Some remarks on 3-partitions of multisets

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Abstract

Partitions play an important role in numerous combinatorial optimization problems. Here we introduce the number of ordered 3-partitions of a multiset M having equal sums denoted by $S(m_1, ..., m_n; \alpha_1, ..., \alpha_n)$, for which we find the generating function and give a useful integral formula. Some recurrence formulae are then established and new integer sequences are added to OEIS, which are related to the number of solutions for the 3-signum equation.

Keywords: multiset; 3-partition of a multiset; generating function; asymptotic formula; 3-signum equation.

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1 Introduction

The *signum equation* for a given sequence of integers is considered in [3], in connection with the Erdös-Surányi problem. In particular, for a given integer $n \geq 2$, the level n solution of this equation represents the number $S(n)$ of ways of choosing $+$ and $-$ such that $\pm 1 \pm 2 \pm 3 \pm \cdots \pm n = 0$. This is also the number of ordered partitions of $\{1, 2, \ldots, n\}$ in two sets with equal sums.

Andrica and Tomescu [4] conjectured an asymptotic formula for $S(n)$:

$$\lim_{\substack{n\to\infty \\ n\equiv 0 \ or \ 3 (mod 4)}} \frac{S(n)}{\frac{2^n}{n\sqrt{n}}} = \sqrt{\frac{6}{\pi}},$$

which was proved by analytic methods by Sullivan [11].

Starting from a problem involving derivatives, Andrica established a generating function which allowed novel approaches in the study of 2-partitions with equal sums for multisets [1]. We refer the reader to [2,3] for connections with Erdös-Suranyi representations, to [10] for general theory of multisets and to [12] for details about generating functions.

This paper is motivated by some recent results on the number of ordered 2-partitions with equal sums for multisets obtained in [5]. The study of 3-partitions of multisets differs essentially from that of 2-partitions. In Section 2 of this paper we investigate the number of ordered 3-partitions of a multiset M having equal sums, for which establish the generating function and a useful integral formula. Some particular instances related to the number of solutions for the 3-signum equation are studied in Section 3, where recurrence formulae are established and some new integer sequences are proposed.

2 3-partitions of multisets with equal sums

Partitions have direct applications to classical combinatorial optimization problems such as Bin Packing Problem (BPP), Multiprocessor Scheduling Problem (MSP) and the 0-1 Multiple Knapsack Problem (MKP) [6].

Of particular interest is the 3-partition problem, one of the famous strongly NP-complete problems [7,8]. Given a positive integer b and a set $[n] = \{1, 2, ..., n\}$ of $n = 3m$ elements, each having a positive integer size a_s, such that $\sum_{s=1}^{n} a_s = mb$. The problem has a solution if there is a partition of N into m subsets, each containing exactly three elements from N, whose sum is exactly b. For example, the set $\{10, 13, 5, 15, 7, 10\}$ can be partitioned into the two sets $\{10, 13, 7\}$, $\{5, 15, 10\}$, each of which sum to 30.

Here we investigate another 3-partition concept of a multiset defined for the real numbers $\alpha_1, ..., \alpha_n$ and the positive integers m_1, ..., m_n, denoted by

$$M = \{\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1 \ times}, \cdots, \underbrace{\alpha_n, \cdots, \alpha_n}_{m_n \ times}\}.$$

We call m_s the *multiplicity* of the element α_s in the multiset M, while the notation $\sigma(M) = \sum_{s=1}^{n} m_s \alpha_s$ represents the *sum* of the elements of M.

Definition 2.1 Denote by $S(m_1, ..., m_n; \alpha_1, ..., \alpha_n)$ the number of ordered 3-partitions of M having equal sums, i.e., the number of triplets (C_1, C_2, C_3) of pairwise disjoint subsets of M such that

(i) $C_1 \cup C_2 \cup C_3 = M$;

(ii) $\sigma(C_1) = \sigma(C_2) = \sigma(C_3) = \frac{1}{3}\sigma(M)$.

