Asian Journal of Control, Vol. 00, No. 0, pp. 1–14, Month 2017
Published online in Wiley InterScience (www.interscience.wiley.com) DOI: 10.1002/asjc.0000

Global projective lag synchronization of fractional order memristor based BAM neural networks with mixed time varying delays

A. Pratap, R. Raja, C. Sowmiya, O. Bagdasar, Jinde Cao, G. Rajchakit

ABSTRACT

This paper addresses Master-Slave synchronization for some memristor-based fractional-order BAM neural networks (MFBNNs) with mixed time varying delays and switching jumps mismatch. Firstly, considering the inherent characteristic of FMNNs, a new type of fractional-order differential inequality is proposed. Secondly, an adaptive switching control scheme is designed to realize the global projective lag synchronization goal of MFBNNs in the sense of Riemann-Liouville derivative. Then, based on a suitable Lyapunov method, under the framework of set-valued map, differential inclusions theory, fractional Barbalat's lemma and proposed control scheme, some new projective lag synchronization criteria for such MFBNNs are obtained. Finally, some numerical examples are presented to illustrate the effectiveness of the proposed theoretical analysis.

Key Words: Projective lag Synchronization; Memristor-based BAM Neural networks; Mixed time-varying delays; Filippov's solutions; adaptive switching control.

I. Introduction

In recent years, numerous researchers have focused on the study of fractional order systems. These represent an extension of the classical integer order differential systems which have been studied for more than three hundred years [1, 2, 3]. Fractional-order calculus has established itself as a powerful tool in describing many processes in fields such as biology, bio-engineering, bio-physics, physics, mechanics, fluid dynamics, economics, ecological systems or optimization.

Manuscript received
A. Pratap and C. Sowmiya are with Department of Mathematics, Alagappa University, Karaikudi-630 004, India.
R. Raja is with Ramanujan Centre for Higher Mathematics, Alagappa University, Karaikudi-630 004, India.
O. Bagdasar is with Department of Electronics, Computing and Mathematics, University of Derby, United Kingdom.
Jinde Cao is with School of Mathematics, Southeast University, Nanjing 211189, China. Corresponding author, e-mail: jdcao@seu.edu.cn.
G. Rajchakit is with Department of Mathematics, Faculty of Science, Maejo University, Chiang Mai, Thailand.

In particular, fractional order dynamical systems are a useful instrument in modelling processes which require infinitely degrees of freedom (in contrast to the classical integer order system), as well as in the description of memory and hereditary properties of various material. As such, numerous results concerning dynamical behaviors of fractional order networks have been investigated, including the stability [4, 5], synchronization [6, 7], stabilization [9], bifurcation [10, 11] and many others. Fractional order nonlinear dynamical systems are used in practical situations, where the discrete time varying delays are unavoidable because of the finite speed of signal transmission and the network parameter fluctuations of the hardware implementation. Usually, the distributed delays are bounded because during the particular period, the signal propagation is distributed. Basic ideas and applications of the bounded distributed delays, as well as suggestions on how to solve simple problems of pattern recognition in a time dependent signal are made in [12].

In 1971, the basic concept of *memristor* (a contraction of memory and resistor) was introduced by Chua [13], as the fourth fundamental basic circuit element along with resistor, inductor and capacitor. Also, Chua established the mathematical proof of the nonlinear relation between charge and flux [13]. In 2008, a Hawlett-Packard research team realized the first memristor, their results being published in Nature [14, 15]. Memristor is a passive two-circuit element with nonlinear relationship between current and voltage. In the construction of memristor based neural networks one can replaces the resistors used in classical artificial neural networks with memristors. In real time, it plays an important role in a novel circuit elements because it simulates the functions of human brain and it had been strong applied to high-speed low-power processors, biological models, pattern recognition, associative memory, filters, improved startup speed of computers or for extending cell phone battery life.

In 1987, *bidirectional associative memory* (BAM) neural networks were postulated by Kosko [17], as an extension of the single-level auto-associative Hebbian circuits to a double-level pattern matched by hetero-associative correlation. The major advantage of this extension is that it recalls and stores pattern pairs regarded as the bidirectionally stable states. This can be successfully applied to many fields such as pattern recognition, artificial intelligence, signal and image transmission or optimization, Some meaningful results have been reported[9, 18, 19]. Currently, the research about combined problems for the synchronization of fractional order nonlinear dynamical systems and memristor based BAM neural network systems has made significant progress in terms of both theory and applications. For example in [9], Wu *et al.* investigate about the Global Mittag-Leffler stabilization of fractional-order bidirectional associative memory neural networks based on the methods of a linear-partial state feedback control techniques, a generalized Gronwall-Bellman inequality approach.

Since the formulation of master-slave synchronization concept in 1990s by Carroll and Pecora, synchronization problems involving fractional order dynamical systems have become a very active research topic, and many applications have been found in areas including nonlinear neural networks, signal processing, bio-engineering, physics-engineering, secure communications or computer networks. Numerous concepts of synchronization have been formulated, such as projective synchronization [21], complete synchronization [22], or anti-synchronization [24]. It is worth noting that many systems cannot be synchronized directly. For example, effective control schemes have been designed for the synchronization of neural network systems, such as state feedback control [34, 35, 37], impulsive control [25, 36], adaptive control [26, 38], hybrid control (combination of adaptive and state feedback controls) [27, 28], or intermittent control [29].

Our paper is motivated by the recent results of Chen *et al.* [21], concerning criteria for the global projective synchronization of *memristor-based fractional-order neural networks* (MFNNs), in the sense of Caputo fractional order derivative, by a hybrid control approach and fractional order inequalities. In this paper, we propose a novel projective lag synchronization method concerning *memristor-based fractional-order BAM neural networks* (MFBNNs) and we derive sufficient conditions ensuring global projective synchronization, global complete synchronization and global anti synchronization. Several related results and techniques have been published in the recent literature. For example, Yu *et al.* [28] investigated the projective synchronization for *fractional neural networks* (FNNs) in the sense of Caputo derivative, while Zhang *et al.* [30], used fractional Barbalet's lemma, Razumikhin stability theorem, and Filippov solutions in the sense of Riemann-Liouville derivative, to formulate several sufficient conditions which ensured the projective synchronization of delayed MFNNs under a switching control scheme.Inspired by the aforementioned discussions, the main contribution of this article is further summarized as below:

- This is the first work that investigates the global projective lag synchronization of fractional order memristor based BAM neural networks with mixed time varying delays in the sense of Riemann-Liouville(R-L) derivatives.
- Here we have used adaptive switching controller, which is proposed by means of switching feature of a memristor.
- By applying the properties of R-L derivative, set-valued map analysis and differential inclusion theory, the projective lag synchronization error system is formulated. And the results are improved when compared to the existing works in the literature.

II. Model Description and Preliminaries

Notations. Throughout this paper, $\mathbb{R}$ is the space of real number, $\mathbb{N}_+$ is the set of positive integer. We denote by $\|p\| = \sum_{i=1}^{n} |p_i|$, which corresponds to the classical notation for the $\|.\|_1$ norm, while $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. The absolute value of a given square matrix $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, is $|A| = (|a_{ij}|)_{n\times n} \in \mathbb{R}^{n\times n}$, while for a given vector $p \in \mathbb{R}^n$ one has $\mathrm{sgn}(p) = [\mathrm{sgn}(p_1), \mathrm{sgn}(p_2), ..., \mathrm{sgn}(p_n)]^T$ is the sign function We also define by $C\Big([-\omega, 0], \mathbb{R}^n\Big)$ the Banach space consisting of the $1-$ order continuous function maps from $[-\omega, 0]$ into $\mathbb{R}^n$. Throughout this paper we consider the Riemann-Liouville (R-L) derivative of order $0 < \alpha < 1$. Here, we first recall some basic definitions of fractional order calculus and some important lemmas (see, e.g., [2]).

Definition II.1 *The Riemann-Liouville derivative of order α for a function $p(t)$ is defined as*

$$\begin{aligned} D^{\alpha}p(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}[D^{-(n-\alpha)}_{t_0,t}p(t)] \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t_0}^{t}(t-s)^{n-\alpha-1}p(s)\,\mathrm{d}s, \end{aligned}$$

where $t \geq t_0$ and n is the positive integer such that $n-1 < \alpha < n$.
Particularly, when $0 < \alpha < 1$,

$$\begin{aligned} D^{\alpha}p(t) &= \frac{\mathrm{d}}{\mathrm{d}t}[D^{-(1-\alpha)}_{t_0,t}p(t)] \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t}(t-s)^{-\alpha}p(s)\,\mathrm{d}s. \end{aligned}$$

Definition II.2 *The Caputo fractional-order derivative of order α for a function $p(t)$ is defined as*

$$\mathbb{D}^{\alpha}p(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{p^n(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

where $t \geq t_0$ and n is the positive integer such that $n-1 < \alpha < n$.
Particularly, when $0 < \alpha < 1$,

$$D^{\alpha}p(t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{p'(s)}{(t-s)^{\alpha}}\,\mathrm{d}s.$$

Remark II.3 *Many authors the initial conditions of fractional derivatives are not easy because the initial conditions of R-L derivative operator are expressed in term of fractional order derivative. In this paper, we set the initial conditions to be zero, as in this case the operators for the Riemann-Liouville (R-L) and Caputo derivatives coincide [3].*

Some properties of the R-L fractional-order derivative are given in the following lemma (see [2]).

Lemma II.4 *Let $\alpha > 0, \beta > 0$, and $w(t),\ u(t) \in C^1[t_0, b]$. The following properties hold:*
(1) $D^{\alpha}D^{-\beta}w(t) = D^{\alpha-\beta}w(t)$;
(2) $D^{\alpha}D^{-\alpha}w(t) = w(t)$;
(3) $D^{\alpha}(w(t) \pm u(t)) = D^{\alpha}w(t) \pm D^{\alpha}u(t)$.

