number theory and cryptography

10.1 Brief theoretical background

Prime number: n prime if his only positive divisors are 1 and itself.

Fundamental theorem of arithmetic (unique factorization theorem):
Every integer $n \geq 1$ is either prime or is the product of prime numbers.
Every integer $n > 1$ can be represented **uniquely** as a product of prime powers

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}, \quad \text{(canonical representation)}$$

where $p_1 < p_2 < ... < p_k$ are primes and a_i are positive integers.

Greatest common divisor: $\gcd(a, b)$ or (a, b)

Least common multiple: $\operatorname{lcm}(a, b)$ or $[a, b]$

Definition: Numbers a and b are **relatively prime** if $\gcd(a, b) = 1$.

Properties of gcd(a,b) and lcm(a,b): If $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}, \quad b = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$ where $p_1 < p_2 < ... < p_k$ are primes and a_i, b_i are non-negative integers.

The following properties hold:

$$\begin{aligned}
a \cdot b &= p_1^{a_1+b_1}\, p_2^{a_2+b_2} \cdots p_k^{a_k+b_k} \\
\gcd(a, b) &= p_1^{\min(a_1,b_1)}\, p_2^{\min(a_2,b_2)} \cdots p_k^{\min(a_k,b_k)} \\
\operatorname{lcm}(a, b) &= p_1^{\max(a_1,b_1)}\, p_2^{\max(a_2,b_2)} \cdots p_k^{\max(a_k,b_k)} \\
a \cdot b &= \operatorname{lcm}(a, b) \cdot \gcd(a, b)
\end{aligned}$$

Euler's totient function $\varphi(n)$: Number of integers $1 \leq k \leq n$ for which $\gcd(n, k) = 1$.

Formula for prime powers: If p is prime, $\gcd(m, n) = 1$ and $k > 1$ then

$$\begin{aligned}
\varphi(p) &= p - 1 \\
\varphi(p^k) &= p^{k-1}(p - 1) \\
\varphi(mn) &= \varphi(m)\varphi(n).
\end{aligned}$$

Formula (Euler): If the factorisation of n is $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ then

$$\varphi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$

Example: $\varphi(36) = \varphi\left(2^2 3^2\right) = 36 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) = 36 \cdot \frac{1}{2} \cdot \frac{2}{3} = 12.$

Modular arithmetic: system of arithmetic for integers (clock arithmetic), where numbers "wrap around" upon reaching a certain value called **modulus**.

Definition: For $n \in \mathbb{N}$, two integers a and b are called **congruent modulo** n:

$$a \equiv b \pmod{n},$$

if their difference $a - b$ is an integer multiple of n.

Operations: If $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n}$, we can define

- Addition: $a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$
  - $2 + 1 \pmod{6} \equiv 3 \pmod{6}$
  - $2 + 4 \pmod{6} \equiv (2 + 4) \pmod{6} = 0 \pmod{6}$
- Substraction $a_1 - a_2 \equiv b_1 - b_2 \pmod{n}$
  - $2 - 1 \pmod{6} \equiv 3 \pmod{6}$
  - $2 - 4 \pmod{6} \equiv (2 - 4) \pmod{6} = 4 \pmod{6}$
- Multiplication $a_1 a_2 \equiv b_1 b_2 \pmod{n}$.
  - $2 \times 2 \pmod{6} \equiv 4 \pmod{6}$
  - $2 \times 3 \pmod{6} \equiv 6 \pmod{6} = 0 \pmod{6}$

Caesar cypher:

- Alphabet: $A = \{A, \ldots, Z\}$ so $n = 26$ (or numbers 0..25).
- m and c are single characters
- encrypt key $e \in \{0, \ldots, 25\}$
- encrypt equation: $c = f(e, m) = m + e \pmod{26}$
- decrypt equation: $m = g(d, c) = c + d \pmod{26}$ $(f = g)$
- clearly $d + e = 0 \pmod{26}$ (both need to be kept secret!)
- Method:

$$\begin{aligned} g(d,c) &= g(d, f(e,m)) = g(d, m + e \pmod{26}) \\ &= ((m + e) \pmod{26}) + d) \pmod{26} = m \end{aligned}$$

- Example: $e = 23$ and $n = 26$, $d = 3$
- look-up table $(e = 3)$:

$$\begin{aligned} plaintext: &\quad A, B, C, D, \ldots \\ cyphertext: &\quad D, E, F, G, \ldots \end{aligned}$$

10.2 Compulsory problems

C 10.1. Find 5 pairs of Pythagorean triples $a, b, c \in \mathbb{N}$ s.t. $a^2 = b^2 + c^2$.