The number $S(m_1, ..., m_n; \alpha_1, ..., \alpha_n)$ is the constant term of the expansion of the Laurent polynomial $F(X, Y) \in \mathbb{Z}[X, Y, X^{-1}, Y^{-1}]$, defined as

$$F(X,Y) = \left(X^{\alpha_1} + Y^{\alpha_1} + \frac{1}{(XY)^{\alpha_1}}\right)^{m_1} \cdots \left(X^{\alpha_n} + Y^{\alpha_n} + \frac{1}{(XY)^{\alpha_n}}\right)^{m_n}. \quad (1)$$

Indeed, assume that in the product $\left(X^{\alpha_s} + Y^{\alpha_s} + \frac{1}{(XY)^{\alpha_s}}\right)^{m_s}$ we have selected c_1^s terms equal to X^{α_s}, c_2^s terms equal to Y^{α_s}, and c_3^s terms equal to $\frac{1}{(XY)^{\alpha_s}}$, with $s = 1, \dots, n$, and notice that in this case we must have $c_1^s + c_2^s + c_3^s = m_s$.

Such a selection contributes to the free term if and only if

$$X^{\sum_{s=1}^{n} c_1^s \alpha_s} \cdot Y^{\sum_{s=1}^{n} c_2^s \alpha_s} \cdot \frac{1}{(XY)^{\sum_{s=1}^{n} c_3^s \alpha_s}} = 1,$$

which is equivalent to

$$\sum_{s=1}^{n} c_1^s \alpha_s = \sum_{j=1}^{n} c_2^s \alpha_s = \sum_{s=1}^{n} c_3^s \alpha_s.$$

This means that the sets

$$C_j = \{\underbrace{\alpha_1, \cdots, \alpha_1}_{c_1^j \ times}, \cdots, \underbrace{\alpha_n, \cdots, \alpha_n}_{c_n^j \ times}\}, \quad j = 1, 2, 3,$$

represent a partition of M which also satisfies property (ii) in Definition 2.1.

Ordering (1) in the increasing order of integer powers, one can write

$$F(X,Y)=\sum_{m\in\mathbb{Z}} P_m(Y)X^m=\sum_{m\in\mathbb{Z}} Q_m(X)Y^m, \tag{2}$$

where $P_m(Y)$ and $Q_m(X)$ are Laurent polynomials. Also, notice that the free term of $F(X,Y)$ is the free term of $P_0(Y)$ and $Q_0(X)$.

Clearly, we can write

$$F(X,Y)=\prod_{s=1}^{n}\left(X^{\alpha_s}+Y^{\alpha_s}+\frac{1}{(XY)^{\alpha_s}}\right)^{m_s}=P_0(Y)+\sum_{m\in\mathbb{Z},j\neq 0} P_j(Y)X^j. \tag{3}$$

Considering $X=\cos t+i\sin t$, in (3) and integrating with respect to t over the interval $[0,2\pi]$, one obtains the following integral representation of the polynomial

$$P_0(Y)=\frac{1}{2\pi}\int_0^{2\pi}\prod_{s=1}^{n}\left(X^{\alpha_s}+Y^{\alpha_s}+\frac{1}{(XY)^{\alpha_s}}\right)^{m_s}\,\mathrm{d}t. \tag{4}$$

Setting $Y=1$ in (3) one obtains

$$F(X,1)=\prod_{s=1}^{n}\left(X^{\alpha_s}+1+\frac{1}{X^{\alpha_s}}\right)^{m_s}=P_0(1)+\sum_{j\in\mathbb{Z},j\neq 0} P_m(1)X^j, \tag{5}$$

which by symmetry in X and X^{-1} gives that

$$P_m(1)=P_{-m}(1),\quad j\in\mathbb{Z}.$$

Also, from (4) we deduce that

$$P_0(1)=\frac{1}{2\pi}\int_0^{2\pi}\prod_{s=1}^{n}\left(X^{\alpha_s}+1+\frac{1}{X^{\alpha_s}}\right)^{m_s}\,\mathrm{d}t. \tag{6}$$