We now consider a class of memristor-based fractional-order BAM neural networks (MFBNNs) models, described as follows:

$$\begin{aligned} D^{\alpha}p_i(t) =& -a_ip_i(t) + \sum_{j=1}^{m}[b_{ji}(p_i(t))]f_j(q_j(t)) \\ &+ \sum_{j=1}^{m}[c_{ji}(p_i(t))]f_j(q_j(t-\tau(t))) \\ &+ \sum_{j=1}^{m}[d_{ji}(p_i(t))]\int_{t-\rho(t)}^{t}f_j(q_j(s))\,\mathrm{d}s \\ &+ J_i,\ t \geq 0, \\ D^{\alpha}q_j(t) =& -u_jq_j(t) + \sum_{i=1}^{n}[v_{ij}(q_j(t))]g_i(p_i(t)) \\ &+ \sum_{i=1}^{n}[w_{ij}(q_j(t))]g_i(p_i(t-\tau(t))) \\ &+ \sum_{i=1}^{n}[z_{ij}(q_j(t))]\int_{t-\rho(t)}^{t}g_i(p_i(s))\,\mathrm{d}s \\ &+ H_j,\ t \geq 0, \end{aligned} \tag{1}$$

where $i = 1, 2, ..., n$ $(n \in \mathbb{N}_+)$ and $j = 1, 2, ..., m$ $(m \in \mathbb{N}_+)$. Let $p(t) = (p_1(t), p_2(t), ..., p_n(t))^T \in \mathbb{R}^n$ and $q(t) = (q_1(t), q_2(t), ..., q_m(t))^T \in \mathbb{R}^m$ denote the state vector of the neurons at time $t > 0$, while $a_i > 0$ and $u_j > 0$ are positive constants. Let $f(q(t)) = ((f_1(q_1(t)), ..., (f_m(q_m(t)))^T \in \mathbb{R}^m$ and $g(p(t)) = ((g_1(p_1(t)), ..., (g_n(p_n(t)))^T \in \mathbb{R}^n$ be continuous feedback functions. Denote by J_i and H_j the bounded input vectors and by $\tau(t) > 0$, $\rho(t)$ the discrete and distributed time varying delays satisfying $0 < \tau(t) \leq \tau$, $\dot{\tau}(t) \leq \sigma < 1$ and $0 < \rho(t) \leq \rho$ (τ and ρ denote the maximum values of the discrete and distributed time delays). The memristor based connection weights are defined as

$$\begin{aligned} b_{ji}(p_i(t)) &= \begin{cases} b_{ji}^{+}, & |p_i(t)| \leq T_i \\ b_{ji}^{-}, & |p_i(t)| > T_i, \end{cases} \\ c_{ji}(p_i(t)) &= \begin{cases} c_{ji}^{+}, & |p_i(t)| \leq T_i \\ c_{ji}^{-}, & |p_i(t)| > T_i, \end{cases} \\ d_{ji}(p_i(t)) &= \begin{cases} d_{ji}^{+}, & |p_i(t)| \leq T_i \\ d_{ji}^{-}, & |p_i(t)| > T_i, \end{cases} \end{aligned}$$

$$v_{ij}(q_j(t)) = \begin{cases} v_{ij}^{+}, & |q_j(t)| \le \tilde{T}_j \\ v_{ij}^{-}, & |q_j(t)| > \tilde{T}_j, \end{cases}$$

$$w_{ij}(q_j(t)) = \begin{cases} w_{ij}^{+}, & |q_j(t)| \le \tilde{T}_j \\ w_{ij}^{-}, & |q_j(t)| > \tilde{T}_j, \end{cases}$$

$$z_{ij}(q_j(t)) = \begin{cases} z_{ij}^{+}, & |q_j(t)| \le \tilde{T}_j \\ z_{ij}^{-}, & |q_j(t)| > \tilde{T}_j, \end{cases}$$

in which switching jumps T_i, $\tilde{T}_j > 0$, b_{ji}^{+}, c_{ji}^{+}, d_{ji}^{+}, v_{ij}^{+}, w_{ij}^{+}, z_{ij}^{+}, b_{ji}^{-}, c_{ji}^{-}, d_{ji}^{-}, v_{ij}^{-}, w_{ij}^{-} and z_{ij}^{-} are constants relating to memristances.

Remark II.5 *In view of the memristor switching rule in the circuit design of Ref [23], the switching states depends on the voltage difference, which is connected to two ports of the memristor. i.e., the values of connection weights can express as the difference between the voltage of the capacitor and network output. But this paper deals with the memristor connection weights that depends on the potential of the state $p_i(t)$, $q_j(t)$ and the voltage of one port, if $p_i(t)$ and $q_j(t)$ is less or more then the switching jump parameter T_i and $\tilde{T}_j$. Based on the above analysis, the memristive connection weights are changed with respect to the state $p_i(t)$ and $q_j(t)$ of the system. It obvious that the memristive connection weights are discontinuous with almost two distinct points of discontinuity $\pm T_i$ and $\pm\tilde{T}_j$, in the memristor model is more general than the existing ones.*

Since the memristive connection weights are discontinuous, the solution in the classical sense may not exist. In this case, solutions of the system can be extended in Filippov's sense [32, 33]. Based on the differential inclusion theory [31] and the properties of set valued maps, it follows that

$$\begin{aligned} D^{\alpha}p_i(t) \in &- a_i p_i(t) + \sum_{j=1}^{m}[\bar{co}(b_{ji}(p_i(t)))]f_j(q_j(t)) \\ &+ \sum_{j=1}^{m}[\bar{co}(c_{ji}(p_i(t)))]f_j(q_j(t-\tau(t))) \\ &+ \sum_{j=1}^{m}[\bar{co}(d_{ji}(p_i(t)))]\int_{t-\rho(t)}^{t} f_j(q_j(s))\,\mathrm{d}s \\ &+ J_i,\ t \ge 0 \\ D^{\alpha}q_j(t) \in &- u_j q_j(t) + \sum_{i=1}^{n}[\bar{co}(v_{ij}(q_j(t)))]g_i(p_i(t)) \\ &+ \sum_{i=1}^{n}[\bar{co}(w_{ij}(q_j(t)))]g_i(p_i(t-\tau(t))) \\ &+ \sum_{i=1}^{n}[\bar{co}(z_{ij}(q_j(t)))]\int_{t-\rho(t)}^{t} g_i(p_i(s))\,\mathrm{d}s + H_j. \end{aligned} \tag{2}$$

We define

$$\bar{co}[b_{ji}(p_i(t))] = \begin{cases} b_{ji}^{+}, & |p_i(t)| < T_i \\ [\underline{b}_{ji}, \bar{b}_{ji}], & |p_i(t)| = T_i \\ b_{ji}^{-}, & |p_i(t)| > T_i, \end{cases}$$

$$\bar{co}[c_{ji}(p_i(t))] = \begin{cases} c_{ji}^{+}, & |p_i(t)| < T_i \\ [\underline{c}_{ji}, \bar{c}_{ji}], & |p_i(t)| = T_i \\ c_{ji}^{-}, & |p_i(t)| > T_i, \end{cases}$$

$$\bar{co}[d_{ji}(p_i(t))] = \begin{cases} d_{ji}^{+}, & |p_i(t)| < T_i \\ [\underline{d}_{ji}, \bar{d}_{ji}], & |p_i(t)| = T_i \\ d_{ji}^{-}, & |p_i(t)| > T_i, \end{cases}$$

$$\bar{co}[v_{ij}(q_j(t))] = \begin{cases} v_{ij}^{+}, & |q_j(t)| < \tilde{T}_j \\ [\underline{v}_{ij}, \bar{v}_{ij}], & |q_j(t)| = \tilde{T}_j \\ v_{ij}^{-}, & |q_j(t)| > \tilde{T}_j, \end{cases}$$

$$\bar{co}[w_{ij}(q_j(t))] = \begin{cases} w_{ij}^{+}, & |q_j(t)| < \tilde{T}_j \\ [\underline{w}_{ij}, \bar{w}_{ij}], & |q_j(t)| = \tilde{T}_j \\ w_{ij}^{-}, & |q_j(t)| > \tilde{T}_j, \end{cases}$$

$$\bar{co}[z_{ij}(q_j(t))] = \begin{cases} z_{ij}^{+}, & |q_j(t)| < \tilde{T}_j \\ [\underline{z}_{ij}, \bar{z}_{ij}], & |q_j(t)| = \tilde{T}_j \\ z_{ij}^{-}, & |q_j(t)| > \tilde{T}_j. \end{cases}$$

In this paper, we set

$$\begin{aligned} \underline{b}_{ji} &= \min\{b_{ji}^{+}, b_{ji}^{-}\},\ \bar{b}_{ji} = \max\{b_{ji}^{+}, b_{ji}^{-}\}, \\ \underline{c}_{ji} &= \min\{c_{ji}^{+}, c_{ji}^{-}\},\ \bar{c}_{ji} = \max\{c_{ji}^{+}, c_{ji}^{-}\}, \\ \underline{d}_{ji} &= \min\{d_{ji}^{+}, d_{ji}^{-}\},\ \bar{d}_{ji} = \max\{d_{ji}^{+}, d_{ji}^{-}\}, \\ \underline{v}_{ij} &= \min\{v_{ij}^{+}, v_{ij}^{-}\},\ \bar{v}_{ij} = \max\{v_{ij}^{+}, v_{ij}^{-}\}, \\ \underline{w}_{ij} &= \min\{w_{ij}^{+}, w_{ij}^{-}\},\ \bar{w}_{ij} = \max\{w_{ij}^{+}, w_{ij}^{-}\}, \\ \bar{z}_{ij} &= \max\{z_{ij}^{+}, z_{ij}^{-}\},\ \underline{z}_{ij} = \min\{z_{ij}^{+}, z_{ij}^{-}\}. \end{aligned}$$

There exist $\hat{b}_{ji}(t) \in [\bar{co}(b_{ji}(p_i(t)))]$, $\hat{c}_{ji}(t) \in [\bar{co}(c_{ji}(p_i(t)))]$, $\hat{d}_{ji}(t) \in [\bar{co}(d_{ji}(p_i(t)))]$, $\hat{v}_{ij}(t) \in [\bar{co}(v_{ij}(q_j(t)))]$, $\hat{w}_{ij}(t) \in [\bar{co}(w_{ij}(q_j(t)))]$ and $\hat{z}_{ij}(t) \in [\bar{co}(z_{ij}(q_j(t)))]$, measurable function such that for $t > 0$, one has

$$\begin{aligned} D^{\alpha}p_i(t) = &- a_i p_i(t) + \sum_{j=1}^{m}\hat{b}_{ji}(t)f_j(q_j(t)) \\ &+ \sum_{j=1}^{m}\hat{c}_{ji}(t)f_j(q_j(t-\tau(t))) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\hat{d}_{ji}(t)\int_{t-\rho(t)}^{t} f_j(q_j(s))\,\mathrm{d}s + J_i,\\
D^{\alpha}q_j(t) &= -u_j q_j(t) + \sum_{i=1}^{n}\hat{v}_{ij}(t)g_i(p_i(t))\\
&+\sum_{i=1}^{n}\hat{w}_{ij}(t)g_i(p_i(t-\tau(t)))\\
&+\sum_{i=1}^{n}\hat{z}_{ij}(t)\int_{t-\rho(t)}^{t} g_i(p_i(s))\,\mathrm{d}s + H_j
\end{aligned} \tag{3}
$$

