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AN EQUIVALENT PROPERTY OF A HILBERT-TYPE INTEGRAL INEQUALITY AND ITS APPLICATIONS

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Making use of complex analytic techniques as well as methods involving weight functions, we study a few equivalent conditions of a Hilbert-type integral inequality with nonhomogeneous kernel and parameters. In the form of applications we deduce a few equivalent conditions of a Hilbert-type integral inequality with homogeneous kernel, and we additionally consider operator expressions.

1. INTRODUCTION

In 1925, Hardy [**6**] proved the following result, which is now very well known as the classical Hardy-Hilbert integral inequality. This states that for positive real numbers p, q with $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and functions $f(x), g(y) \geq 0$, with

$$0 < \int_0^\infty f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(y)dy < \infty,$$

we have

$$(1) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}},$$

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where the constant factor

$$\frac{\pi}{\sin(\pi/p)}$$

is the best possible.

For $p = q = 2$, (1) yields the well known Hilbert integral inequality. Both (1), as well as Hilbert's integral inequality play an important role in analysis and its applications (cf. [**7**], [**16**]).

In 1934, Hardy et al. established the following extension of (1):
If $k_1(x, y)$ is a nonnegative homogeneous function of degree -1, and one defines

$$k_p = \int_0^\infty k_1(u, 1)u^{-\frac{1}{p}}du \in \mathbb{R}_+ := (0, \infty),$$

then we have the following Hardy-Hilbert-type integral inequality:

$$\int_0^\infty \int_0^\infty k_1(x, y)f(x)g(y)dxdy < k_p \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \tag{2}$$

where the constant factor k_p is the best possible (cf. [**7**], Theorem 319).

Additionally, the following Hilbert-type integral inequality with nonhomogeneous kernel holds true:
If $h(u) > 0, \phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in \mathbb{R}_+$, then

$$\begin{aligned} &\int_0^\infty \int_0^\infty h(xy)f(x)g(y)dxdy \\ &< \phi\left(\frac{1}{p}\right) \left(\int_0^\infty x^{p-2}f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \end{aligned} \tag{3}$$

where the constant factor $\phi\left(\frac{1}{p}\right)$ is the best possible (cf. [**7**], Theorem 350).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang established an extension of Hilbert's integral inequality, namely the following (cf. [**19**], [**20**]):

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \\ &< B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_0^\infty x^{1-\lambda}f^2(x)dx \int_0^\infty y^{1-\lambda}g^2(y)dy\right)^{\frac{1}{2}}, \end{aligned} \tag{4}$$

where the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ is the best possible ($B(u, v)$ is the beta function).

In 2004, by introducing two pairs of conjugate exponents (p, q) and (r, s), Yang [**21**] proved the following extension of (1):
If $\lambda > 0$, $p, r > 1$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1$, and $f(x), g(y) \geq 0$, satisfy

$$0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx < \infty \;\; \textit{and} \;\; 0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy < \infty,$$

then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy$$

$$(5) \qquad < \frac{\pi}{\lambda \sin(\pi/r)} \left[\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y) dy \right]^{\frac{1}{q}},$$

where the constant factor

$$\frac{\pi}{\lambda \sin(\pi/r)}$$

is the best possible. For $\lambda = 1, r = q, s = p$, (5) reduces to (1).

In 2005, the paper [**22**] also provided an extension of (1) and (4) with the kernel $\frac{1}{(x+y)^\lambda}$ and two pairs of conjugate exponents. Krnić et al. [**1**], [**2**], [**3**], [**8**], [**10**], [**11**], [**14**], [**18**], [**30**], proved some extensions and particular cases of (1), (2) and (3) with parameters. In 2009, Yang established an extension of (2) and (5), namely the following (cf. [**23**], [**24**]):
If $\lambda_1 + \lambda_2 = \lambda \in \mathbb{R}$, $k_\lambda(x, y)$ is a nonnegative homogeneous function of degree $-\lambda$, satisfying

$$k_\lambda(ux, uy) = u^{-\lambda} k_\lambda(x, y) \ (u, x, y > 0),$$

and

$$k(\lambda_1) = \int_0^\infty k_\lambda(u, 1) u^{\lambda_1 - 1} du \in \mathbb{R}_+,$$

then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dxdy$$

$$(6) \qquad < k(\lambda_1) \left[\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\lambda_2)-1} g^q(y) dy \right]^{\frac{1}{q}},$$

where the constant factor $k(\lambda_1)$ is the best possible.
For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (6) reduces to (2), while for $\lambda > 0$, $\lambda_1 = \frac{\lambda}{r}$, $\lambda_2 = \frac{\lambda}{s}$, $k_\lambda(x, y) = \frac{1}{x^\lambda + y^\lambda}$, (6) reduces to (5).