Since $X^{\alpha_s}+1+\frac{1}{X^{\alpha_s}}=1+2\cos\alpha_s t$, we have

$$P_0(1)=\frac{1}{2\pi}\int_0^{2\pi}\prod_{s=1}^{n}(1+2\cos\alpha_s t)^{m_s}\,\mathrm{d}t. \tag{7}$$

Note that

$$P_0(1)=S(m_1,...,m_n;\alpha_1,...,\alpha_n)+R(m_1,...,m_n;\alpha_1,...,\alpha_n), \tag{8}$$

where $R(m_1, ..., m_n; \alpha_1, ..., \alpha_n)$ is the sum of the coefficients different from the free term of $P_0(Y)$. This is also equivalent to finding the number of solutions of the 3-signum equation for a multiset

$$\sum_{s=1}^{n} \left(\sum_{j=1}^{m_k} \varepsilon_{s,j} \alpha_s \right) = 0, \tag{9}$$

where $\varepsilon_{s,j} \in \{-1, 0, 1\}$, and corresponds to $P_0(1)$. Furthermore, setting $X = 1$ in (5) we obtain $F(1,1) = 3^{m_1+\cdots+m_n} = \sum_{m\in\mathbb{Z}} P_m(1)$, that is the sum of all the coefficients in all polynomials is $3^{m_1+\cdots+m_n}$.

3 3-partitions with equal sums of the set $\{1, \ldots, n\}$

When $\alpha_s = s$ and $m_s = 1$ for $s = 1, \ldots, n$ one obtains

$$F_n(X,Y) = \prod_{s=1}^{n} \left(X^s + Y^s + \frac{1}{(XY)^s} \right) = \sum_{m\in\mathbb{Z}} P_{n,m}(Y) X^m. \tag{10}$$

The computation of polynomials $P_{n,m}(Y)$ can be done recursively.

Theorem 3.1 *The following recurrence is valid for $m \in \mathbb{Z}$ and $n \geq 1$.*

$$P_{n,m}(Y) = P_{n-1,m-n}(Y) + Y^n P_{n-1,m}(Y) + Y^{-n} P_{n-1,m+n}(Y). \tag{11}$$

Also, for $m = 0$ we have

$$P_{n,0}(Y) = P_{n-1,-n}(Y) + Y^n P_{n-1,0}(Y) + Y^{-n} P_{n-1,n}(Y). \tag{12}$$

Proof. The following formula can be established.

$$\begin{aligned} F_n(X,Y) &= F_{n-1}(X,Y) \left(X^n + Y^n + \frac{1}{(XY)^n} \right) \\ &= \left(\sum_{m\in\mathbb{Z}} P_{n-1,m}(Y) X^m \right) \left(X^n + Y^n + \frac{1}{(XY)^n} \right) \\ &= \sum_{m\in\mathbb{Z}} \left(P_{n-1,m-n}(Y) + Y^n P_{n-1,m}(Y) + Y^{-n} P_{n-1,m+n}(Y) \right) X^m. \end{aligned}$$

□

From simple computations we obtain the numbers in Table 1.