In this work, the slave system is described by

$$
\begin{aligned}
D^{\alpha}\tilde{p}_i(t) =& -a_i\tilde{p}_i(t) + \sum_{j=1}^{m}[b_{ji}(\tilde{p}_i(t))]f_j(\tilde{q}_j(t))\\
&+\sum_{j=1}^{m}[c_{ji}(\tilde{p}_i(t))]f_j(\tilde{q}_j(t-\tau(t)))\\
&+\sum_{j=1}^{m}[d_{ji}(\tilde{p}_i(t))]\int_{t-\rho(t)}^{t} f_j(\tilde{q}_j(s))\,\mathrm{d}s\\
&+J_i+\zeta_i(t),\ t\geq 0\\
D^{\alpha}\tilde{q}_j(t) =& -u_j\tilde{q}_j(t) + \sum_{i=1}^{n}[v_{ij}(\tilde{q}_j(t))]g_i(\tilde{p}_i(t))\\
&+\sum_{i=1}^{n}[w_{ij}(\tilde{q}_j(t))]g_i(\tilde{p}_i(t-\tau(t)))\\
&+\sum_{i=1}^{n}[z_{ij}(\tilde{q}_j(t))]\int_{t-\rho(t)}^{t} g_i(\tilde{p}_i(s))\,\mathrm{d}s\\
&+H_j+\varsigma_j(t),\ t\geq 0,
\end{aligned} \tag{4}
$$

for $i=1,2,...,n,\ j=1,2,...,m$, where $\zeta_i(t)$ and $\varsigma_j(t)$ denote the control inputs. Consider the state variable of the slave system $\tilde{p}(t)=(\tilde{p}_1(t),\tilde{p}_2(t),...,\tilde{p}_n(t))^T\in\mathbb{R}^n$, $\tilde{q}(t)=(\tilde{q}_1(t),\tilde{q}_2(t),...,\tilde{q}_m(t))^T\in\mathbb{R}^m$. The system parameters can be defined as

$$
\begin{aligned}
b_{ji}(\tilde{p}_i(t)) &= \begin{cases} b_{ji}^{+}, & |\tilde{p}_i(t)|\leq \acute{T}_i\\ b_{ji}^{-}, & |\tilde{p}_i(t)|> \acute{T}_i,\end{cases}\\
c_{ji}(\tilde{p}_i(t)) &= \begin{cases} c_{ji}^{+}, & |\tilde{p}_i(t)|\leq \acute{T}_i\\ c_{ji}^{-}, & |\tilde{p}_i(t)|> \acute{T}_i,\end{cases}\\
d_{ji}(\tilde{p}_i(t)) &= \begin{cases} d_{ji}^{+}, & |\tilde{p}_i(t)|\leq \acute{T}_i\\ d_{ji}^{-}, & |\tilde{p}_i(t)|> \acute{T}_i,\end{cases}\\
v_{ij}(\tilde{q}_j(t)) &= \begin{cases} v_{ij}^{+}, & |\tilde{q}_j(t)|\leq \acute{\acute{T}}_j\\ v_{ij}^{-}, & |\tilde{q}_j(t)|> \acute{\acute{T}}_j,\end{cases}\\
w_{ij}(\tilde{q}_j(t)) &= \begin{cases} w_{ij}^{+}, & |\tilde{q}_j(t)|\leq \acute{\acute{T}}_j\\ w_{ij}^{-}, & |\tilde{q}_j(t)|> \acute{\acute{T}}_j,\end{cases}\\
z_{ij}(\tilde{q}_j(t)) &= \begin{cases} z_{ij}^{+}, & |\tilde{q}_j(t)|\leq \acute{\acute{T}}_j\\ z_{ij}^{-}, & |\tilde{q}_j(t)|> \acute{\acute{T}}_j.\end{cases}
\end{aligned}
$$

Next, we apply the differential inclusion theory, in the sense of Filippov.

The set valued map of the slave system can be described as

$$
\begin{aligned}
D^{\alpha}\tilde{p}_i(t) \in& -a_i\tilde{p}_i(t) + \sum_{j=1}^{m}[\bar{co}(b_{ji}(\tilde{p}_i(t)))]f_j(\tilde{q}_j(t))\\
&+\sum_{j=1}^{m}[\bar{co}(c_{ji}(\tilde{p}_i(t)))]f_j(\tilde{q}_j(t-\tau(t)))\\
&+\sum_{j=1}^{m}[\bar{co}(d_{ji}(\tilde{p}_i(t)))]\int_{t-\rho(t)}^{t} f_j(\tilde{q}_j(s))\,\mathrm{d}s\\
&+J_i+\mathbb{H}[\zeta_i(t)],\ t\geq 0\\
D^{\alpha}\tilde{q}_j(t) \in& -u_j\tilde{q}_j(t) + \sum_{i=1}^{n}[\bar{co}(v_{ij}(\tilde{q}_j(t)))]g_i(\tilde{p}_i(t))\\
&+\sum_{i=1}^{n}[\bar{co}(w_{ij}(\tilde{q}_j(t)))]g_i(\tilde{p}_i(t-\tau(t)))\\
&+\sum_{i=1}^{n}[\bar{co}(z_{ij}(\tilde{q}_j(t)))]\int_{t-\rho(t)}^{t} g_i(\tilde{p}_i(s))\,\mathrm{d}s\\
&+H_j+\mathbb{H}[\varsigma_j(t)],\ t\geq 0.
\end{aligned} \tag{5}
$$

We define

$$
\begin{aligned}
\bar{co}[b_{ji}(\tilde{p}_i(t))] &= \begin{cases} b_{ji}^{+}, & |\tilde{p}_i(t)|< \acute{T}_i\\ [\underline{b}_{ji},\bar{b}_{ji}], & |\tilde{p}_i(t)|= \acute{T}_i\\ b_{ji}^{-}, & |\tilde{p}_i(t)|> \acute{T}_i,\end{cases}\\
\bar{co}[c_{ji}(\tilde{p}_i(t))] &= \begin{cases} c_{ji}^{+}, & |\tilde{p}_i(t)|< \acute{T}_i\\ [\underline{c}_{ji},\bar{c}_{ji}], & |\tilde{p}_i(t)|= \acute{T}_i\\ c_{ji}^{-}, & |\tilde{p}_i(t)|> \acute{T}_i,\end{cases}\\
\bar{co}[d_{ji}(\tilde{p}_i(t))] &= \begin{cases} d_{ji}^{+}, & |\tilde{p}_i(t)|< \acute{T}_i\\ [\underline{d}_{ji},\bar{d}_{ji}], & |\tilde{p}_i(t)|= \acute{T}_i\\ d_{ji}^{-}, & |\tilde{p}_i(t)|> \acute{T}_i,\end{cases}\\
\bar{co}[v_{ij}(\tilde{q}_j(t))] &= \begin{cases} v_{ij}^{+}, & |\tilde{q}_j(t)|< \acute{\acute{T}}_j\\ [\underline{v}_{ij},\bar{v}_{ij}], & |\tilde{q}_j(t)|= \acute{\acute{T}}_j\\ v_{ij}^{-}, & |\tilde{q}_j(t)|> \acute{\acute{T}}_j,\end{cases}\\
\bar{co}[w_{ij}(\tilde{q}_j(t))] &= \begin{cases} w_{ij}^{+}, & |\tilde{q}_j(t)|< \acute{\acute{T}}_j\\ [\underline{w}_{ij},\bar{w}_{ij}], & |\tilde{q}_j(t)|= \acute{\acute{T}}_j\\ w_{ij}^{-}, & |\tilde{q}_j(t)|> \acute{\acute{T}}_j,\end{cases}
\end{aligned}
$$

$$\bar{co}[z_{ij}(\tilde{q}_j(t))] = \begin{cases} z_{ij}^{+}, & |\tilde{q}_j(t)| < \acute{\tilde{T}}_j \\ [\underline{z}_{ij}, \bar{z}_{ij}], & |\tilde{q}_j(t)| = \acute{\tilde{T}}_j \\ z_{ij}^{-}, & |\tilde{q}_j(t)| > \acute{\tilde{T}}_j. \end{cases}$$

All other parameter are defined as neural slave system. The initial conditions assumed for the Master-Slave system are described as follows $p_i(s) = \hat{\sigma}_i(s),\ q_j(s) = \hat{\varrho}_j(s),\ \tilde{p}_i(s) = \tilde{\sigma}_i(s),\ \tilde{q}_j(s) = \tilde{\varrho}_j(s)$, respectively, $s \in [-\omega, 0],\ \hat{\sigma}_i(s),\ \hat{\varrho}_j(s),\ \tilde{\sigma}_i(s),\ \tilde{\varrho}_j(s) \in C\Big([-\omega, 0], \mathbb{R}\Big)$. Here $\omega = \max\{\tau^*, \rho^*\}\ for\ i = 1, 2, ..., n, j = 1, 2, ..., m$.
Then there exist $\breve{b}_{ji}(t) \in [\bar{co}(b_{ji}(\tilde{p}_i(t)))]$, $\check{c}_{ji}(t) \in [\bar{co}(c_{ji}(\tilde{p}_i(t)))]$ $\check{d}_{ji}(t) \in [\bar{co}(d_{ji}(\tilde{p}_i(t)))]$, $\zeta_i^*(t) \in \mathbb{H}[\zeta_i(t)]$, $\check{v}_{ij}(t) \in [\bar{co}(v_{ij}(\tilde{q}_j(t)))]$, $\breve{w}_{ij}(t) \in [\bar{co}(w_{ij}(\tilde{q}_j(t)))]$, $\check{z}_{ij}(t) \in [\bar{co}(z_{ij}(\tilde{q}_j(t)))]$ and $\varsigma_j^*(t) \in \mathbb{H}[\varsigma_j(t)]$ the measurable function such that

$$\begin{aligned} D^\alpha \tilde{p}_i(t) = &- a_i \tilde{p}_i(t) + \sum_{j=1}^{m} \breve{b}_{ji}(t) f_j(\tilde{q}_j(t)) \\ &+ \sum_{j=1}^{m} \check{c}_{ji}(t) f_j(\tilde{q}_j(t - \tau(t))) \\ &+ \sum_{j=1}^{m} \check{d}_{ji}(t) \int_{t-\rho(t)}^{t} f_j(\tilde{q}_j(s))\, \mathrm{d}s \\ &+ J_i + \zeta_i^*(t),\ t \geq 0, \\ D^\alpha \tilde{q}_j(t) = &- u_j \tilde{q}_j(t) + \sum_{i=1}^{n} \check{v}_{ij}(t) g_i(\tilde{p}_i(t)) \\ &+ \sum_{i=1}^{n} \breve{w}_{ij}(t) g_i(\tilde{p}_i(t - \tau(t))) \\ &+ \sum_{i=1}^{n} \check{z}_{ij}(t) \int_{t-\rho(t)}^{t} g_i(\tilde{p}_i(s))\, \mathrm{d}s \\ &+ H_j + \varsigma_j^*(t),\ t \geq 0. \end{aligned} \quad (6)$$

Definition II.6 *For any solution of the Master-Slave system with different initial values is said to be a projective lag synchronization, if*

$$\lim_{t\to\infty} \|\tilde{p}(t) - \lambda p(t - \vartheta)\| + \lim_{t\to\infty} \|\tilde{q}(t) - \lambda q(t - \vartheta)\| = 0.$$

Where λ is the projective coefficients and ϑ is the projective transmittal delay.

