A SCALED POWER PRODUCT RECURRENCE EXAMINED USING MATRIX METHODS

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Abstract. A scaled power product recurrence is examined here via a matrix approach which both recovers and extends some recent results found using standard difference equations theory alone. Closed forms for the associated sequence terms are derived for a range of recursion parameter conditions in which so called Catalan polynomials are integral to the process.

1. Introduction

1.1. **Background.** Let $c \in \mathbb{Z}^+$ be an arbitrary scaling variable. Consider, given $z_0 = a$, $z_1 = b$, the scaled power product recurrence

$$z_n = c(z_{n-1})^p (z_{n-2})^q, \qquad n \geq 2, \tag{1.1}$$

which defines a sequence $\{z_n\}_{n=0}^{\infty} = \{z_n\}_0^{\infty} = \{z_n(a,b,p,q;c)\}_0^{\infty}$ with first few terms

$$\begin{aligned}\{z_n(a,b,p,q;c)\}_0^{\infty} \\ = \ \ & \{a, b, a^q b^p c, a^{pq} b^{p^2+q} c^{p+1}, a^{p^2q+q^2} b^{p^3+2pq} c^{p^2+p+q+1}, \\ & a^{p^3q+2pq^2} b^{p^4+3p^2q+q^2} c^{p^3+p^2+p(2q+1)+q+1}, \ldots\}.\end{aligned} \tag{1.2}$$

The non-scaled $c = 1$ version $\{z_n(a,b,p,q;1)\}_0^{\infty}$ has its origins in an observation by M. W. Bunder from the mid-1970s. The reader is referred to background information on this, and subsequent related work, in a forerunner article by the present authors [4].

This paper enhances [4] in which closed forms for $z_n(a,b,p,q;c)$ were found, using standard difference equations theory, according to root types of the characteristic equation $0 = \lambda^2 - p\lambda - q$ associated with the variable $t_n = \ln(z_n)$ that was introduced to underpin a formulation process. We summarize these as follows:

Case I. For non-degenerate characteristic roots ($p^2 + 4q \neq 0$) it was shown that, subject to the constraints $p + q = 1$, $q \neq -1$ (or $p \neq 2$), then writing $p(q) = 1 - q$,

$$z_n(a,b,p(q),q;c) = a^{N_a(q,n)} b^{N_b(q,n)} c^{N_c(q,n)}, \qquad n \geq 0, \tag{1.3}$$

where

$$N_a(q,n) = \frac{q+(-q)^n}{1+q},$$
$$N_b(q,n) = \frac{1-(-q)^n}{1+q},$$
$$N_c(q,n) = \frac{1}{1+q}\left(n - \frac{[1-(-q)^n]}{1+q}\right). \tag{1.4}$$

Case II. The degenerate characteristic roots case ($p^2+4q=0$) was shown to correspond to particular values $q=-1$ and $p=2$ (with the Case I condition $p+q=1$ still holding), whereupon the sequence delivered by (1.1) has

$$z_n(a,b,2,-1;c) = a^{D_a(n)} b^{D_b(n)} c^{D_c(n)}, \qquad n \geq 0, \tag{1.5}$$

as its general term, for which

$$D_a(n) = 1-n,$$
$$D_b(n) = n,$$
$$D_c(n) = n(n-1)/2. \tag{1.6}$$

1.2. **This Paper.** It was shown in [4] how—subject to the balanced-power condition $p+q=1$—recurrence parameters dictate the nature of each solution type, as indicated, and a fundamental link between them was highlighted (that is to say, the exponent functions of (1.6) are the limiting case $q \to -1$ of those in (1.4)). In this paper we adopt an entirely different approach to the analysis of the sequence $\{z_n(a,b,p,q;c)\}_0^\infty$ using matrices. In doing so Cases I and II are recovered naturally by the methodology, and a new case is also examinable which is subject only to the constraint $p+q \neq 1$; the form of $z_n(a,b,p,q;c)$ in this instance emerges in terms of so called Catalan polynomials, and its derivation provides a framework for the rest of the paper in which the results of Cases I and II follow in consequence. We will see, therefore, that the work presented in [4] is both validated and extended here.