Additionally, the following extension of (3) was proved:

$$\int_0^\infty \int_0^\infty h(xy) f(x) g(y) dxdy$$

$$(7) \qquad < \phi(\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}},$$

where the constant factor $\phi(\sigma)$ is the best possible (cf. [**25**]).
For $\sigma = \frac{1}{p}$, (7) reduces to (3). Some equivalent inequalities of (6) and (7) were constructed in [**24**]. In 2013, Yang [**25**] also studied the equivalence of (6) and (7) by adding a condition $h(u) = k_\lambda(u, 1)$. In 2017, Hong [**9**] studied an equivalent

condition for (6) involving certain parameters, and some further related results were established in [**4**], [**5**], [**15**], [**29**], [**28**].

In the present paper, making use of complex analytic techniques as well as methods involving weight functions, we study a few equivalent conditions of a Hilbert-type integral inequality with the nonhomogeneous kernel

$$\frac{1}{\prod_{k=1}^{s}[(xy)^{\lambda}+c_k]}\ (c_k>0)$$

and a best possible constant factor. In the form of applications we deduce a few equivalent conditions of a Hilbert-type integral inequality with homogeneous kernel. We also consider some operator expressions.

2. SOME LEMMAS

Lemma 1. *(cf. [**26**]) If $\mathbb{C}$ is the set of complex numbers and $\mathbb{C}_\infty=\mathbb{C}\cup\{\infty\}$,*

$$z_k\in\mathbb{C}\backslash\{z\mid Re(z)\geq 0,\quad Im(z)=0\}\quad (k=1,2,\ldots,n)$$

are different points, the function $f(z)$ is analytic in $\mathbb{C}_\infty$ except for z_i $(i=1,2,\ldots,n)$, and $z=\infty$ is a zero point of $f(z)$ whose order is not less than 1, then for $\alpha\in\mathbb{R}$, we have

$$\int_0^\infty f(x)x^{\alpha-1}dx=\frac{2\pi i}{1-e^{2\pi\alpha i}}\sum_{k=1}^{n}Re(s)[f(z)z^{\alpha-1},\,z_k],$$

where $0<Im(\ln z)=\arg z<2\pi$. In particular, if z_k $(k=1,\ldots,n)$ are all poles of order 1, setting

$$\varphi_k(z)=(z-z_k)f(z)\ (\varphi_k(z_k)\neq 0),$$

then

$$\int_0^\infty f(x)x^{\alpha-1}dx=\frac{\pi}{\sin\pi\alpha}\sum_{k=1}^{n}(-z_k)^{\alpha-1}\varphi_k(z_k). \tag{8}$$

Example 2. *For $s\in\mathbb{N}=\{1,2,\ldots\}$ and $0<c_1\leq\cdots\leq c_s$, $0<\sigma<s\lambda,\varepsilon>0$, we set*

$$h(u):=\frac{1}{\prod_{k=1}^{s}(u^{\lambda}+c_k)},\quad (u>0),$$

and

$$\widetilde{c}_k=c_k+(k-1)\varepsilon\ (k=1,\ldots,s).$$

By (8), for $z_k=-\widetilde{c}_k$, we derive that

$$\begin{aligned}\widetilde{k}_s(\sigma) &= \int_0^\infty\frac{1}{\prod_{k=1}^{s}(t^{\lambda}+\widetilde{c}_k)}t^{\sigma-1}dt\\ &= \frac{1}{\lambda}\int_0^\infty\frac{1}{\prod_{k=1}^{s}(u+\widetilde{c}_k)}u^{\frac{\sigma}{\lambda}-1}du\\ &= \frac{\pi}{\lambda\sin\frac{\pi\sigma}{\lambda}}\sum_{k=1}^{s}\frac{\widetilde{c}_k^{\frac{\sigma}{\lambda}-1}}{\prod_{j=1(j\neq k)}^{s}(\widetilde{c}_j-\widetilde{c}_k)}.\end{aligned}$$