Remark II.7 *For $\vartheta = 0$, the Master-Slave system achieves globally projective synchronization. For $\vartheta = 0,\ \lambda = 1$, the Master-Slave system achieves globally complete synchronization; also, for the values $\vartheta = 0,\ \lambda = -1$, the Master-Slave system attains globally anti-synchronization.*

The following assumptions ensure the existence and uniqueness of solution of the master-slave system.
($\mathbf{H1}$). The activation functions f_j and g_i are Lipschitz-continuous on $\mathbb{R}$ and $f_j(0)$ and $g_i(0)$ are bounded, i.e., there exist Lipschitz constants $L_i > 0$ and $\tilde{L}_j > 0$ such that

$$\begin{aligned} |g_i(\tilde{p}) - g_i(p)| &\leq L_i |\tilde{p} - p|, \\ |f_j(\tilde{q}) - f_j(q)| &\leq \tilde{L}_j |\tilde{q} - q|, \end{aligned}$$

for all $p,\ q,\ \tilde{p},\ \tilde{q} \in \mathbb{R}$ and $i = 1, 2, ..., n,\ j = 1, 2, ..., m$.
($\mathbf{H2}$). For all $p,\ q \in \mathbb{R}$, there exist constants $L_i > 0$ and $\tilde{L}_j > 0$ such that

$$|g_i(p)| \leq L_i,\ |f_j(q)| \leq \tilde{L}_j,$$

for $i = 1, 2, ..., n,\ j = 1, 2, ..., m$.
The following fractional Barbalat's lemma (see, e.g., [20]) plays to critical role in our main theorems.

Lemma II.8 *If $\int_{t_0}^{t} \Lambda(s)\, \mathrm{d}s$ has finite limit as $t \to \infty$, $D^\alpha \Lambda(t)$ is bounded, then $\lim_{t\to\infty} \Lambda(t) = 0$, where $0 < \alpha < 1$.*

III. Synchronization results

In this section, we use suitable Lyapunov functions to derive sufficient conditions ensuring global projective lag synchronization, based on adaptive switching control scheme.

Let $r_i(t) = \tilde{p}_i(t) - \lambda p_i(t - \vartheta)$ and $\tilde{r}_j(t) = \tilde{q}_j(t) - \lambda q_j(t - \vartheta)$. Considering the adaptive switching control given in the slave system (4), it follows that

$$\begin{aligned} \zeta_i(t) &= \begin{cases} -\eta_i(t) r_i(t) - \beta_i \operatorname{sgn}(r_i(t)) + (\lambda - 1) J_i \\ \dot{\eta}_i(t) = \kappa_i |r_i(t)|, \end{cases} \\ \varsigma_j(t) &= \begin{cases} -\mu_j(t) \tilde{r}_j(t) - \gamma_j \operatorname{sgn}(\tilde{r}_j(t)) + (\lambda - 1) H_j \\ \dot{\mu}_j(t) = \iota_j |\tilde{r}_j(t)|, \end{cases} \end{aligned} \quad (7)$$

where $\beta_i,\ \kappa_i,\ \iota_j$ and γ_j denotes the positive gains, $\eta_i(t)$ and $\mu_j(t)$ denotes the adaptive constants.

The dynamical error systems (3) and (6) can now be expressed in the following form

$$\begin{aligned} D^\alpha r_i(t) = &- a_i r_i(t) + \sum_{j=1}^{m} \Big[\breve{b}_{ji}(t) f_j(\tilde{q}_j(t)) \\ &- \lambda \hat{b}_{ji}(t - \vartheta) f_j(q_j(t - \vartheta)) \Big] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\Big[\check{c}_{ji}(t)f_j(\tilde{q}_j(t-\tau(t)))\\
&-\lambda\hat{c}_{ji}(t-\vartheta)f_j(q_j(t-\vartheta-\tau(t-\vartheta)))\Big]\\
&+\sum_{j=1}^{m}\Big[\check{d}_{ji}(t)\int_{t-\rho(t)}^{t}f_j(\tilde{q}_j(s))\,\mathrm{d}s\\
&-\lambda\hat{d}_{ji}((t-\vartheta))\int_{t-\rho(t)}^{t}f_j(\tilde{q}_j(s-\vartheta))\,\mathrm{d}s\Big]\\
&-\eta_i(t)r_i(t)-\beta_i\chi_i(t), \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
D^{\alpha}\tilde{r}_j(t)=&-u_j\tilde{r}_j(t)+\sum_{i=1}^{n}\Big[\check{v}_{ij}(t)g_i(\tilde{p}_i(t))\\
&-\lambda\hat{v}_{ij}(t-\vartheta)g_i(p_i(t-\vartheta))\Big]\\
&+\sum_{i=1}^{n}\Big[\check{w}_{ij}(t)g_i(\tilde{p}_i(t-\tau(t)))\\
&-\lambda\hat{w}_{ij}(t-\vartheta)g_i(p_i(t-\vartheta-\tau(t-\vartheta)))\Big]\\
&+\sum_{i=1}^{n}\Big[\check{z}_{ij}(t)\int_{t-\rho(t)}^{t}g_i(\tilde{p}_i(s))\,\mathrm{d}s\\
&-\lambda\hat{z}_{ij}(t-\vartheta)\int_{t-\rho(t)}^{t}g_i(p_i(s-\vartheta))\,\mathrm{d}s\\
&-\mu_j(t)\tilde{r}_j(t)-\gamma_j\nu_j(t), \qquad (9)
\end{aligned}
$$

where

$$
\chi_i(t)\in\mathbb{H}[\mathrm{sgn}(r_i(t))]=\begin{cases}-1, & r_i(t)>0\\ [-1,1], & r_i(t)=0\\ 1, & r_i(t)>0,\end{cases}
$$

$$
\nu_j(t)\in\mathbb{H}[\mathrm{sgn}(\tilde{r}_j(t))]=\begin{cases}-1, & \tilde{r}_i(t)>0\\ [-1,1], & \tilde{r}_i(t)=0\\ 1, & \tilde{r}_i(t)>0.\end{cases}
$$

Next we shall prove to the following lemma, needed by the proof of our main theorem.

Lemma III.1 *Assume that* $\hat{b}_{ji}(t)\in[\bar{co}(b_{ji}(p_i(t)))]$, $\hat{c}_{ji}(t)\in[\bar{co}(c_{ji}(p_i(t)))]$, $\hat{d}_{ji}(t)\in[\bar{co}(d_{ji}(p_i(t)))]$, $\hat{v}_{ij}(t)\in[\bar{co}(v_{ij}(q_j(t)))]$, $\hat{w}_{ij}(t)\in[\bar{co}(w_{ij}(q_j(t)))]$, $\hat{z}_{ij}(t)\in[\bar{co}(z_{ij}(q_j(t)))]$, $\check{b}_{ji}(t)\in[\bar{co}(b_{ji}(\tilde{p}_i(t)))]$, $\check{c}_{ji}(t)\in[\bar{co}(c_{ji}(\tilde{p}_i(t)))]$, $\check{d}_{ji}(t)\in[\bar{co}(d_{ji}(\tilde{p}_i(t)))]$, $\check{v}_{ij}(t)\in[\bar{co}(v_{ij}(\tilde{q}_j(t)))]$, $\check{w}_{ij}(t)\in[\bar{co}(w_{ij}(\tilde{q}_j(t)))]$ *and* $\check{z}_{ij}(t)\in[\bar{co}(z_{ij}(\tilde{q}_j(t)))]$.
By the assumptions $H(1)$ *and* $H(2)$*, the following conditions are satisfied:*

$$
\begin{aligned}
&\Big|\check{b}_{ji}(t)f_j(\tilde{q}_j(t))-\lambda\hat{b}_{ji}(t-\vartheta))f_j(q_j(t-\vartheta))\Big|\\
\le& b^*_{ji}\bar{L}_j|\tilde{r}_j(t)|+\Phi_1,\\
&\Big|\check{c}_{ji}(t)f_j(\tilde{q}_j(t-\tau(t)))-\lambda\hat{c}_{ji}(t-\vartheta))\\
&\times f_j(q_j(t-\vartheta-\tau(t-\vartheta)))\Big|\\
\le& c^*_{ji}\bar{L}_j|\tilde{r}_j(t-\tau(t))|+\Phi_2,\\
&\Big|\check{d}_{ji}(t)\int_{t-\rho(t)}^{t}f_j(\tilde{q}_j(s))\,\mathrm{d}s\\
&-\lambda\hat{d}_{ji}((t-\vartheta))\int_{t-\rho(t)}^{t}f_j(\tilde{q}_j(s-\vartheta))\,\mathrm{d}s\Big|\\
\le& d^*_{ji}\bar{L}_j\int_{t-\rho(t)}^{t}|\tilde{r}_j(s))|\,\mathrm{d}s+\Phi_3,\\
&\Big|\check{v}_{ij}(t)g_i(\tilde{p}_i(t))\\
&-\lambda\hat{v}_{ij}(t-\vartheta))g_i(p_i(t-\vartheta))\Big|\\
\le& v^*_{ij}L_i|r_i(t)|+\Psi_1,\\
&\Big|\check{w}_{ij}(t)g_i(\tilde{p}_i(t-\tau(t)))\\
&-\lambda\hat{w}_{ij}(t-\vartheta))g_i(p_i(t-\vartheta-\tau(t-\vartheta)))\Big|\\
\le& w^*_{ij}L_i|r_i(t-\tau(t))|+\Psi_2,\\
&\Big|\check{z}_{ij}(t)\int_{t-\rho(t)}^{t}g_i(\tilde{p}_i(s))\,\mathrm{d}s\\
&-\lambda\hat{z}_{ij}(t-\vartheta))\int_{t-\rho(t)}^{t}g_i(p_i(s-\vartheta))\,\mathrm{d}s\Big|\\
\le& z^*_{ij}L_i\int_{t-\rho(t)}^{t}|r_i(s))|\,\mathrm{d}s+\Psi_3,
\end{aligned}
$$

where $i=1,2,...,n$ *and* $j=1,2,...,m$, λ *is the projective coefficients,* ϑ *is the projective transmittal delay, with the notations* $b^*_{ji}=\max\{|b^+_{ji}|,\ |b^-_{ji}|\}$, $c^*_{ji}=\max\{|c^+_{ji}|,\ |c^-_{ji}|\}$, $d^*_{ji}=\max\{|d^+_{ji}|,\ |d^-_{ji}|\}$, $v^*_{ij}=\max\{|v^+_{ij}|,\ |v^-_{ij}|\}$, $w^*_{ij}=\max\{|w^+_{ij}|,\ |w^-_{ij}|\}$, $z^*_{ij}=\max\{|z^+_{ij}|,\ |z^-_{ij}|\}$. *We also have*
$\Phi_1=\max\{|b^+_{ji}-b^-_{ji}|\bar{L}_j,\ |b^+_{ji}-b^-_{ji}||\lambda|\bar{L}_j,0\}$,
$\Phi_2=\max\{|c^+_{ji}-c^-_{ji}|\bar{L}_j,\ |c^+_{ji}-c^-_{ji}||\lambda|\bar{L}_j,0\}$,
$\Phi_3=\max\{|d^+_{ji}-d^-_{ji}|\rho\bar{L}_j,\ |d^+_{ji}-d^-_{ji}||\lambda|\rho\bar{L}_j,0\}$,
$\Psi_1=\max\{|v^+_{ij}-v^-_{ij}|L_i,\ |v^+_{ij}-v^-_{ij}||\lambda|L_i,0\}$,
$\Psi_2=\max\{|w^+_{ij}-w^-_{ij}|L_i,\ |w^+_{ij}-w^-_{ij}||\lambda|L_i,0\}$,
$\Psi_3=\max\{|z^+_{ij}-z^-_{ij}|\rho L_i,\ |z^+_{ij}-z^-_{ij}||\lambda|\rho L_i,0\}$.