2. Results and Analysis

2.1. **Basic Formulation.** We write

$$z_n(a,b,p,q;c) = a^{\alpha_n(p,q)} b^{\beta_n(p,q)} c^{\gamma_n(p,q)}, \qquad n \geq 0, \tag{2.1}$$

as the solution to (1.1) at the outset, and proceed accordingly. Our aim is to provide as complete a description of the power functions $\alpha_n(p,q), \beta_n(p,q)$

and $\gamma_n(p,q)$ as possible. Combining (1.1) and (2.1) then

$$\begin{aligned} z_n &= c(z_{n-1})^p(z_{n-2})^q \\ &= c[a^{\alpha_{n-1}(p,q)}b^{\beta_{n-1}(p,q)}c^{\gamma_{n-1}(p,q)}]^p[a^{\alpha_{n-2}(p,q)}b^{\beta_{n-2}(p,q)}c^{\gamma_{n-2}(p,q)}]^q, \end{aligned} \tag{2.2}$$

in turn giving individual recurrences

$$\begin{aligned} \alpha_n(p,q) &= p\alpha_{n-1}(p,q)+q\alpha_{n-2}(p,q), \\ \beta_n(p,q) &= p\beta_{n-1}(p,q)+q\beta_{n-2}(p,q), \\ \gamma_n(p,q) &= p\gamma_{n-1}(p,q)+q\gamma_{n-2}(p,q)+1, \end{aligned} \tag{2.3}$$

which we express as

$$\mathbf{F}_n(p,q)=\mathbf{H}(p,q)\mathbf{F}_{n-1}(p,q)+\mathbf{K}, \tag{2.4}$$

where

$$\begin{aligned} \mathbf{F}_n(p,q) &= \begin{pmatrix} \alpha_n(p,q) & \beta_n(p,q) & \gamma_n(p,q) \\ \alpha_{n-1}(p,q) & \beta_{n-1}(p,q) & \gamma_{n-1}(p,q) \end{pmatrix}, \\ \mathbf{K} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{2.5}$$

are 2×3 matrices, and $\mathbf{H}(p,q)$ is the 2-square matrix

$$\mathbf{H}(p,q)=\begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}. \tag{2.6}$$

Equation (2.4) yields $\mathbf{F}_n(p,q) = \mathbf{H}(p,q)[\mathbf{H}(p,q)\mathbf{F}_{n-2}(p,q)+\mathbf{K}]+\mathbf{K} = \mathbf{H}^2(p,q)\mathbf{F}_{n-2}(p,q)+\mathbf{H}(p,q)\mathbf{K}+\mathbf{K}$ and, after another application, $\mathbf{F}_n(p,q) = \mathbf{H}^2(p,q)[\mathbf{H}(p,q)\mathbf{F}_{n-3}(p,q)+\mathbf{K}]+\mathbf{H}(p,q)\mathbf{K}+\mathbf{K} = \mathbf{H}^3(p,q)\mathbf{F}_{n-3}(p,q)+\mathbf{H}^2(p,q)\mathbf{K}+\mathbf{H}(p,q)\mathbf{K}+\mathbf{K}$. The procedure is exhausted after application $n-2$, at which point

$$\mathbf{F}_n(p,q)=\mathbf{H}^{n-1}(p,q)\mathbf{S}+\mathbf{T}_n(p,q)\mathbf{K}, \tag{2.7}$$

with

$$\mathbf{S}=\mathbf{F}_1(p,q)=\begin{pmatrix} \alpha_1(p,q) & \beta_1(p,q) & \gamma_1(p,q) \\ \alpha_0(p,q) & \beta_0(p,q) & \gamma_0(p,q) \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.8}$$

and (denoting the 2×2 identity matrix as $\mathbf{I}_2$)

$$\mathbf{T}_n(p,q)=\mathbf{H}^{n-2}(p,q)+\mathbf{H}^{n-3}(p,q)+\cdots+\mathbf{H}^2(p,q)+\mathbf{H}(p,q)+\mathbf{I}_2. \tag{2.9}$$