C 10.2. Factorize the numbers 96, 144, 286, 777 and 1001.

C 10.3. Find $\gcd(144, 96)$, $\gcd(1001, 777)$ and $\gcd(1001, 286)$.

C 10.4. Find $\text{lcm}(144, 96)$, $\text{lcm}(1001, 777)$ and $\text{lcm}(1001, 286)$.

C 10.5. Compute (find results in the set $\{0, \ldots, n-1\}$ for $\pmod n$):

$$5+2 \pmod 2, \quad 3\times 5 \pmod 2$$
$$3+7 \pmod 5, \quad 3-7 \pmod 5, \quad 3\times 7 \pmod 5.$$
$$3+7 \pmod{11}, \quad 3-7 \pmod{11}, \quad 3\times 7 \pmod{11}.$$

C 10.6. Find the $\varphi(n)$ function for the numbers $n = 11, 21, 24, 36, 49, 81, 100$.

C 10.7. Code the message "I LIKE MATHS" using a Caesar cipher with key $e = 10$.

10.3 Supplementary problems

S 10.1. Find the largest prime number smaller than 1000.

S 10.2. Goldbach conjecture:
For n even there are a, b primes s.t. $n = a + b$. Check this results for $n \leq 100$.

S 10.3. Find 7 pairs of twin primes.

S 10.4. Find 5 Mersenne primes of the form $M_p = 2^p - 1$.

S 10.5. Factorize the numbers 703, 779, 968, 1002, 1440, 1547, 1763, 2261.

S 10.6. Find $\gcd(1440, 968)$, $\gcd(1002, 703)$, $\gcd(2261, 1547)$ and $\gcd(779, 1763)$.

S 10.7. Find $\text{lcm}(1440, 968)$, $\text{lcm}(1002, 703)$, $\text{lcm}(2261, 1547)$ and $\gcd(779, 1763)$.

S 10.8. Solve the following equations (in the set $\{0, \ldots, n-1\}$ for $\pmod n$):

$$1+x=0 \pmod 2, \quad 3\times x = 1 \pmod 2$$
$$3+x=1 \pmod 8, \quad 3-x=6 \pmod 8, \quad 4\times x=7 \pmod 8.$$
$$5+x=3 \pmod{11}, \quad 3-x=7 \pmod{11}, \quad 4\times x=8 \pmod{11}.$$

S 10.9. Find the $\varphi(n)$ function for the numbers $n = 19, 31, 28, 48, 144, 169, 1001$.

S 10.10. Find the RSA key d for the public key $e = 5$, for $n = 65$.

10.4 Answers for Tutorial 10

Compulsory problems

C 10.1. (3,4,5), (5,12,13), (7, 24, 25), (8, 15, 17), (9, 40, 41).

C 10.2. $96 = 2^5 \cdot 3^1$; $144 = 2^4 \cdot 3^2$; $286 = 2 \cdot 11 \cdot 13$; $777 = 3 \cdot 7 \cdot 37$; $1001 = 7 \cdot 11 \cdot 13$.

C 10.3.

$$\begin{aligned} \gcd(96, 144) &= 2^4 \cdot 3 = 48; \\ \gcd(1001, 777) &= 7; \\ \gcd(1001, 286) &= 11 \cdot 13 = 143. \end{aligned}$$

C 10.4.

$$\begin{aligned} \operatorname{lcm}(96, 144) &= 2^5 \cdot 3^2 = 288; \\ \operatorname{lcm}(1001, 777) &= 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 = 111111; \\ \operatorname{lcm}(1001, 286) &= 2 \cdot 7 \cdot 11 \cdot 13 = 2002. \end{aligned}$$

C 10.5.

$$\begin{aligned} &1 \pmod 2, \quad 1 \pmod 2 \\ &0 \pmod 5, \quad 1 \pmod 5, \quad 1 \pmod 5. \\ &10 \pmod{11}, \quad 7 \pmod{11}, \quad 10 \pmod{11}. \end{aligned}$$

C 10.6. One just needs to factorise the numbers and apply the multiplicity of $\varphi(n)$.