Setting $\mu = s\lambda - \sigma (> 0)$, we obtain that

$$\begin{aligned} 0 &< \widetilde{k}_s(\sigma) = \frac{1}{\lambda}\int_0^\infty \frac{1}{\prod_{k=1}^s (u+\widetilde{c}_k)} u^{\frac{\sigma}{\lambda}-1} du \\ &\le \frac{1}{\lambda}\int_0^\infty \frac{1}{(u+c_1)^s} u^{\frac{\sigma}{\lambda}-1} du \\ &= \frac{1}{\lambda c_1^{\mu/\lambda}}\int_0^\infty \frac{1}{(v+1)^s} v^{\frac{\sigma}{\lambda}-1} dv \\ &= \frac{1}{\lambda c_1^{\mu/\lambda}} B\left(\frac{\sigma}{\lambda}, \frac{\mu}{\lambda}\right) < \infty, \end{aligned}$$

and by Levi's theorem (cf. [**12**]), it follows that

$$\begin{aligned} k_s(\sigma) &= \int_0^\infty \frac{t^{\sigma-1}}{\prod_{k=1}^s (t^\lambda + c_k)} dt = \lim_{\varepsilon\to 0^+}\int_0^\infty \frac{t^{\sigma-1}}{\prod_{k=1}^s (t^\lambda + \widetilde{c}_k)} dt \\ &= \lim_{\varepsilon\to 0^+} \widetilde{k}_s(\sigma) = \frac{\pi}{\lambda \sin\frac{\pi\sigma}{\lambda}} \sum_{k=1}^s \frac{c_k^{\frac{\sigma}{\lambda}-1}}{\prod_{j=1(j\neq k)}^s (c_j - c_k)} \in \mathbb{R}_+. \end{aligned} \tag{9}$$

In particular:
(i) for $s = 1$, we obtain

$$k_1(\sigma) = \frac{1}{\lambda}\int_0^\infty \frac{u^{(\sigma/\lambda)-1}}{u+c_1} du = \frac{\pi}{\lambda c_1^{\mu/\lambda} \sin\left(\frac{\pi\sigma}{\lambda}\right)};$$

(ii) for $s = 2$, we get that

$$\begin{aligned} k_2(\sigma) &= \int_0^\infty \frac{1}{(t^\lambda + c_1)(t^\lambda + c_2)} t^{\sigma-1} dt \\ &= \frac{\pi}{\lambda \sin\frac{\pi\sigma}{\lambda}} \frac{c_1^{\frac{\sigma}{\lambda}-1} - c_2^{\frac{\sigma}{\lambda}-1}}{c_2 - c_1}; \end{aligned}$$

(iii) for $c_s = \cdots = c_1$ in (9), we have

$$k(\sigma) := \int_0^\infty \frac{t^{\sigma-1}}{(t^\lambda + c_1)^s} dt = \frac{s}{\lambda c_1^{\mu/\lambda}} B\left(\frac{\sigma}{\lambda}, \frac{\mu}{\lambda}\right).$$

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $s \in \mathbb{N}$, $0 < c_1 \le \cdots \le c_s$, $0 < \sigma < s\lambda$, $\sigma_1 \in \mathbb{R}$, then for $n \in \mathbb{N}$, we define the following two expressions:

$$I_1 := \int_1^\infty \left\{ \int_0^1 \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} x^{\sigma + \frac{1}{pn} - 1} dx \right\} y^{\sigma_1 - \frac{1}{qn} - 1} dy, \tag{10}$$

$$I_2 := \int_0^1 \left\{ \int_1^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} x^{\sigma - \frac{1}{pn} - 1} dx \right\} y^{\sigma_1 + \frac{1}{qn} - 1} dy. \tag{11}$$