With the fundamentals of our strategy finished, we partition results according to cases alluded to in Section 1.2. First, however, we introduce the Catalan polynomials on which our closed forms for the exponent functions $\alpha_n(p,q), \beta_n(p,q)$ and $\gamma_n(p,q)$ of a, b and c are based. They are a particular family of polynomials—the initial ones being $P_0(x)=P_1(x)=1$, $P_2(x)=1-x$, $P_3(x)=1-2x$, $P_4(x)=1-3x+x^2$, $P_5(x)=1-4x+3x^2$, $P_6(x)=1-5x+6x^2-x^3$, $P_7(x)=1-6x+10x^2-4x^3$, and so on—with $P_n(x)=\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}(-x)^i$ for $n\geq 0$.

2.2. **The Case** $p+q\neq 1$. From (2.9) we obtain almost immediately

$$\mathbf{T}_n(p,q) = [\mathbf{H}(p,q)-\mathbf{I}_2]^{-1}[\mathbf{H}^{n-1}(p,q)-\mathbf{I}_2]. \tag{2.10}$$

Now

$$[\mathbf{H}(p,q)-\mathbf{I}_2]^{-1} = \frac{1}{p+q-1}\begin{pmatrix} 1 & q \\ 1 & 1-p \end{pmatrix}, \tag{2.11}$$

which requires $\mathbf{H}(p,q)-\mathbf{I}_2$ be non-singular, or $p+q\neq 1$. With this in mind we formulate the matrix $\mathbf{H}^{n-1}(p,q)$ as

$$\begin{aligned}\mathbf{H}^{n-1}(p,q) &= \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{n-1} \\ &= p^{n-1}\begin{pmatrix} P_{n-1}(-q/p^2) & (q/p)P_{n-2}(-q/p^2) \\ (1/p)P_{n-2}(-q/p^2) & (q/p^2)P_{n-3}(-q/p^2) \end{pmatrix}\end{aligned} \tag{2.12}$$

by appeal to the fact that any matrix of the form $\binom{1\ \ x}{y\ \ 0}$ has an nth power which is expressible in terms of Catalan polynomials as

$$\begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix}^n = \begin{pmatrix} P_n(-xy) & xP_{n-1}(-xy) \\ yP_{n-1}(-xy) & xyP_{n-2}(-xy) \end{pmatrix}, \qquad n\geq 2. \tag{2.13}$$

Thus,

$$\begin{aligned}&\mathbf{H}^{n-1}(p,q)-\mathbf{I}_2 \\ &= \begin{pmatrix} p^{n-1}P_{n-1}(-q/p^2)-1 & p^{n-2}qP_{n-2}(-q/p^2) \\ p^{n-2}P_{n-2}(-q/p^2) & p^{n-3}qP_{n-3}(-q/p^2)-1 \end{pmatrix}\end{aligned} \tag{2.14}$$

which, together with (2.11), gives $\mathbf{T}_n(p,q)$ (2.10) as

$$\mathbf{T}_n(p,q) = \frac{1}{p+q-1}\begin{pmatrix} T_1(p,q,n) & T_2(p,q,n) \\ T_3(p,q,n) & T_4(p,q,n) \end{pmatrix}, \tag{2.15}$$

where

$$\begin{aligned} T_1(p,q,n) &= p^{n-1}P_{n-1}(-q/p^2)+p^{n-2}qP_{n-2}(-q/p^2)-1, \\ T_2(p,q,n) &= p^{n-2}qP_{n-2}(-q/p^2)+q[p^{n-3}qP_{n-3}(-q/p^2)-1], \\ T_3(p,q,n) &= p^{n-1}P_{n-1}(-q/p^2)+(1-p)p^{n-2}P_{n-2}(-q/p^2)-1, \\ T_4(p,q,n) &= p^{n-2}qP_{n-2}(-q/p^2) \\ &\qquad + (1-p)[p^{n-3}qP_{n-3}(-q/p^2)-1]. \end{aligned} \tag{2.16}$$

Remark 2.1. We note that (2.13) has been visible in some recent work by the authors—see [1, Eq. (2.5), p. 351] where the result has been deployed in a sufficiency argument for cross-family member equality within a certain class of polynomial families, and [3, Eq. (I.1), p. 176] where it drives the proof of an invariance property for the particular matrix characterizing such families; it has proven to be a most useful result to have formulated.