$$\begin{aligned} \varphi(11) &= 10; \\ \varphi(21) &= \varphi(3 \cdot 7) = \varphi(3) \cdot \varphi(7) = 2 \cdot 6 = 12; \\ \varphi(21) &= \varphi(3 \cdot 7) = \varphi(3) \cdot \varphi(7) = 2 \cdot 6 = 12; \\ \varphi(24) &= \varphi(2^3 \cdot 3) = \varphi(2^3) \cdot \varphi(3) = 4 \cdot 2 = 8; \\ \varphi(36) &= \varphi(2^2 \cdot 3^2) = \varphi(2^2) \cdot \varphi(3^2) = 2 \cdot 6 = 12; \\ \varphi(49) &= \varphi(7^2) = 49(1 - 1/7) = 42; \\ \varphi(81) &= \varphi(3^4) = 81(1 - 1/3) = 54; \\ \varphi(100) &= \varphi(2^2 \cdot 5^2) = \varphi(2^2) \cdot \varphi(5^2) = 2 \cdot 20 = 40; \end{aligned}$$

C 10.7. Each letter in the string needs to be shifted to the right by 10 positions.
If the transformed letter is after Z, we use the order:
$A, B, C, D, \ldots, X, Y, Z, A, B, C, \ldots$

Supplementary problems

S 10.1. 997.

S 10.2. Example: $4 = 2+2; 6 = 3+3; 8 = 3+5; 10 = 3+7; 12 = 5+7; 14 = 3+11; 16 = 3+13; 18 = 5+13; 20 = 3+17; \ldots; 96 = 7+89; 98 = 19+79; 100 = 3+97$.

S 10.3. (3,5), (5,7), (11,13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73), (101, 103).

S 10.4. 3, 7, 31, 127, 8191.

S 10.5.

$$\begin{aligned}
703 &= 19 \cdot 37;\\
779 &= 19 \cdot 41;\\
968 &= 2^3 \cdot 11^2;\\
1002 &= 2 \cdot 3 \cdot 167;\\
1440 &= 2^5 \cdot 3^2 \cdot 5;\\
1547 &= 7 \cdot 13 \cdot 17;\\
1763 &= 41 \cdot 43;\\
2261 &= 7 \cdot 17 \cdot 19;
\end{aligned}$$

S 10.6.

$$\begin{aligned}
\gcd(1440, 968) &= 2^3 = 8;\\
\gcd(1002, 703) &= 1;\\
\gcd(2261, 1547) &= 7 \cdot 17 = 119;\\
\gcd(779, 1763) &= 41.
\end{aligned}$$

S 10.7.

$$\begin{aligned}
\mathrm{lcm}(1440, 968) &= 2^5 \cdot 3^2 \cdot 5 \cdot 11^2;\\
\mathrm{lcm}(1002, 703) &= 2 \cdot 3 \cdot 19 \cdot 37 \cdot 167;\\
\mathrm{lcm}(2261, 1547) &= 7 \cdot 13 \cdot 17 \cdot 19;\\
\mathrm{lcm}(779, 1763) &= 19 \cdot 41 \cdot 43.
\end{aligned}$$

S 10.8.

$$\begin{aligned}
&x = 1 \pmod 2, \quad x = 1 \pmod 2\\
&x = 6 \pmod 8, \quad x = 5 \pmod 8, \quad \text{there is no such } x.\\
&x = 9 \pmod{11}, \quad x = 7 \pmod{11}, \quad x = 2 \pmod{11}.
\end{aligned}$$

S 10.9. One just needs to factorise the numbers and apply the multiplicity of $\varphi(n)$.

$$\begin{aligned}
\varphi(11) &= 10; \\
\varphi(21) &= \varphi(3 \cdot 7) = \varphi(3) \cdot \varphi(7) = 2 \cdot 6 = 12; \\
\varphi(21) &= \varphi(3 \cdot 7) = \varphi(3) \cdot \varphi(7) = 2 \cdot 6 = 12; \\
\varphi(24) &= \varphi(2^3 \cdot 3) = \varphi(2^3) \cdot \varphi(3) = 4 \cdot 2 = 8; \\
\varphi(36) &= \varphi(2^2 \cdot 3^2) = \varphi(2^2) \cdot \varphi(3^2) = 2 \cdot 6 = 12; \\
\varphi(49) &= \varphi(7^2) = 49(1 - 1/7) = 42; \\
\varphi(81) &= \varphi(3^4) = 81(1 - 1/3) = 54; \\
\varphi(100) &= \varphi(2^2 \cdot 5^2) = \varphi(2^2) \cdot \varphi(5^2) = 2 \cdot 20 = 40;
\end{aligned}$$

S 10.10. $n = 65$ then $p = 5$, $q = 13$ and $\varphi(65) = 48$. To find d one needs to solve $5 * d = 1 \pmod{48}$ which gives $d = 30$.

11 Elements of Calculus

11.1 Brief theoretical background

Sequence: ordered list of objects. Discrete function $x : \mathbb{N} \to X$, $x(n) = x_n$.