Setting $u = xy$ in (10) and (11), by Fubini's theorem (cf. [**12**]), we obtain

$$\begin{aligned}
I_1 &= \int_1^\infty \left[\int_0^y \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} \left(\frac{u}{y}\right)^{\sigma+\frac{1}{pn}-1} \frac{1}{y} du\right] y^{\sigma_1-\frac{1}{qn}-1} dy \\
&= \int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1} \left[\int_0^y \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1} du\right] dy \\
&= \int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1} dy \int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1} du \\
&\quad + \int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1} \int_1^y \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1} du dy \\
&= \int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1} dy \int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1} du \\
&\quad + \int_1^\infty \left[\int_u^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1} dy\right] \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1} du, \qquad (12)
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_0^1 \left\{\int_y^\infty \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} \left(\frac{u}{y}\right)^{\sigma-\frac{1}{pn}-1} \frac{1}{y} du\right\} y^{\sigma_1+\frac{1}{qn}-1} dy \\
&= \int_0^1 y^{(\sigma_1-\sigma)+\frac{1}{n}-1} \left[\int_y^\infty \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma-\frac{1}{pn}-1} du\right] dy \\
&= \int_0^1 y^{(\sigma_1-\sigma)+\frac{1}{n}-1} dy \int_y^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma-\frac{1}{pn}-1} du \\
&\quad + \int_0^1 y^{(\sigma_1-\sigma)+\frac{1}{n}-1} \int_1^\infty \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma-\frac{1}{pn}-1} du dy \\
&= \int_0^1 \left[\int_0^u y^{(\sigma_1-\sigma)+\frac{1}{n}-1} dy\right] \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma-\frac{1}{pn}-1} du \\
&\quad + \int_0^1 y^{(\sigma_1-\sigma)+\frac{1}{n}-1} dy \int_1^\infty \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma-\frac{1}{pn}-1} du. \qquad (13)
\end{aligned}$$

In what follows we suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, s \in \mathbb{N}, 0 < c_1 \le \cdots \le c_s$, $\sigma, \mu > 0$, $\sigma + \mu = s\lambda$, $\sigma_1 \in \mathbb{R}$.

Lemma 3. *If there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in $(0, \infty)$, the following inequality*

$$\begin{aligned}
I &:= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx dy \\
&\le M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy\right]^{\frac{1}{q}} \qquad (14)
\end{aligned}$$

holds true, then we have $\sigma_1 = \sigma$. In this case, it follows that $M \ge k_s(\sigma)$.

Proof. If $\sigma_1 < \sigma$, then for $n > \frac{1}{\sigma - \sigma_1}$ $(n \in \mathbb{N})$, we set two functions

$$f_n(x) := \begin{cases} 0, \ 0 < x < 1 \\ x^{\sigma - \frac{1}{pn} - 1}, \ x \geq 1 \end{cases}, \quad g_n(y) := \begin{cases} y^{\sigma_1 + \frac{1}{qn} - 1}, \ 0 < y \leq 1 \\ 0, \ y > 1 \end{cases}.$$

Hence, we obtain that

$$\begin{aligned} J_2 &:= \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y) dy\right]^{\frac{1}{q}} \\ &= \left(\int_1^\infty x^{-\frac{1}{n}-1} dx\right)^{\frac{1}{p}} \left(\int_0^1 y^{\frac{1}{n}-1} dy\right)^{\frac{1}{q}} = n. \end{aligned}$$

By (13) and (14), we have

$$\begin{aligned} &\int_0^1 \left[\int_0^u y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy\right] \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{pn} - 1} du \\ (15) \qquad &\leq I_2 = \int_0^\infty \int_0^\infty \frac{f_n(x) g_n(y)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx dy \leq M J_2 = Mn. \end{aligned}$$

Since $(\sigma_1 - \sigma) + \frac{1}{n} < 0$, it follows that for any $u \in (0, 1)$,

$$\int_0^u y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy = \infty.$$

By (15), in view of

$$\frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma - \frac{1}{pn} - 1} > 0, \quad u \in (0, 1),$$

we deduce that $\infty \leq Mn < \infty$, which is a contradiction.

If $\sigma_1 > \sigma$, then for $n > \frac{1}{\sigma_1 - \sigma}$ $(n \in \mathbb{N})$, we set

$$\widetilde{f}_n(x) := \begin{cases} x^{\sigma + \frac{1}{pn} - 1}, \ 0 < x \leq 1 \\ 0, \ x > 1 \end{cases}, \quad \widetilde{g}_n(y) := \begin{cases} 0, \ 0 < y < 1 \\ y^{\sigma_1 - \frac{1}{qn} - 1}, \ y \geq 1 \end{cases}.$$

Hence, we derive that

$$\begin{aligned} \widetilde{J}_2 \ &: \ = \left[\int_0^\infty x^{p(1-\sigma)-1} \widetilde{f}_n^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} \widetilde{g}_n^q(y) dy\right]^{\frac{1}{q}} \\ &= \left(\int_0^1 x^{\frac{1}{n}-1} dx\right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\frac{1}{n}-1} dy\right)^{\frac{1}{q}} = n. \end{aligned}$$