Proof. The proof of this result is divided into four cases.

(**1**). For $|\tilde{p}_i(t)|\le\acute{T}_i$, $|p_i(t-\vartheta)|\le T_i$ and $|\tilde{q}_j(t)|\le\acute{\tilde{T}}_j$, $|q_j(t-\vartheta)|\le\tilde{T}_j$, we get

$$
\Big|\check{b}_{ji}(t)f_j(\tilde{q}_j(t))-\lambda\hat{b}_{ji}(t-\vartheta))f_j(q_j(t-\vartheta))\Big|
$$

$$=\Big|b_{ji}^{+}\Big(f_j(\tilde{q}_j(t))-\lambda f_j(q_j(t-\vartheta))\Big)\Big|$$
$$\le|b_{ji}^{+}|\tilde{L}_j\Big|\tilde{q}_j(t)-\lambda q_j(t-\vartheta))\Big|$$
$$\le b_{ji}^{*}\tilde{L}_j|\tilde{r}_j(t)|,$$

and

$$\Big|\check{v}_{ij}(t)g_i(\tilde{p}_i(t))-\lambda\hat{v}_{ij}(t-\vartheta))g_i(p_i(t-\vartheta))\Big|$$
$$=\Big|v_{ij}^{+}\Big(g_i(\tilde{p}_i(t))-\lambda g_i(p_i(t-\vartheta))\Big)\Big|$$
$$\le|v_{ij}^{+}|L_i\Big|\tilde{p}_i(t)-\lambda p_i(t-\vartheta))\Big|$$
$$\le v_{ij}^{*}L_i|r_i(t)|.$$

(**2**). For $|\tilde{p}_i(t)|\ge\acute{T}_i$, $|p_i(t-\vartheta)|\ge T_i$ and $|\tilde{q}_j(t)|\ge\acute{\tilde{T}}_j$, $|q_j(t-\vartheta)|\ge\tilde{T}_j$, we get

$$\Big|\check{b}_{ji}(t)f_j(\tilde{q}_j(t))-\lambda\hat{b}_{ji}(t-\vartheta))f_j(q_j(t-\vartheta))\Big|$$
$$=\Big|b_{ji}^{-}\Big(f_j(\tilde{q}_j(t))-\lambda f_j(q_j(t-\vartheta))\Big)\Big|$$
$$\le|b_{ji}^{-}|\tilde{L}_j\Big|\tilde{q}_j(t)-\lambda q_j(t-\vartheta))\Big|$$
$$\le b_{ji}^{*}\tilde{L}_j|\tilde{r}_j(t)|,$$

and

$$\Big|\check{v}_{ij}(t)g_i(\tilde{p}_i(t))-\lambda\hat{v}_{ij}(t-\vartheta))g_i(p_i(t-\vartheta))\Big|$$
$$=\Big|v_{ij}^{-}\Big(g_i(\tilde{p}_i(t))-\lambda g_i(p_i(t-\vartheta))\Big)\Big|$$
$$\le|v_{ij}^{-}|L_i\Big|\tilde{p}_i(t)-\lambda p_i(t-\vartheta))\Big|$$
$$\le v_{ij}^{*}L_i|r_i(t)|.$$

From above two cases, Similarly we have to prove that

$$\Big|\check{c}_{ji}(t)f_j(\tilde{q}_j(t-\tau(t)))-\lambda\hat{c}_{ji}(t-\vartheta))$$
$$\times f_j(q_j(t-\vartheta-\tau(t-\vartheta)))\Big|\le c_{ji}^{*}\bar{L}_j|\tilde{r}_j(t-\tau(t))|,$$
$$\Big|\check{d}_{ji}(t)\int_{t-\rho(t)}^{t}f_j(\tilde{q}_j(s))\,\mathrm{d}s-\lambda\hat{d}_{ji}(t-\vartheta)$$
$$\times\int_{t-\rho(t)}^{t}f_j(\tilde{q}_j(s-\vartheta))\,\mathrm{d}s\Big|\le d_{ji}^{*}\bar{L}_j\int_{t-\rho(t)}^{t}|\tilde{r}_j(s))|\,\mathrm{d}s,$$

and

$$\Big|\check{w}_{ij}(t)g_i(\tilde{p}_i(t-\tau(t)))-\lambda\hat{w}_{ij}(t-\vartheta))$$
$$\times g_i(p_i(t-\vartheta-\tau(t-\vartheta)))\Big|\le w_{ij}^{*}L_i|r_i(t-\tau(t))|,$$

$$\Big|\check{z}_{ij}(t)\int_{t-\rho(t)}^{t}g_i(\tilde{p}_i(s))\,\mathrm{d}s-\lambda\hat{w}_{ij}(t-\vartheta))$$
$$\times\int_{t-\rho(t)}^{t}g_i(p_i(s-\vartheta))\,\mathrm{d}s\Big|$$

$$\le z_{ij}^{*}L_i\int_{t-\rho(t)}^{t}|r_i(s))|\,\mathrm{d}s.$$

(**3**). For $|\tilde{p}_i(t)|>\acute{T}_i$, $|p_i(t-\vartheta)|\le T_i$ and $|\tilde{q}_j(t)|>\acute{\tilde{T}}_j$, $|q_j(t-\vartheta)|\le\tilde{T}_j$, we get

$$\Big|\check{b}_{ji}(t)f_j(\tilde{q}_j(t))-\lambda\hat{b}_{ji}(t-\vartheta))f_j(q_j(t-\vartheta))\Big|$$
$$\Big|\check{b}_{ji}(t)f_j(\tilde{q}_j(t))-\lambda\hat{b}_{ji}(t-\vartheta))f_j(q_j(t-\vartheta))$$
$$+\lambda\check{b}_{ji}(t)f_j(q_j(t-\vartheta))-\lambda\check{b}_{ji}(t)f_j(q_j(t-\vartheta))\Big|$$
$$\le|\check{b}_{ji}(t)|\Big|f_j(\tilde{q}_j(t))-\lambda f_j(q_j(t-\vartheta))\Big|+$$
$$|\lambda|\Big|\check{b}_{ji}(t)-\hat{b}_{ji}(t-\vartheta)\Big||f_j(q_j(t-\vartheta))|$$
$$\le|b_{ji}^{-}|\tilde{L}_j\Big|\tilde{q}_j(t)-\lambda(q_j(t-\vartheta))\Big|+|b_{ji}^{-}-b_{ji}^{+}||\lambda|\tilde{L}_j$$
$$\le b_{ji}^{*}\tilde{L}_j|\tilde{r}_j(t)|+|b_{ji}^{+}-b_{ji}^{-}||\lambda|\tilde{L}_j,$$

and

$$\Big|\check{v}_{ij}(t)g_i(\tilde{p}_i(t))-\lambda\hat{v}_{ij}(t-\vartheta))]g_i(p_i(t-\vartheta))\Big|$$
$$=\Big|\check{v}_{ij}(t)g_i(\tilde{p}_i(t))-\lambda\hat{v}_{ij}(t-\vartheta))g_i(p_i(t-\vartheta))$$
$$+\lambda\check{v}_{ij}(t)g_i(p_i(t-\vartheta))-\lambda\check{v}_{ij}(t)g_i(p_i(t-\vartheta))\Big|$$
$$\le|\check{v}_{ij}(t)|\Big|g_i(\tilde{p}_i(t))-\lambda g_i(p_i(t-\vartheta))\Big|+$$
$$|\lambda|\Big|\check{v}_{ij}(t)-\hat{v}_{ij}(t-\vartheta)\Big||g_i(p_i(t-\vartheta))|$$
$$\le|v_{ij}^{-}|L_i\Big|\tilde{p}_i(t)-\lambda(p_i(t-\vartheta))\Big|+|v_{ij}^{-}-v_{ij}^{+}||\lambda|L_i$$
$$\le v_{ij}^{*}L_i|\tilde{r}_j(t)|+|v_{ij}^{+}-v_{ij}^{-}||\lambda|L_i.$$

Similarly, we have

$$\Big|\check{c}_{ji}(t)f_j(\tilde{q}_j(t-\tau(t)))-\lambda\hat{c}_{ji}(t-\vartheta))$$
$$\times f_j(q_j(t-\vartheta-\tau(t-\vartheta)))\Big|$$
$$\le c_{ji}^{*}\bar{L}_j|\tilde{r}_j(t-\tau(t))|+|c_{ji}^{+}-c_{ji}^{-}||\lambda|\tilde{L}_j$$
$$\Big|\check{d}_{ji}(t)\int_{t-\rho(t)}^{t}f_j(\tilde{q}_j(s))\,\mathrm{d}s-\lambda\hat{d}_{ji}(t-\vartheta)$$
$$\times\int_{t-\rho(t)}^{t}f_j(\tilde{q}_j(s-\vartheta))\,\mathrm{d}s\Big|$$
$$\le d_{ji}^{*}\bar{L}_j\int_{t-\rho(t)}^{t}|\tilde{r}_j(s))|\,\mathrm{d}s+\rho|d_{ji}^{+}-d_{ji}^{-}||\lambda|\tilde{L}_j,$$

and

$$\Big|\check{w}_{ij}(t)g_i(\tilde{p}_i(t-\tau(t))) - \lambda\hat{w}_{ij}(t-\vartheta)) \times g_i(p_i(t-\vartheta-\tau(t-\vartheta)))\Big| \le w^*_{ij}L_i|r_i(t-\tau(t))| + |v^+_{ij} - v^-_{ij}||\lambda|L_i,$$

$$\Big|\check{z}_{ij}(t)\int_{t-\rho(t)}^{t} g_i(\tilde{p}_i(s))\,\mathrm{d}s - \lambda\hat{w}_{ij}(t-\vartheta)) \times \int_{t-\rho(t)}^{t} g_i(p_i(s-\vartheta))\,\mathrm{d}s\Big| \le z^*_{ij}L_i\int_{t-\rho(t)}^{t}|r_i(s))|\,\mathrm{d}s + \rho|z^+_{ij} - z^-_{ij}||\lambda|L_i.$$