Limit of a sequence: value to which sequence terms **"get close to eventually"**.

- We say that the sequence **converges** to the limit
- If a sequence does not converge it is **divergent**

Limit of a function: Suppose $f : R \to R$ is defined on the real line and $p, L \in R$. We say "the limit of f as x approaches p is L" and write

$$\lim_{x \to p} f(x) = L,$$

if for every convergent sequence $x_n \to p$ we have $f(x_n) \to L$.

Continuous function: A function $f : R \to R$ is continuous if

- for every point x and convergent sequence $x_n \to x$ we have $f(x_n) \to f(x)$.
- the graph is a single unbroken curve with no "holes" or "jumps".

The intermediate value theorem:
If f is continuous on $[a, b]$ and $f(a)f(b) < 0$ then there is $c \in [a, b]$ s.t. $f(c) = 0$.

Derivative: measure of how a function changes as its input changes.
For x and $x + h$ we define the ratio

$$m = \frac{\Delta f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - (x)} = \frac{f(x+h) - f(x)}{h}.$$

The derivative of the function $f : \mathbb{R} \to \mathbb{R}$ at x is the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation of elementary functions

- powers: $f(x) = x^n (n \in \mathbb{N}) : \quad f'(x) = nx^{n-1}$
- polynomials: $(ax^2 + bx + c)' = 2ax + b$ (a, b, c: constants)
- trigonometric functions:

$$f(x) = \sin(x) : \quad f'(x) = \cos(x)$$
$$f(x) = \cos(x) : \quad f'(x) = -\sin(x)$$

Definite integral: Let the function f defined on $[a, b]$. The definite integral denoted by

$$\int_a^b f(x)\, dx$$

represents the area of the region in the xy-plane bounded by:

- the graph of f
- the x-axis,
- vertical lines $x = a$ and $x = b$.

Fundamental theorem of calculus: If f is a continuous real-valued function defined on $[a, b]$ and a function F s.t. $F'(x) = f(x)$, then the definite integral of f over $[a, b]$ is

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Integration of elementary functions

- powers: $f(x) = x^n (n \in \mathbb{N}) : \quad F(x) = \int f(x)\mathrm{d}x = \frac{x^{n+1}}{n+1} + C$ (constant)
- polynomials: $f(x) = ax + b; \quad F(x) = \int f(x)\mathrm{d}x = a\frac{x^2}{2} + bx + C$ (a, b, C: constants)
- trigonometric functions:

$$f(x) = \sin(x) : \quad F(x) = \int f(x)\mathrm{d}x = -\cos(x) + C$$
$$f(x) = \cos(x) : \quad F(x) = \int f(x)\mathrm{d}x = \sin(x) + C$$

Iterative method: trapezium rule
The region under the graph of function $f : [a, b] \to \mathbb{R}$ is approximated as a trapezoid

$$\int_a^b f(x)\, dx \approx (b - a)\frac{f(a) + f(b)}{2}.$$

Numerical implementation: Dividing the interval $[a, b]$ into N intervals we define the

- **Step:** $h = \frac{b-a}{N}$ (step);
- **Partition:** $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$,...,$x_N = a + Nh = b$.

We apply the trapezium rule and obtain the following approximation for the area

$$\int_a^b f(x)\, dx \approx \frac{h}{2}\sum_{k=1}^{N} (f(x_{k-1}) + f(x_k))$$
$$= \frac{b-a}{2N}(f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{N-1}) + f(x_N)).$$

11.2 Compulsory problems

C 11.1. Find the limit of the following sequences

$$\begin{aligned}
&(1)\quad x_n = 1; \quad (2)\quad x_n = 1 + 1/n; \quad (3)\quad x_n = 1/n^2; \quad (4)\quad x_n = 1/2^n;\\
&(5)\quad 0.9, 0.99, 0.999, 0.9999, ...;\\
&(6)\quad 0.23, 0.23, 0.2323, 0.232323, ...;\\
&(7)\quad 0.423, 0.42323, 0.4232323, 0.423232323, ...
\end{aligned}$$

C 11.2. Prove that $f(x) = 2x^2 + 2x - 5$ has a root in the intervals $[1,2]$ and [-3,-2].

C 11.3. Prove that the cubic function $f(x) = 2x^3 + 2x^2 - 3$ has a root $x_0 \in [0,1]$.

C 11.4. Differentiate the following functions: (1) $f(x) = 2x + 1$; (2) $f(x) = 2x^2 + 3$.