By (12) and (14), we have

$$\begin{aligned} &\int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy \int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma + \frac{1}{pn} - 1} du \\ (16) \qquad &\leq I_1 = \int_0^\infty \int_0^\infty \frac{\widetilde{f}_n(x) \widetilde{g}_n(y)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx dy \leq M \widetilde{J}_2 = Mn. \end{aligned}$$

Since $(\sigma_1 - \sigma) - \frac{1}{n} > 0$, it follows that

$$\int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1} dy = \infty.$$

By (16), in view of

$$\int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1} du > 0,$$

we have $\infty \le Mn < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, we reduce (12) and then apply (16) as follows:

$$\begin{aligned} \frac{1}{n} I_1 &= \frac{1}{n} \left[\int_1^\infty y^{-\frac{1}{n}-1} dy \int_0^1 \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1} du \right. \\ &\quad \left. + \int_1^\infty \left(\int_u^\infty y^{-\frac{1}{n}-1} dy \right) \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1} du \right] \\ &= \int_0^1 \frac{u^{\sigma+\frac{1}{pn}-1}}{\prod_{k=1}^s (u^\lambda + c_k)} du + \int_1^\infty \frac{u^{\sigma-\frac{1}{qn}-1}}{\prod_{k=1}^s (u^\lambda + c_k)} du \\ &\le \frac{1}{n} M \widetilde{J}_2 = M. \end{aligned}$$

Since the sequence

$$\left\{ \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1} \right\}_{n=1}^\infty \quad \left(resp.\ \left\{ \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma-\frac{1}{qn}-1} \right\}_{n=1}^\infty \right)$$

is nonnegative and increasing in $(0,1)$ (resp. $(1,\infty)$), by Levi's theorem (cf. [**12**]), we deduce that

$$\begin{aligned} k_s(\sigma) &= \int_0^1 \lim_{n\to\infty} \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma+\frac{1}{pn}-1} du \\ &\quad + \int_1^\infty \lim_{n\to\infty} \frac{1}{\prod_{k=1}^s (u^\lambda + c_k)} u^{\sigma-\frac{1}{qn}-1} du \\ &= \lim_{n\to\infty} \left[\int_0^1 \frac{u^{\sigma+\frac{1}{pn}-1} du}{\prod_{k=1}^s (u^\lambda + c_k)} + \int_1^\infty \frac{u^{\sigma-\frac{1}{qn}-1} du}{\prod_{k=1}^s (u^\lambda + c_k)} \right] \le M < \infty. \end{aligned}$$

This completes the proof of the lemma. □

3. MAIN RESULTS

Theorem 4. *The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following inequality:

$$\begin{aligned} J &:= \left\{\int_0^\infty y^{p\sigma_1 - 1}\left[\int_0^\infty \frac{f(x)}{\prod_{k=1}^s[(xy)^\lambda + c_k]}dx\right]^p dy\right\}^{\frac{1}{p}} \\ &< M\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}; \end{aligned} \tag{17}$$

(ii) there exists a constant M, such that for any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy < \infty,$$

we have the following Hilbert-type integral inequality with nonhomogeneous kernel:

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\prod_{k=1}^s[(xy)^\lambda + c_k]}dxdy \\ &< M\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}; \end{aligned} \tag{18}$$

(iii) $\sigma_1 = \sigma$.

If Condition (iii) is satisfied, then $M \geq k_s(\sigma)$ and the constant factor $M = k_s(\sigma)$ in (17) and (18) is the best possible.

Proof. $(i) \Rightarrow (ii)$. By Hölder's inequality (cf. [**13**]), we have

$$\begin{aligned} I &= \int_0^\infty \left\{y^{\sigma_1 - \frac{1}{p}}\int_0^\infty \frac{f(x)}{\prod_{k=1}^s[(xy)^\lambda + c_k]}dx\right\}\left(y^{\frac{1}{p}-\sigma_1}g(y)\right)dy \\ &\leq J\left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{19}$$

Then by (17), we derive (18).

$(ii) \Rightarrow (iii)$. By Lemma 1, we have $\sigma_1 = \sigma$.