(**4**). For $|\tilde{p}_i(t)| \le \acute{T}_i$, $|p_i(t-\vartheta)| > T_i$ and $|\tilde{q}_j(t)| \le \acute{\tilde{T}}_j$, $|q_j(t-\vartheta)| > \tilde{T}_j$, we get

$$\begin{aligned}&\Big|\check{b}_{ji}(t)f_j(\tilde{q}_j(t)) - \lambda\hat{b}_{ji}(t-\vartheta))f_j(q_j(t-\vartheta))\Big|\\ =&\Big|\check{b}_{ji}(t)f_j(\tilde{q}_j(t)) - \lambda\hat{b}_{ji}(t-\vartheta))f_j(q_j(t-\vartheta)) + \hat{b}_{ji}(t-\vartheta))f_j(\tilde{q}_j(t)) - \hat{b}_{ji}(t-\vartheta))f_j(\tilde{q}_j(t))\Big|\\ \le&|\hat{b}_{ji}(t-\vartheta))|\Big|f_j(\tilde{q}_j(t)) - \lambda f_j(q_j(t-\vartheta))\Big| + \Big|\check{b}_{ji}(t) - \hat{b}_{ji}(t-\vartheta))\Big||f_j(\tilde{q}_j(t))|\\ \le&|b^+_{ji}|\tilde{L}_j\Big|\tilde{q}_j(t) - \lambda(q_j(t-\vartheta))\Big| + |b^+_{ji} - b^-_{ji}|\tilde{L}_j\\ \le&b^*_{ji}\tilde{L}_j|\tilde{r}_j(t)| + |b^+_{ji} - b^-_{ji}|\tilde{L}_j,\end{aligned}$$

and

$$\begin{aligned}&\Big|\check{v}_{ij}(t)g_i(\tilde{p}_i(t)) - \lambda\hat{v}_{ij}(t-\vartheta))g_i(p_i(t-\vartheta))\Big|\\ =&\Big|\check{v}_{ij}(t)g_i(\tilde{p}_i(t)) - \lambda\hat{v}_{ij}(t-\vartheta))g_i(p_i(t-\vartheta)) + \hat{v}_{ij}(t-\vartheta))g_i(\tilde{p}_i(t)) - \hat{v}_{ij}(t-\vartheta))g_i(\tilde{p}_i(t))\Big|\\ \le&|\hat{v}_{ij}(t-\vartheta))|\Big|g_i(\tilde{p}_i(t)) - \lambda g_i(p_i(t-\vartheta))\Big| + \Big|\check{v}_{ij}(t) - \hat{v}_{ij}(t-\vartheta))\Big||g_i(\tilde{p}_i(t))|\\ \le&|v^+_{ij}|L_i\Big|\tilde{p}_i(t) - \lambda(p_i(t-\vartheta))\Big| + |v^+_{ij} - v^-_{ij}|L_i\\ \le&v^*_{ij}L_i|r_i(t)| + |v^+_{ij} - v^-_{ij}|L_i.\end{aligned}$$

Similarly, we have

$$\Big|\check{c}_{ji}(t)f_j(\tilde{q}_j(t-\tau(t))) - \lambda\hat{c}_{ji}(t-\vartheta)) \times f_j(q_j(t-\vartheta-\tau(t-\vartheta)))\Big| \le c^*_{ji}\bar{L}_j|\tilde{r}_j(t-\tau(t))| + |c^+_{ji} - c^-_{ji}|\tilde{L}_j$$

$$\Big|\check{d}_{ji}(t)\int_{t-\rho(t)}^{t} f_j(\tilde{q}_j(s))\,\mathrm{d}s - \lambda\hat{d}_{ji}(t-\vartheta)\int_{t-\rho(t)}^{t} f_j(\tilde{q}_j(s-\vartheta))\,\mathrm{d}s\Big| \le d^*_{ji}\bar{L}_j\int_{t-\rho(t)}^{t}|\tilde{r}_j(s))|\,\mathrm{d}s + \rho|d^+_{ji} - d^-_{ji}|\tilde{L}_j,$$

and

$$\Big|\check{w}_{ij}(t)g_i(\tilde{p}_i(t-\tau(t))) - \hat{w}_{ij}(t-\vartheta))\lambda \times g_i(p_i(t-\vartheta-\tau(t-\vartheta)))\Big| \le w^*_{ij}L_i|r_i(t-\tau(t))| + |v^+_{ij} - v^-_{ij}|L_i,$$

$$\Big|\check{z}_{ij}(t)\int_{t-\rho(t)}^{t} g_i(\tilde{p}_i(s))\,\mathrm{d}s - \hat{w}_{ij}(t-\vartheta)) \times \int_{t-\rho(t)}^{t} \lambda g_i(p_i(s-\vartheta))\,\mathrm{d}s\Big| \le z^*_{ij}L_i\int_{t-\rho(t)}^{t}|r_i(s))|\,\mathrm{d}s + \rho|z^+_{ij} - z^-_{ij}|L_i.$$

This completes the proof of the lemma. Based on the above lemma and the adaptive switching controller, the sufficient conditions are derived to the following results.

Theorem III.2 *Assume that $H(1)$ and $H(2)$ hold. If the following conditions are satisfied, then the Master-Slave systems are globally projective lag synchronization based on the switching controller* (7)

$$\Omega_1 = \min_{1\le i\le n}\Big\{a_i + \eta_i - \sum_{j=1}^{m}\Big(v^*_{ij} + \frac{w^*_{ij}}{1-\sigma} + \rho z^*_{ij}\Big)L_i\Big\} > 0,$$

$$\Omega_2 = \min_{1\le i\le n}\Big(\beta_i - [\sum_{j=1}^{m}\Psi_1 + \Psi_2 + \Psi_3]\Big) > 0,$$

$$\Theta_1 = \min_{1\le j\le m}\Big\{u_j + \mu_j - \sum_{i=1}^{n}\Big(b^*_{ji} + \frac{c^*_{ji}}{1-\sigma} + \rho d^*_{ji}\Big)\tilde{L}_j\Big\} > 0,$$

$$\Theta_2 = \min_{1\le j\le m}\Big(\gamma_j - [\sum_{i=1}^{n}\Phi_1 + \Phi_2 + \Phi_3]\Big) > 0.$$

Here b^*_{ji}, c^*_{ji}, d^*_{ji}, v^*_{ij}, w^*_{ij}, z^*_{ij}, Ψ_1, Ψ_2, Ψ_3, Φ_1, Φ_2 and Φ_3 are those already defined in Lemma $III.1$, while η_i, β_i, μ_j and γ_j denote appropriate positive constants.

Proof. Choose the Lyapunov-like function defined by $V(t) = V_1(t) + V_2(t) + V_3(t)$, where

$$\begin{aligned}
V_1(t) =& D^{-(1-\alpha)}\Big(\sum_{i=1}^{n}|r_i(t)| + \sum_{j=1}^{m}|\tilde{r}_j(t)|\Big),\\
V_2(t) =& \sum_{i=1}^{n}\frac{1}{2\kappa_i}(\eta_i(t)-\eta_i)^2 + \sum_{j=1}^{m}\frac{1}{2\iota_j}(\mu_j(t)-\mu_j)^2\\
V_3(t) =& \frac{1}{1-\sigma}\sum_{j=1}^{m}\Big(\sum_{i=1}^{n}c_{ji}^{*}\tilde{L}_j\Big)\int_{t-\tau(t)}^{t}|\tilde{r}_j(s)|\,\mathrm{d}s\\
&+\sum_{j=1}^{m}\sum_{i=1}^{n}d_{ji}^{*}\tilde{L}_j\int_{-\rho(t)}^{0}\int_{t+s}^{t}|\tilde{r}_j(s)|\,\mathrm{d}s\\
&+\frac{1}{1-\sigma}\sum_{i=1}^{n}\Big(\sum_{j=1}^{m}w_{ij}^{*}L_i\Big)\int_{t-\tau(t)}^{t}|r_i(s)|\,\mathrm{d}s\\
&+\sum_{i=1}^{n}\sum_{j=1}^{m}z_{ij}^{*}\mathbb{L}_i\int_{-\rho(t)}^{0}\int_{t+s}^{t}|r_i(s)|\,\mathrm{d}s
\end{aligned}$$

By calculating the derivative of the Lyapunov-like function based on the Riemann-Liouville derivative definition along the solutions of the error system (8) and (9), we get

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t), \tag{10}$$

where

$$\begin{aligned}
\dot{V}_1(t) =& D^{\alpha}\Big[\sum_{i=1}^{n}|r_i(t)| + \sum_{j=1}^{m}|\tilde{r}_j(t)|\Big]\\
\le& \sum_{i=1}^{n}\Big\{-a_i|r_i(t)| + \sum_{j=1}^{m}\Big[b_{ji}^{*}\tilde{L}_j|\tilde{r}_j(t)| + \Phi_1\Big]\\
&+\sum_{j=1}^{m}\Big[c_{ji}^{*}\tilde{L}_j|\tilde{r}_j(\tau(t))| + \Phi_2\Big] + \sum_{j=1}^{m}\Big[d_{ji}^{*}\tilde{L}_j\\
&\times\int_{t-\rho(t)}^{t}|\tilde{r}_j(s)|\,\mathrm{d}s + \Phi_3\Big] - \eta_i(t)|r_i(t)| - \beta_i\chi_i(t)\Big\}\\
&+\sum_{j=1}^{m}\Big\{-u_j|\tilde{r}_j(t)| + \sum_{i=1}^{n}\Big[v_{ij}^{*}L_i|r_i(t)| + \Psi_1\Big]\\
&+\sum_{i=1}^{n}\Big[w_{ij}^{*}L_i|r_i(t-\tau(t))| + \Psi_2\Big]\sum_{i=1}^{n}+\Big[z_{ij}^{*}L_i\\
&\times\int_{t-\rho(t)}^{t}|r_i(s)|\,\mathrm{d}s + \Psi_3\Big] - \mu_j(t)|\tilde{r}_j(t)| - \gamma_j\nu_j(t)\Big\}
\end{aligned}$$

where $\chi_i(t) = \mathrm{sgn}(r_i(t))$, if $r_i(t) \neq 0$, while $\epsilon_i(t)$ can be arbitrarily chosen in $[-1, 1]$, if $r_i(t) = 0$ and $\nu_j(t) = \mathrm{sgn}(\tilde{r}_j(t))$, while for $\tilde{r}_j(t) \neq 0$, $\tilde{\epsilon}_j(t)$ can be arbitrary chosen in $[-1, 1]$, if $\tilde{r}_j(t) = 0$.

$$\begin{aligned}
\dot{V}_2(t) =& \sum_{i=1}^{n}\frac{2}{2\kappa_i}(\eta_i(t)-\eta_i)\dot{\eta}_i(t)\\
&+\sum_{j=1}^{m}\frac{2}{2\iota_j}(\mu_j(t)-\mu_j)\dot{\mu}_j(t)\\
=& \sum_{i=1}^{n}\eta_i(t)|r_i(t)| - \sum_{i=1}^{n}\eta_i|r_i(t)|\\
&+\sum_{j=1}^{m}\mu_j(t)|\tilde{r}_j(t)| - \sum_{j=1}^{m}\mu_j|\tilde{r}_j(t)|
\end{aligned}$$

and

$$\begin{aligned}
\dot{V}_3(t) \le& -\sum_{i=1}^{n}\sum_{j=1}^{m}w_{ij}^{*}L_i|r_i(t-\tau(t))|\\
&+\frac{1}{1-\sigma}\sum_{i=1}^{n}\sum_{j=1}^{m}w_{ij}^{*}L_i|r_i(t)|\\
&+\sum_{i=1}^{n}\sum_{j=1}^{m}z_{ij}^{*}L_i\rho|r_i(t)|\\
&-\sum_{i=1}^{n}\sum_{j=1}^{m}z_{ij}^{*}L_i\int_{t-\rho(t)}^{t}|r_i(s)|\,\mathrm{d}s\\
&+\sum_{j=1}^{m}\sum_{i=1}^{n}c_{ji}^{*}\tilde{L}_j|\tilde{r}_j(t-\tau(t))|\\
&+\frac{1}{1-\sigma}\sum_{j=1}^{m}\sum_{i=1}^{n}c_{ji}^{*}\tilde{L}_j|\tilde{r}_j(t)|\\
&+\sum_{j=1}^{m}\sum_{i=1}^{n}d_{ji}^{*}\tilde{L}_j\rho|\tilde{r}_j(t)|\\
&-\sum_{j=1}^{m}\sum_{i=1}^{n}d_{ji}^{*}\tilde{L}_j\int_{t-\rho(t)}^{t}|\tilde{r}_j(s)|\,\mathrm{d}s.
\end{aligned}$$