$P_{2,0}(Y)$	Y^3
$P_{3,0}(Y)$	$\frac{2}{Y^3} + Y^6$
$P_{4,0}(Y)$	$\frac{2}{Y^5} + \frac{2}{Y^2} + 2\,Y + Y^{10}$
$P_{5,0}(Y)$	$\frac{2}{Y^6} + \frac{2}{Y^3} + 6 + 2\,Y^3 + 2\,Y^6 + Y^{15}$
$P_{6,0}(Y)$	$\frac{2}{Y^9} + \frac{4}{Y^6} + \frac{4}{Y^3} + 6 + 8\,Y^3 + 6\,Y^6 + 2\,Y^9 + 2\,Y^{12} + Y^{21}$
$P_{7,0}(Y)$	$\frac{8}{Y^{14}} + \frac{4}{Y^{11}} + \frac{6}{Y^8} + \frac{10}{Y^5} + \frac{8}{Y^2} + 10\,Y + 8\,Y^4 + 14\,Y^7 + 8\,Y^{10} +$ $6\,Y^{13} + 2\,Y^{16} + 2\,Y^{19} + Y^{28}$
$P_{8,0}(Y)$	$\frac{4}{Y^{18}} + \frac{6}{Y^{15}} + \frac{10}{Y^{12}} + \frac{18}{Y^9} + \frac{22}{Y^6} + \frac{22}{Y^3} + 18 + 22\,Y^3 + 16\,Y^6 + 18\,Y^9 +$ $18\,Y^{12} + 14\,Y^{15} + 8\,Y^{18} + 6\,Y^{21} + 2\,Y^{24} + 2\,Y^{27} + Y^{36}$

Table 1
Polynomials $P_{n,0}(Y)$ and their coefficients for $n = 2, 3, 4, 5, 6, 7, 8$.

Setting $Y = 1$ in (10) one obtains

$$F_n(X, 1) = \prod_{s=1}^{n} \left(X^s + 1 + \frac{1}{X^s} \right) = \sum_{m \in \mathbb{Z}} P_{n,m}(1) X^m. \tag{13}$$

By the symmetry in X and X^{-1}, we obtain $P_{n,-m}(1) = P_{n,m}(1)$ for $m \in \mathbb{Z}$. Also, by (12) we obtain the recurrence generating the sequence $\{P_{n,0}(1)\}_{n \geq 1}$:

$$P_{n,0}(1) = P_{n-1,-n}(1) + P_{n-1,0}(1) + P_{n-1,n}(1) = P_{n-1,0}(1) + 2P_{n-1,n}(1). \tag{14}$$

Sequence $P_{n,0}(1)$ has provided new context for the OEIS sequence A007576:

$$1, 1, 3, 7, 15, 35, 87, 217, 547, 1417, 3735, 9911, 26513, 71581, 194681, 532481, \ldots$$

By applying (7) to this case, one obtains the integral formula

$$P_{n,0}(1) = S_3(n) + R_3(n) = \frac{1}{2\pi} \int_0^{2\pi} \prod_{s=1}^{n} (1 + 2\cos st)\, \mathrm{d}t, \tag{15}$$

where $S_3(n) = S(1, \ldots, 1; 1, \ldots, n)$ and $R_3(n) = R(1, \ldots, 1; 1, \ldots, n)$.

The free term $S_3(n)$ of (13) has been added by us to OEIS as A317577:

$$0, 0, 0, 0, 6, 6, 0, 18, 54, 0, 258, 612, 0, 3570, 8880, 0, 55764, 142368, 0, 947946,$$

For $n = 3k + 1$, the number $\frac{n(n+1)}{2}$ is not divisible by 3, hence $S_3(n) = 0$. The following identity holds $S_3(n) = 6 \cdot a(n)$, where $a(n)$ is sequence A112972. This is also the third row of the triangle $T(n,k)$ indexed as A275714 in OEIS.

The sequence $R_3(n)$ is new, and has the numerical values

$$1, 1, 3, 7, 9, 29, 87, 199, 493, 1417, 3477, 9299, 26513, 68011, 185801, 532481, \ldots$$

Recall that $P_{n,0}(1)$ (15) represents the free term in the expansions (10) and (13), hence corresponds to the number of solutions of the 3-signum equation

$$\varepsilon_1 \cdot 1 + \varepsilon_2 \cdot 2 + \cdots + \varepsilon_n \cdot n = 0, \qquad (16)$$

where $\varepsilon_s \in \{-1, 0, 1\}$, $s = 1, \ldots, n$.