$(iii) \Rightarrow (i)$. Setting $u = xy$ for $y > 0$, we obtain the following weight function

$$\begin{aligned} \omega(\sigma, y) &:= y^{\sigma} \int_0^{\infty} \frac{1}{\prod_{k=1}^{s}[(xy)^{\lambda} + c_k]} x^{\sigma-1} dx \\ &= \int_0^{\infty} \frac{1}{\prod_{k=1}^{s}(u^{\lambda} + c_k)} u^{\sigma-1} du = k_s(\sigma). \end{aligned} \tag{20}$$

By Hölder's weighed inequality and (20), we have

$$\begin{aligned} & \left\{ \int_0^{\infty} \frac{1}{\prod_{k=1}^{s}[(xy)^{\lambda} + c_k]} f(x) dx \right\}^p \\ = & \left\{ \int_0^{\infty} \frac{1}{\prod_{k=1}^{s}[(xy)^{\lambda} + c_k]} \left[\frac{y^{(\sigma-1)/p}}{x^{(\sigma-1)/q}} f(x) \right] \left[\frac{x^{(\sigma-1)/q}}{y^{(\sigma-1)/p}} \right] dx \right\}^p \\ \leq & \int_0^{\infty} \frac{1}{\prod_{k=1}^{s}[(xy)^{\lambda} + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \\ & \times \left\{ \int_0^{\infty} \frac{1}{\prod_{k=1}^{s}[(xy)^{\lambda} + c_k]} \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} dx \right\}^{p/q} \\ = & \left[\frac{\omega(\sigma, y)}{y^{q(\sigma-1)+1}} \right]^{p-1} \int_0^{\infty} \frac{1}{\prod_{k=1}^{s}[(xy)^{\lambda} + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \\ = & \frac{(k_s(\sigma))^{p-1}}{y^{p\sigma-1}} \int_0^{\infty} \frac{1}{\prod_{k=1}^{s}[(xy)^{\lambda} + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \end{aligned} \tag{21}$$

If (21) assumes the form of equality for some $y \in (0, \infty)$, then (cf. [**13**]) there exist constants A and B, such that they are not both zero, and

$$A \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) = B \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} \text{ a.e. in } \mathbb{R}_+.$$

We suppose that $A \neq 0$ (otherwise $B = A = 0$). Then it follows that

$$x^{p(1-\sigma)-1} f^p(x) = y^{q(1-\sigma)} \frac{B}{Ax} \text{ a.e. in } \mathbb{R}_+,$$

which contradicts the fact that

$$0 < \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx < \infty.$$

Hence, (21) assumes the form of strict inequality.

For $\sigma_1 = \sigma$, by Fubini's theorem, we have

$$\begin{aligned} J &< (k_s(\sigma))^{\frac{1}{q}} \left\{ \int_0^\infty \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx dy \right\}^{\frac{1}{p}} \\ &= (k_s(\sigma))^{\frac{1}{q}} \left\{ \int_0^\infty \left[\int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} \frac{y^{\sigma-1}}{x^{(\sigma-1)(p-1)}} dy \right] f^p(x) dx \right\}^{\frac{1}{p}} \\ &= (k_s(\sigma))^{\frac{1}{q}} \left[\int_0^\infty \omega(\sigma, x) x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \\ &= k_s(\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned}$$

Setting $M \geq k_s(\sigma)$, then (17) follows.

Therefore, the conditions (i), (ii) and (iii) are equivalent.

When Condition (iii) is satisfied, if there exists a constant $M < k_s(\sigma)$, such that (18) is valid, then by Lemma 3, we have $M \geq k_s(\sigma)$. By this contradiction it follows that the constant factor $M = k_s(\sigma)$ in (18) is the best possible. The constant factor $M = k_s(\sigma)$ in (17) is still the best possible. Otherwise, by (19) (for $\sigma_1 = \sigma$), we would conclude that the constant factor $M = k_s(\sigma)$ in (18) is not the best possible. □

Setting $y = \frac{1}{Y}$, $G(Y) = Y^{s\lambda-2} g\left(\frac{1}{Y}\right), \mu_1 = s\lambda - \sigma_1$ in Theorem 4, then replacing Y (respectively $G(Y)$) by y (respectively $g(y)$), we deduce the following result.