C 11.5. Integrate the functions: (1) $f(x) = 1$; (2) $f(x) = 2x + 3$; (3) $f(x) = 2x^2 + 3$.

C 11.6. Evaluate $\int_0^2 f(x)dx$ for $f(x) = x^2 + 2$ using **Fundamental theorem of calculus**.

C 11.7. Evaluate $\int_0^2 f(x)dx$ for $f(x) = x^2 + 2$ using the trapezium rule for $2, 4$ intervals.

11.3 Supplementary problems

S 11.1. Find the limit of the following sequences

$$\begin{aligned}
&(1)\quad x_n = \frac{2n}{n-4};\\
&(2)\quad x_n = (-1)^n \frac{2^n}{n!};\\
&(3)\quad x_n = n^2 - 5n;\\
&(4)\quad x_n = \sqrt{n+1} - \sqrt{n}.
\end{aligned}$$

S 11.2. Find the number of real roots of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 8x^3 + 12x^2 - 2x - 3$.

S 11.3. Prove that the function $f(x) = x^2 - 3\cos\left(\frac{\pi}{2}x\right) + 2$ has a root $x_0 \in [0,1]$.

S 11.4. Differentiate the function $f(x) = 2x^3 + 7x + 3$.

S 11.5. Integrate the functions: (1) $f(x) = 2x + 7\cos(x) + 3\sin(x)$; (2) $f(x) = 2x^3 + 1$.

S 11.6. Find the area between the curves $f(x) = -x^2 + 6x - 2$ and $g(x) = 2x + 1$.

S 11.7. Evaluate the integral $\int_0^2 f(x)dx$ for $f(x) = x^3 + 4x^2 - 5x - 2$ exactly and by using the trapezium rule for $2, 4, 8$. What can you say about the result ?

11.4 Answers for Tutorial 11

Compulsory problems

C 11.1. (1) 0; (2) 0; (3) 0; (4) 0; (5) 1; (6) 23/99; (7) 423/990.

C 11.2. $f(1)f(2) = (-1)7 < 0;\quad f(-3)f(-2) = (7)(-1) < 0.$

C 11.3. $f(0)f(1) = (-3)1 < 0.$

C 11.4. (1) $f'(x) = 2$; (2) $f'(x) = 4x$.

C 11.5. (1) $\int f(x)dx = x + C$; (2) $\int f(x)dx = x^2 + 3x + C$; (3) $\int f(x)dx = \frac{2x^3}{3} + 3x + C$.

C 11.6. $F(x) = \int f(x)dx = \frac{x^3}{3} + 2x + C$. The integral is therefore given by

$$\int_0^2 f(x)dx = F(2) - F(0) = 8/3 + 4 + C - C = 20/3.$$

C 11.7. When $N = 2$ we have $h = 2/2 = 1$ and the formula gives the approximation

$$T_2 = \frac{1}{2}(f(0) + 2f(1) + f(2)) = \frac{1}{2}(2 + 2 \cdot 3 + 6) = 7.$$

When $N = 4$ we have $h = 2/4 = 1/2$ and the formula gives the approximation

$$T_4 = \frac{1}{4}(f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + 2f\left(\frac{3}{2}\right) + f(2)) = \frac{1}{4}(2 + 2\frac{9}{4} + 2 \cdot 3 + 2\frac{17}{4} + 6) = 27/4.$$

Supplementary problems

S 11.1. (1) 2; (2) 0; (3) ∞; (4) 0.

S 11.2. $f(-2) = -11; f(-1) = 3; f(0) = -3; f(1) = 15.$
We have $f(-2)f(-1) < 0$, $f(-1)f(0) < 0$ and $f(0)f(1) < 0$.

S 11.3. $f(0)f(1) = (-1)3 < 0.$

S 11.4. $f'(x) = 6x^2 + 7.$

S 11.5. (1) $\int f(x)dx = x^2 + 7\sin(x) - 3\cos(x)$; (2) $\int f(x)dx = x^4/2 + C$.

S 11.6. We first solve $f(x) = g(x)$ and obtain the solutions $x_1 = 1$, $x_2 = 3$.
$A = \int_1^3 f(x) - g(x)dx = \int_1^3 -(x^2 - 4x + 3)dx = -(x^3/3 - 4x^2 + 3x + C)|_1^3 = 4/3.$

S 11.7. $T_2 = 3$; $T_4 = 1.25$; $T_8 = 0.81$; $T_{16} = 0.70$; $T_{32} = 0.67$; $\int_0^2 f(x)dx = 2/3$.
Conclusion: precision always gets close to the actual result (eventually).