As the monomials in the $F_n(X,Y)$ expansion have the form $X^\alpha Y^\beta (XY)^{-\gamma}$, a term is independent of X if and only if $\alpha = \gamma$. For a given n, we have to enumerate all the partitions A, B, C of $[n]$ having the property $\sigma(A) = \sigma(C)$. The problem is equivalent to finding all triplets (α, β, γ) such that $\alpha, \beta, \gamma \geq 0$, $\alpha = \gamma$ and $\alpha + \beta + \gamma = \sigma([n])$.

For example, when $[n] = \{1, 2, 3, 4\}$ we have $\sigma([n]) = 10$. Table 2 presents all such partitions and the possible configurations ε_s, $s = 1, \ldots, 4$ with (16). This also clearly illustrates that we have $P_{4,0}(1) = 7$.

α	β	γ	A	B	C	ε_1	ε_2	ε_3	ε_4	Multiplicity
0	10	0	$\emptyset$	$[n]$	$\emptyset$	0	0	0	0	1
3	4	3	$\{1,2\}$	$\{4\}$	$\{3\}$	1	1	-1	0	1
			$\{3\}$	$\{4\}$	$\{1,2\}$	-1	-1	1	0	1
4	2	4	$\{1,3\}$	$\{2\}$	$\{4\}$	1	0	1	-1	1
			$\{4\}$	$\{2\}$	$\{1,3\}$	-1	0	-1	1	1
5	5	5	$\{1,4\}$	$\emptyset$	$\{2,3\}$	1	-1	-1	1	1
			$\{2,3\}$	$\emptyset$	$\{1,4\}$	-1	1	1	-1	1

Table 2
Partitions of $[n]$ into 3 subsets when $n = 4$ and $k = 3$.

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References

[1] Andrica, D., *A combinatorial result concerning the product of two or more derivatives*, Bull. Cal. Math. Soc., **92**(4), 299-304(2000).

[2] Andrica, D., Ionascu, E.J., *Some unexpected connections between Analysis and Combinatorics*, in "Mathematics without boundaries. Topics in pure Mathematics", Th.M. Rassias and P. Pardalos, Eds., Springer, 2014, pp.1-20.

[3] Andrica, D., Ionascu, E.J., *The signum equation for Erdos-Suranyi sequences*, INTEGERS 15A, 1–9(2015).

[4] Andrica, D., Tomescu, I., *On an integer sequence related to a product of trigonometric fuctions, and its combinatorial relevance*, J.Integer Sequence, **5**(2002), Article 02.2.4.

[5] Bagdasar, O., Andrica, D., *New results and conjectures on 2-partitions of multisets*, 2017 7th International Conference on Modeling, Simulation, and Applied Optimization (ICMSAO), Sharjah, 2017, pp. 1-5. doi: 10.1109/ICMSAO.2017.7934928

[6] Dell'Amico, M., Martello, S., *Reduction of the Three-Partition Problem*, Journal of Combinatorial Optimization, Vol.**3**, No.1, 17-30(1999).

[7] Garey, M.R., Johnson D. S., *Complexity results for multiprocessor scheduling under resource constraints*, SIAM Journal on Computing, 4(1975), 397–411.

[8] Garey, M.R., Johnson D. S., *Computers and Intractability; A Guide to the Theory of NP-Completeness*, Freeman: San Francisco (1979).

[9] The On-Line Encyclopedia of Integer Sequences, http://oeis.org, OEIS Foundation Inc. 2011.

[10] Stanley, R.P.,*Weyl groups, the hard Lefschetz theorem and the Sperner property*, SIAM J. Algebraic and Discrete Methods 1 (1980), 168-184.

[11] Sullivan, B.D., *On a Conjecture of Andrica and Tomescu*, J.Integer Sequence 16(2013), Article 13.3.1.

[12] H. Wilf, *Generatingfunctionology*, Academic Press, New York, 1994.