Corollary 5. *The following conditions are equivalent:*

(i) There exists a constant M, *such that for any* $f(x) \geq 0$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following integral inequality:

$$\begin{aligned} &\left\{ \int_0^\infty y^{p\mu_1 - 1} \left[\int_0^\infty \frac{f(x)}{\prod_{k=1}^s (x^\lambda + c_k y^\lambda)} dx \right]^p dy \right\}^{\frac{1}{p}} \\ &< M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}; \end{aligned} \tag{22}$$

(ii) There exists a constant M, *such that for any* $f(x), g(y) \geq 0$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1-\mu_1)-1} g^q(y) dy < \infty,$$

we have the following Hilbert-type integral inequality with homogeneous kernel:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\prod_{k=1}^s (x^\lambda + c_k y^\lambda)} dxdy$$

$$(23) \qquad < M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\mu_1)-1} g^q(y)dy\right]^{\frac{1}{q}};$$

(iii) $\mu_1 = \mu$.

If Condition (iii) holds, then we have $M \geq k_s(\sigma)$, *and the constant factor* $M = k_s(\sigma)$ *in (22) and (23) is the best possible.*

Remark 6. *On the other hand, setting* $y = \frac{1}{Y}$, $G(Y) = Y^{s\lambda-2} g(\frac{1}{Y}), \sigma_1 = s\lambda - \mu_1$, *in Corollary 5, then replacing* Y *(resp.* $G(Y)$*) by* y *(resp.* $g(y)$*), we deduce Theorem 4. Hence, Theorem 4 and Corollary 5 are equivalent.*

4. OPERATOR EXPRESSIONS

We set the following functions:
$\varphi(x) := x^{p(1-\sigma)-1}, \psi(y) := y^{q(1-\sigma)-1}, \phi(y) := y^{q(1-\mu)-1}$, wherefrom,

$$\psi^{1-p}(y) = y^{p\sigma-1}, \phi^{1-p}(y) = y^{p\mu-1} \ (x, y \in \mathbb{R}_+).$$

Define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbb{R}_+) = \left\{ f : \|f\|_{p,\varphi} := \left(\int_0^\infty \varphi(x)|f(x)|^p dx\right)^{\frac{1}{p}} < \infty \right\},$$

$$L_{q,\psi}(\mathbb{R}_+) = \left\{ g : \|g\|_{q,\psi} := \left(\int_0^\infty \psi(y)|g(y)|^q dy\right)^{\frac{1}{q}} < \infty \right\},$$

$$L_{q,\phi}(\mathbb{R}_+) = \left\{ g : \|g\|_{q,\phi} := \left(\int_0^\infty \phi(y)|g(y)|^q dy\right)^{\frac{1}{q}} < \infty \right\},$$

$$L_{p,\psi^{1-p}}(\mathbb{R}_+) = \left\{ h : \|h\|_{p,\psi^{1-p}} = \left(\int_0^\infty \psi^{1-p}(y)|h(y)|^p dy\right)^{\frac{1}{p}} < \infty \right\},$$

$$L_{q,\phi^{1-p}}(\mathbb{R}_+) = \left\{ h : \|h\|_{p,\phi^{1-p}} = \left(\int_0^\infty \phi^{1-p}(y)|h(y)|^p dy\right)^{\frac{1}{p}} < \infty \right\}.$$

(a) In view of Theorem 4 (setting $\sigma_1 = \sigma$), for $f \in L_{p,\varphi}(\mathbb{R}_+)$, setting

$$h_1(y) := \int_0^\infty \frac{1}{\prod_{k=1}^s [(xy)^\lambda + c_k]} f(x)dx \ (y \in \mathbb{R}_+),$$

by (17), we have

$$(24) \qquad \|h_1\|_{p,\psi^{1-p}} = \left(\int_0^\infty \psi^{1-p}(y) h_1^p(y)dy\right)^{\frac{1}{p}} < M\|f\|_{p,\varphi} < \infty.$$

Definition 7. *Define a Hilbert-type integral operator with nonhomogeneous kernel* $T^{(1)}: L_{p,\varphi}(\mathbb{R}_+) \to L_{p,\psi^{1-p}}(\mathbb{R}_+)$ *as follows: For any* $f \in L_{p,\varphi}(\mathbb{R}_+)$, *there exists a unique representation* $T^{(1)}f = h_1 \in L_{p,\psi^{1-p}}(\mathbb{R}_+)$, *satisfying* $T^{(1)}f(y) = h_1(y)$, *for any* $y \in \mathbb{R}_+$.