Moving on,

$$\mathbf{T}_n(p,q)\mathbf{K} = \frac{1}{p+q-1}\begin{pmatrix} 0 & 0 & T_1(p,q,n) \\ 0 & 0 & T_3(p,q,n) \end{pmatrix} \tag{2.17}$$

by (2.5) and (2.15), with

$$\mathbf{H}^{n-1}(p,q)\mathbf{S} = \begin{pmatrix} p^{n-2}qP_{n-2}(-q/p^2) & p^{n-1}P_{n-1}(-q/p^2) & 0 \\ p^{n-3}qP_{n-3}(-q/p^2) & p^{n-2}P_{n-2}(-q/p^2) & 0 \end{pmatrix} \tag{2.18}$$

by (2.12) and (2.8), so that (2.7) reads

$$\mathbf{F}_n(p,q) =$$

$$\begin{pmatrix} p^{n-2}qP_{n-2}(-q/p^2) & p^{n-1}P_{n-1}(-q/p^2) & T_1(p,q,n)/(p+q-1) \\ p^{n-3}qP_{n-3}(-q/p^2) & p^{n-2}P_{n-2}(-q/p^2) & T_3(p,q,n)/(p+q-1) \end{pmatrix} \tag{2.19}$$

from which, by term comparison with $\mathbf{F}_n(p,q)$ in (2.5), we can write down the desired exponent functions $\alpha_n(p,q)$ and $\beta_n(p,q)$ of (2.1) (beyond those of the initial value terms z_0, z_1) immediately as

$$\begin{aligned} \alpha_n(p,q) &= p^{n-2}qP_{n-2}(-q/p^2), \\ \beta_n(p,q) &= p^{n-1}P_{n-1}(-q/p^2), \qquad n \geq 2, \end{aligned} \tag{2.20}$$

while

$$\begin{aligned} \gamma_n(p,q) &= T_1(p,q,n)/(p+q-1) \\ &= [p^{n-1}P_{n-1}(-q/p^2) + p^{n-2}qP_{n-2}(-q/p^2) - 1]/(p+q-1) \\ &= [\alpha_n(p,q) + \beta_n(p,q) - 1]/(p+q-1), \end{aligned} \tag{2.21}$$

by (2.16) and (2.20). With reference to (2.1) it is interesting to see that $\alpha_n(p,q)$ and $\beta_n(p,q)$ have seemingly independent closed forms, but $\gamma_n(p,q)$ (that functional exponent of the recurrence scalar multiplier c) exhibits dependency on both. This has not been observed before.

Example 2.1. By way of an example we verify these representations for value $n = 5$. Noting that $P_3(x) = 1-2x$ and $P_4(x) = 1-3x+x^2$ it is readily seen that, from (2.20), $\alpha_5(p,q) = p^3qP_3(-q/p^2) = p^3q\cdot(p^2+2q)/p^2 = p^3q+2pq^2$ and $\beta_5(p,q) = p^4P_4(-q/p^2) = p^4\cdot(p^4+3p^2q+q^2)/p^4 = p^4+3p^2q+q^2$, which agree with those powers of a, b in the term $z_5(a,b,p,q;c)$ of (1.2). Furthermore, from (1.2) we find that $(p+q-1)\gamma_5(p,q) = (p+q-1)[p^3+p^2+p(2q+1)+q+1] = \cdots = \tau_1(p,q)+\tau_2(p,q)-1$ after a little algebra, where $\tau_1(p,q) = p^3q+2pq^2 = \alpha_5(p,q)$ and $\tau_2(p,q) = p^4+3p^2q+q^2 = \beta_5(p,q)$; this confirms that that $(p+q-1)\gamma_5(p,q) = \alpha_5(p,q)+\beta_5(p,q)-1$ as required by (2.21).