According to Eq. (10) and the hypotheses of our theorem, it follows that

$$\begin{aligned}
\dot{V}(t) \;\le&\; -\Omega_1\sum_{i=1}^{n}|r_i(t)| - \Theta_1|\sum_{j=1}^{m}\tilde{r}_j(t)| - (\Omega_2+\Theta_2)\\
\le&\; -\Upsilon\Big[\sum_{i=1}^{n}|r_i(t)| + \sum_{j=1}^{m}|\tilde{r}_j(t)|\Big] \le -\Upsilon\Lambda(t),
\end{aligned}$$

where $\Lambda(t) = \sum_{i=1}^{n}|r_i(t)| + \sum_{j=1}^{m}|\tilde{r}_j(t)|$ and $\Upsilon = \min\{\Omega_1,\ \Theta_1\}$. As $V(t) + \Upsilon\int_{t_0}^{t}\Lambda(s)\,\mathrm{d}s \le V(t_0)$, it follows that $\lim_{t\to\infty}\Lambda(t)$ is bounded, hence $D^{\alpha}|r_i(t)|$ and $D^{\alpha}|\tilde{r}_j(t)|$ is also bounded. According to the fractional Barbalat's Lemma we have $\lim_{t\to\infty}\Lambda(t)=0$, therefore it follows that

$$\lim_{t\to\infty}\|r(t)\|=0 \; and \; \lim_{t\to\infty}\|\tilde{r}(t)\|=0.$$

Hence, the Master-Slave systems based on the switching controller (7) are globally projective lag synchronized. This completes the proof of our main theorem.

In the particular case when the master-slave systems are without distributed delay, we have some interesting consequences. The following corollary follows directly from Theorem $III.2$.

Corollary III.3 *Suppose that Assumptions $H(1)$ and $H(2)$ hold. The Master-Slave systems based on the switching controller (7) are globally projective lag synchronization, if the following conditions hold:*

$$\Omega_1=\min_{1\le i\le n}\Big\{a_i+\eta_i-\sum_{j=1}^{m}\Big(v^*_{ij}+\frac{w^*_{ij}}{1-\sigma}\Big)L_i\Big\}>0,$$

$$\Upsilon_2=\min_{1\le i\le n}\Big(\beta_i-\sum_{j=1}^{m}\{\Psi_1+\Psi_2\}\Big)>0,$$

$$\Theta_3=\min_{1\le j\le m}\Big\{u_j+\mu_j-\sum_{i=1}^{n}\Big(b^*_{ji}+\frac{c^*_{ji}}{1-\sigma}\Big)\tilde{L}_j\Big\}>0,$$

$$\Lambda_4=\min_{1\le j\le m}\Big(\gamma_j-\sum_{i=1}^{n}\{\Phi_1+\Phi_2\}\Big)>0.$$

The parameters $b^*_{ji},\ c^*_{ji},\ v^*_{ij},\ w^*_{ij},\ \Psi_1,\ \Psi_2,\ \Phi_1$ and Φ_2 have been defined in Lemma $III.1$, while $\eta_i,\ \beta_i,\ \mu_j$ and γ_j denote appropriate positive scalars.

Remark III.4 *Many of the results have been investigated the fractional order synchronization of memristor based neural networks to non-linear activations which satisfies the conditions: $g_i(0)=0,\ f_j(0)=0,\ g_i(\pm T_i)=f_j(\pm\tilde{T}_j)=g_i(\pm\acute{T}_i)=f_j(\pm\acute{\tilde{T}}_j)=0,\ T_i=\acute{T}_i$ and $\tilde{T}_i=\acute{\tilde{T}}_i$, where $T_i,\ \tilde{T}_j,\ \tilde{T}_i,\ \acute{\tilde{T}}_j$ are parameter switching jumps, see Ref [6, 7, 8, 21]. But in this paper, the above assumptions are to be restricted and the major improvements are listed as below:*

1. *Assume the switching jump parameter of MFBNNs in (1) is different from switching jump parameters of MFBNNs in (4), that is $g_i(\pm T_i)\neq 0,\ f_j(\pm\tilde{T}_j)\neq 0,\ g_i(\pm T_i)\neq g_i(\pm\acute{T}_i)$ and $f_j(\pm\tilde{T}_j)\neq f_j(\pm\acute{\tilde{T}}_j)$, where $T_i,\ \tilde{T}_j,\ \acute{T}_i,\ \acute{\tilde{T}}$ are switching parameter jumps.*
2. *The mismatch switching parameter jumps are unavoidable in real life applications because the noises, external disturbance or other human factors.*
3. *Based on the restricted assumption, the system (1) can be treated as discontinuous dynamical system. Additionally, the switching jumps depend on the Master-Slave synchronization condition, which leads to switching jumps when they are relatively small.*
4. *This article is associated with the synchronization errors, projective coefficient, transmitted delay and switching parameter jumps.*

IV. Numerical simulations

In this section, a numerical examples is given to illustrate the theoretical results obtained above.

Example IV.1 *In system (1), choose $\alpha=0.97$, $p=(p_1,p_2)^T$, $q=(q_1,\ q_2)^T$, $f(q)=(f_1(q_1),\ f_2(q_2))^T$, $g(p)=(g_1(p_1),\ g_2(p_2))^T$, $f_j(q_j)=\tanh(q_j)$, $g_i(p_i)=\tanh(p_i)$, $\tau(t)=\frac{e^t}{1+e^t}$, $\rho(t)=0.1\sin t+1$, $\sigma=0.3<1$, $\tau=2$, $\rho=2$, $J=(0\ \ 0)^T$, $H=(0\ \ 0)^T$, $a_1=0.5$, $a_2=1$, $u_1=1.75$, $u_2=1.75$ and*

$$b_{11}(p_1(t))=\begin{cases}-1.5, & |p_1(t)|\le 0.1\\ -0.95, & |p_1(t)|>0.1,\end{cases}$$

$$b_{12}(p_2(t))=\begin{cases}-2.35, & |p_2(t)|\le 0.2\\ -1.72, & |p_2(t)|>0.2,\end{cases}$$

$$b_{21}(p_1(t))=\begin{cases}1.75, & |p_1(t)|\le 0.1\\ 0.98, & |p_1(t)|>0.1,\end{cases}$$

$$b_{22}(p_2(t))=\begin{cases}-1.25, & |p_2(t)|\le 0.2\\ -0.48, & |p_2(t)|>0.2,\end{cases}$$

$$c_{11}(p_1(t))=\begin{cases}-0.87, & |p_1(t)|\le 0.1\\ -0.48, & |p_1(t)|>0.1,\end{cases}$$

$$c_{12}(p_2(t))=\begin{cases}-1.53, & |p_2(t)|\le 0.2\\ -1.2, & |p_2(t)|>0.2,\end{cases}$$

$$c_{21}(p_1(t))=\begin{cases}1.52, & |p_1(t)|\le 0.1\\ 1.08, & |p_1(t)|>0.1,\end{cases}$$

$$c_{22}(p_2(t)) = \begin{cases} -1.65, & |p_2(t)| \le 0.2 \\ -1.48, & |p_2(t)| > 0.2, \end{cases}$$

$$d_{11}(p_1(t)) = \begin{cases} 2.85, & |p_1(t)| \le 0.1 \\ 2.01, & |p_1(t)| > 0.1, \end{cases}$$

$$d_{12}(p_2(t)) = \begin{cases} -1.85, & |p_2(t)| \le 0.2 \\ -1.42, & |p_2(t)| > 0.2, \end{cases}$$

$$d_{21}(p_1(t)) = \begin{cases} 1.65, & |p_1(t)| \le 0.1 \\ 0.91, & |p_1(t)| > 0.1, \end{cases}$$

$$d_{22}(p_2(t)) = \begin{cases} 1.45, & |p_2(t)| \le 0.2 \\ 0.92, & |p_2(t)| > 0.2, \end{cases}$$

$$v_{11}(q_1(t)) = \begin{cases} 0.43, & |q_1(t)| \le 0.1 \\ 0.73, & |q_1(t)| > 0.1, \end{cases}$$

$$v_{12}(q_2(t)) = \begin{cases} 0.63, & |q_2(t)| \le 0.2 \\ 0.78, & |q_2(t)| > 0.2, \end{cases}$$

$$v_{21}(q_1(t)) = \begin{cases} -0.73, & |q_1(t)| \le 0.1 \\ -0.37, & |q_1(t)| > 0.1, \end{cases}$$

$$v_{22}(q_1(t)) = \begin{cases} -0.53, & |q_1(t)| \le 0.2 \\ -0.32, & |q_1(t)| > 0.2, \end{cases}$$

$$w_{11}(q_1(t)) = \begin{cases} -0.58, & |q_1(t)| \le 0.1 \\ -0.23, & |q_1(t)| > 0.1, \end{cases}$$

$$w_{12}(q_2(t)) = \begin{cases} 0.83, & |q_2(t)| \le 0.2 \\ 0.58, & |q_2(t)| > 0.2, \end{cases}$$

$$w_{21}(q_1(t)) = \begin{cases} 0.91, & |q_1(t)| \le 0.1 \\ 0.67, & |q_1(t)| > 0.1, \end{cases}$$

$$w_{22}(q_1(t)) = \begin{cases} -1.1, & |q_2(t)| \le 0.2 \\ -0.77, & |q_2(t)| > 0.2, \end{cases}$$

$$z_{11}(q_1(t)) = \begin{cases} 0.73, & |q_1(t)| \le 0.1 \\ 0.43, & |q_1(t)| > 0.1, \end{cases}$$

$$z_{12}(q_2(t)) = \begin{cases} -0.93, & |q_2(t)| \le 0.2 \\ -0.62, & |q_2(t)| > 0.2, \end{cases}$$

$$z_{21}(q_1(t)) = \begin{cases} -0.83, & |q_1(t)| \le 0.1 \\ -0.37, & |q_1(t)| > 0.1, \end{cases}$$

$$z_{22}(q_2(t)) = \begin{cases} 1.03, & |q_1(t)| \le 0.2 \\ 0.87, & |q_1(t)| > 0.2. \end{cases}$$

In the switching control scheme (7) *we can choose the values* $\eta_1(0) = 0.09,\ \eta_2(0) = 0.02,\ \mu_1(0) = 0.04,\ \mu_2(0) = 0.02,\ k_i = \iota_j = 0.1,\ \lambda = 2,\ \eta_1 = 5.8,\ \eta_2 = 6,\ \mu_1 = 10.4,\ \mu_2 = 8.5$ *and* $L_1 = L_2 = \tilde{L}_1 = \tilde{L}_2 = 1,\ \beta_i = 4.7, \gamma_j = 8.5$ *for* $i, j = 1, 2$. *By simple manipulation of* $\Phi_1,\ \Phi_2,\ \Phi_3,\ \Psi_1,\ \Psi_2$ *and* Ψ_3, *we get* $\Phi_1 = 3.08,\ \Phi_2 = 1.76,\ \Phi_3 = 3.36,\ \Psi_1 = 1.44,\ \Psi_2 = 1.32$ *and* $\Psi_3 = 1.84$.