In view of (24), it follows that

$$\|T^{(1)}f\|_{p,\psi^{1-p}} = \|h_1\|_{p,\psi^{1-p}} \leq M\|f\|_{p,\varphi},$$

and then the operator $T^{(1)}$ is bounded satisfying

$$\|T^{(1)}\| = \sup_{f(\neq 0)\in L_{p,\varphi}(\mathbb{R}_+)} \frac{\|T^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq M.$$

If we define the formal inner product of $T^{(1)}f$ and g as follows:

$$(T^{(1)}f, g) := \int_0^\infty \left\{ \int_0^\infty \frac{f(x)}{\prod_{k=1}^s [(xy)^\lambda + c_k]} dx \right\} g(y) dy,$$

then we can rewrite Theorem 4 as follows:

Theorem 8. *The following conditions are equivalent:*

(i) There exists a constant M, *such that for any* $f(x) \geq 0$, $f \in L_{p,\varphi}(\mathbb{R}_+)$, $\|f\|_{p,\varphi} > 0$, *we have the following inequality:*

$$\|T^{(1)}f\|_{p,\psi^{1-p}} < M\|f\|_{p,\varphi};$$

(ii) there exists a constant M, *such that for any* $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbb{R}_+)$, $g \in L_{q,\psi}(\mathbb{R}_+)$, $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$, *we have the following inequality:*

$$(T^{(1)}f, g) < M\|f\|_{p,\varphi}\|g\|_{q,\psi}.$$

We still have $\|T^{(1)}\| = k_s(\sigma) \leq M$.

(b) In view of Corollary 5 (with $\mu_1 = \mu$), for $f \in L_{p,\varphi}(\mathbb{R}_+)$, setting

$$h_2(y) := \int_0^\infty \frac{f(x)}{\prod_{k=1}^s (x^\lambda + c_k y^\lambda)} dx$$

defined for every $y \in \mathbb{R}_+$, by (22) we have

$$\|h_2\|_{p,\phi^{1-p}} = \left(\int_0^\infty \phi^{1-p}(y) h_2^p(y) dy \right)^{\frac{1}{p}} < M\|f\|_{p,\varphi} < \infty. \tag{25}$$

Definition 9. *Define a Hilbert-type integral operator with the homogeneous kernel* $T^{(2)}: L_{p,\varphi}(\mathbb{R}_+) \to L_{p,\phi^{1-p}}(\mathbb{R}_+)$ *as follows: For any* $f \in L_{p,\varphi}(\mathbb{R})$, *there exists a unique representation* $T^{(2)}f = h_2 \in L_{p,\phi^{1-p}}(\mathbb{R}_+)$, *satisfying* $T^{(2)}f(y) = h_2(y)$, *for any* $y \in \mathbb{R}_+$.

In view of (25), it follows that

$$\|T^{(2)}f\|_{p,\phi^{1-p}} = \|h_2\|_{p,\phi^{1-p}} \leq M\|f\|_{p,\varphi},$$

and then the operator $T^{(2)}$ is bounded satisfying

$$\|T^{(2)}\| = \sup_{f(\neq 0)\in L_{p,\varphi}(\mathbb{R}_+)} \frac{\|T^{(2)}f\|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq M.$$

If we define the formal inner product of $T^{(2)}f$ and g as follows:

$$(T^{(2)}f, g) := \int_0^\infty \left[\int_0^\infty \frac{f(x)}{\prod_{k=1}^s (x^\lambda + c_k y^\lambda)} dx\right] g(y)dy,$$

then we can rewrite Corollary 5 as below:

Corollary 10. *The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \geq 0$, $f \in L_{p,\varphi}(\mathbb{R}_+)$, $\|f\|_{p,\varphi} > 0$, we have the following inequality:

$$\|T^{(2)}f\|_{p,\phi^{1-p}} < M\|f\|_{p,\varphi};$$

(ii) there exists a constant M, such that for any $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbb{R}_+)$, $g \in L_{q,\phi}(\mathbb{R}_+), \|f\|_{p,\varphi}, \|g\|_{q,\phi} > 0$, we have the following inequality:

$$(T^{(2)}f, g) < M\|f\|_{p,\varphi}\|g\|_{q,\phi}.$$

We still have $\|T^{(2)}\| = k_s(\sigma) \leq M$.

Remark 11. *Theorem 8 and Corollary 10 are equivalent.*

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