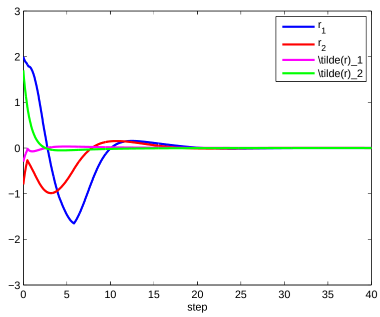

Fig. 1. Projective lag synchronization error evolution of $r_i(t)$ and $\tilde{r}_j(t)$.

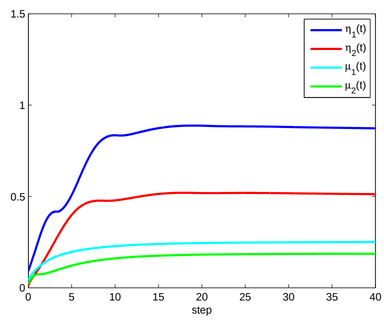

Fig. 2. The dynamical coupling strengthes of the switching controller (8).

It is easy to check,

$$\min_i \left\{ a_i + \eta_i + \frac{\xi_i}{1-\sigma} - \sum_{j=1}^{2} \left(v_{ij}^* + \frac{w_{ij}^*}{1-\sigma} + \rho z_{ij}^* \right) L_i \right\} > 0,$$

$$\min_i \left\{ \beta_i - \sum_{j=1}^{2} \{\Psi_1 + \Psi_2 + \Psi_3\} \right\} > 0,$$

$$\min_j \left\{ u_j + \mu_j + \frac{\theta_j}{1-\sigma}\right.$$

$$-\sum_{i=1}^{2}\Big(b^*_{ji}+\frac{c^*_{ji}}{1-\sigma}+\rho d^*_{ji}\Big)\tilde{L}_j\Big\}>0,$$

$$\min_j\Big\{\gamma_j-\sum_{i=1}^{2}\{\Phi_1+\Phi_2+\Phi_3\}\Big\}>0.$$

for $(i,j=1,2)$. *According to Theorem III.2, the Master-Slave systems are globally projective lag synchronization based on the switching controller* (7). *Next, we choose the initial conditions of MFBNNs master system and slave system as* $(p_1(t),p_2(t),q_1(t),q_2(t))=(-0.6,1.7,1.9,-0.5)$ *and* $(\tilde{p}_1(t),\tilde{p}_2(t),\tilde{q}_1(t),\tilde{q}_2(t))=(1.4,0.9,1.6,1.2)$. *Figure.1 displays the projective lag synchronization evolution for state variables, while Figure.2 shows that the adaptive control gains tends to positive number.*

V. Conclusion

This paper discussed a class of global projective lag synchronization of MFBNNs with mixed time varying delays. By employing the fractional Barbalat's lemma, differential inclusion theory and suitable Lyapunov-type methods, some new sufficient criteria is derived to ensure that the synchronization error dynamical system converges to zero, based on adaptive switching control schemes. Finally, a numerical simulation have been provided, to illustrate the effectiveness and feasibility of our main theoretical results. Basically, the bilayer structured competitive type neural networks contains two kinds of state variables such as long-term memory and short-term memory and it played a significant role in broad spectrum of potential application in different fields. However, the proposed method can be easily extended to global projective lag synchronization of fractional order competitive type neural networks with discontinuous activation and we will consider the interesting issue for future work.

Acknowledgements

This work was jointly supported with the financial support of RUSA - Phase 2.0 grant sanctioned vide letter No. F.24-51/2014-U (TNMulti-Gen), Dept.of Education Govt. of India, Thailand research grant fund (RSA5980019), the Jiangsu Provincial Key Laboratory of Networked Collective Intelligence under Grant No. BM2017002.

REFERENCES

1. Diethelm K., The Analysis of Fractional Differential Equations, *Springer, Berlin, Germany* 2010.
2. Kilbas A., Srivastava A., Trujillo J., Theory and applications of fractional differential equations, *Elsevier Science Limited*, 2006.
3. Podlubny I., Fractional differential equations, San Diego, *California: Academic Press*, 1999.
4. Ren F., Feng C., Cao J., Mittag-Leffler stability and generalized Mittag-Leffler stability of fractional-order gene regulatory networks, *Neurocomputing*, vol.160, pp. 185-190, 2015.
5. Shi, Y., Cao J., Chen, G., Exponential stability of complex-valued memristor-based neural networks with time-varying delays, *Applied Mathematics and Computation*, vol. 313, 222-234, 2017.
6. Bao H., Park Ju H., Cao J., Adaptive synchronization of fractional-order memristor-based neural networks with time delay, *Nonlinear Dynamics*, vol. 31, pp. 343-1354, 2015.
7. Cao, J., Li, R., Fixed-time synchronization of delayed memristor-based recurrent neural networks, Science China Information Sciences, vol. 60, no. 3, pp. 032201, 2017.
8. Velmurugan, G., Rakkiyappan, R., Hybrid projective synchronization of fractional-order memristor-based neural networks with time delays. *Nonlinear Dynamics*, vol. 83, pp. 419-432, 2016.
9. Wu A., Zeng Z., Song X., Global Mittag-Leffler stabilization of fractionalorder bidirectional associative memory neural networks. *Neurocomputing*, vol.177, pp. 489-496, 2016.
10. Huang C., Cao J., Xiao M., Hybrid control on bifurcation for a delayed fractional gene regulatory network, *Chaos Solitons and Fractals*, vol. 87, pp. 19-29, 2016.
11. Xiao M., Zheng W., Jiang G., Cao J., Undamped oscillations generated by Hopf bifurcations in fractional order recurrent neural networks with Caputo derivative, *IEEE Transactions on Neural Networks and Learning Systems*, vol. 26, pp. 3201-3214, 2015.
12. Tank D., Hopfield J., Neural computation by concentrating information in time, *Proceedings of the National Academy of Sciences*, vol. 84, pp. 1896-1900, 1987.
13. Chua, L., Memristor-the missing circuit element, *IEEE Transactions on Circuit Theory*, vol. 18, pp. 507-519, 1971.
14. Strukov D., Snider G., Stewart D., Williams R., The missing memristor found, *Nature* , vol.453, pp. 80-83, 2008.
15. Tour J., He T., The fourth element, Nature vol. 453, pp. 42-43, 2008.

16. Shahverdiev E.M., Sivaprakasam S., Shore K.A., Lag synchronization in time delayed systems, *Physics Letters A*, vol. 292, pp. 320-324, 2002.
17. Kosko B., Adaptive bidirectional associative memories, *Applied Optics*, vol. 26, pp. 4947-4960, 1987.
18. Cao J., Wan Y., Matrix measure strategies for stability and synchronization of inertial BAM neural network with time delays, *Neural Networks*, vol.53, pp. 165-172, 2014.
19. Tu Z., Wang L., Zha Z., Jian J., Global dissipativity of a class of BAM neural networks with time-varying and unbound delays, *Communications in Nonlinear Science and Numerical Simulation*, vol.18 no.9, pp. 2562-2570, 2013.
20. Wang F., Yang Y., Hu M., Asymptotic stability of delayed fractional-order neural networks with impulsive effects, *Neurocomputing*, vol. 154, pp. 239-244, 2015.
21. Bao H., Cao. J., Projective synchronization of fractional-order memristor-based neural networks, *Neural Networks*, vol. 63, pp. 1-9, 2015.
22. Li, N., Cao, J., Lag synchronization of memristor-based coupled neural networks via ω-measure, *IEEE Transactions on Neural Networks and Learning Systems*, vol. 27, no, 3, pp. 686-697, 2016.
23. Li, N., Cao, J., New synchronization criteria for memristor-based networks: Adaptive control and feedback control schemes, *Neural Networks*, vol. 61, pp. 1-9, 2015.
24. Wu A., Zeng Z., Anti-synchronization control of a class of memristive recurrent neural networks, *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, pp. 373-385, 2013.
25. Mathiyalagan K., Park J. H., Sakthivel R., Synchronization for delayed memristive BAM neural networks using impulsive control with random nonlinearities, *Applied Mathematics and Computation*, vol. 259, pp. 967-979, 2015.
26. Wang, L., Shen, Y., Yin, Q., Zhang, G., Adaptive synchronization of memristor-based neural networks with time-varying delays. *IEEE Transactions on Neural Networks and Learning Systems*, vol. 26, no.9, pp. 2033-2042, 2015.
27. Cheng Z., Cao J., Hybrid control of Hopf bifurcation in complex networks with delays, *Neurocomputing*, vol. 131, 164-170, 2014
28. Yu J., Hu C., Jiang H., Fan X., Projective synchronization for fractional neural networks, Neural Networks, vol. 49, pp. 87-95, 2014.
29. Zhang L., Yang Y., Wang F., Lag synchronization for fractional-order memristive neural networks via period intermittent control, *Nonlinear Dyn*, vol. 89, no. 1, pp. 367-381, 2017.
30. Zhang L., Yang Y., Wang. F, Projective synchronization of fractional-order memristive neural networks with switching jumps mismatch, *Physica A*, vol. 471, pp. 402-415, 2017.
31. Aubin J., Cellina A., Differential inclusions, Germany, *Berlin: Springer-Verlag*, 1984.
32. Filippov A. F., Differential equations with discontinuous right-hand sides, *Dordrecht: Kluwer*, 1988.
33. Henderson O., Ouahab A., Fractional functional differential inclusions with finite delay, *Nonlinear Analysis: Theory, Methods and Applications*, vol.70, no. 5, pp. 2091-2105, 2009.
34. A. Pratap, R. Raja, C. Sowmiya, O. Bagdasar, J. Cao, G. Rajchak, Robust generalized Mittag-Leffler synchronization of fractional order neural networks with discontinuous activation and impulses, Neural networks 103 (2018) 128-141.
35. Yang. X., Cao. J., Yu. W., Exponential synchronization of memristive CohenGrossberg neural networks with mixed delays, *Cognitive Neurodynamics*, vol.8, pp. 239-249, 2015.
36. Yang. X., Cao. J., Qiu. J., pth moment exponential stochastic synchronization of coupled memristor-based neural networks with mixed delays via delayed impulsive control, *Neural Networks*, vol.65, pp. 80-91, 2015.
37. Yang. X., Ho. D., Synchronization of Delayed Memristive Neural Networks: Robust Analysis Approach, *IEEE Transactions on Cybernetics*, vol.46, no. 12. pp. 3377-3387, 2016.
38. Yang. X., Cao. J., Exponential synchronization of memristive neural networks with delays: Interval matrix method, *IEEE Transactions on neural networs and Learning systems*, vol. 28, no. 8. pp. 1878-1888, 2017.