Some Further Remarks. Returning briefly to (2.19) then, as a matter of completeness, we need to check that $T_3(p,q,n) = T_1(p,q,n-1)$, which reduces to showing $0 = p^{n-3}qP_{n-3}(-q/p^2) + p^{n-1}P_{n-2}(-q/p^2) -$

$p^{n-1}P_{n-1}(-q/p^2)$ using (2.16). This is readily forthcoming from the known linear recurrence equation

$$0 = xP_n(x) - P_{n+1}(x) + P_{n+2}(x) \tag{2.22}$$

for the Catalan polynomials [2, Eq. (A1), p. 116] when evaluated at $x = -q/p^2$.

We remark also that recursions for Catalan polynomials give rise to analogues for the exponent functions $\alpha_n(p,q), \beta_n(p,q)$ and $\gamma_n(p,q)$ (or combinations thereof) in the light of (2.20) and (2.21). A simple illustration is provided by the non-linear identity [2, Eq. (A4), p. 116]

$$x^n = P_n^2(x) - P_{n+1}(x)P_{n-1}(x) \tag{2.23}$$

which, on using the relations of (2.20) separately, yields recurrences

$$\alpha_n^2(p,q) - \alpha_{n+1}(p,q)\alpha_{n-1}(p,q) = (-q)^n \tag{2.24}$$

and

$$\beta_n^2(p,q) - \beta_{n+1}(p,q)\beta_{n-1}(p,q) = (-q)^{n-1}, \tag{2.25}$$

both having been validated extensively by computer for $n \geq 1$. As an example of these, for $n = 4$ (i) $\alpha_4^2(p,q) - \alpha_5(p,q)\alpha_3(p,q) = (p^2q + q^2)^2 - (p^3q + 2pq^2)(pq) = \cdots = q^4 = (-q)^4$, and (ii) $\beta_4^2(p,q) - \beta_5(p,q)\beta_3(p,q) = (p^3 + 2pq)^2 - (p^4 + 3p^2q + q^2)(p^2 + q) = \cdots = -q^3 = (-q)^3$.

We now proceed to recover, from our methodology, those (balanced-power) Case I and Case II formulations described in Section 1.1.

2.3. **The Case $p + q = 1$, $p \neq 2$ (or $q \neq -1$): Case I.** Consider the case for which $p + q = 1$. Writing $q(p) = 1 - p$ the matrix $\mathbf{H}(p) = \mathbf{H}(p, q(p))$ (2.6) is

$$\mathbf{H}(p) = \begin{pmatrix} p & 1-p \\ 1 & 0 \end{pmatrix}, \tag{2.26}$$

which may be decomposed conveniently as

$$\mathbf{H}(p) = \mathbf{U}(p)\mathbf{D}(p)\mathbf{U}^{-1}(p), \tag{2.27}$$

where

$$\mathbf{U}(p) = \begin{pmatrix} p-1 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.28}$$

and $\mathbf{D}(p)$ is the diagonal matrix

$$\mathbf{D}(p) = \begin{pmatrix} p-1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.29}$$

Noting, therefore, that $p \neq 2$ (otherwise $\mathbf{U}(p)$ is singular) we are dealing with Case I introduced at the start of the paper, the results from which we will reproduce here.

First, from (2.9) we have $\mathbf{T}_n(p) = \mathbf{T}_n(p, q(p)) = \sum_{i=0}^{n-2} \mathbf{H}^i(p) = \sum_{i=0}^{n-2} [\mathbf{U}(p)\, \mathbf{D}(p)\mathbf{U}^{-1}(p)]^i$ (by (2.27)) $= \mathbf{U}(p)\mathbf{X}(p)\mathbf{U}^{-1}(p)$, where

$$\begin{aligned} \mathbf{X}(p) &= \sum_{i=0}^{n-2} \mathbf{D}^i(p) \\ &= \begin{pmatrix} \sum_{i=0}^{n-2}(p-1)^i & 0 \\ 0 & \sum_{i=0}^{n-2} 1 \end{pmatrix} \\ &= \begin{pmatrix} f_n(p) & 0 \\ 0 & n-1 \end{pmatrix}, \end{aligned} \tag{2.30}$$

denoting by $f_n(p)$ the geometric series

$$f_n(p) = \sum_{i=0}^{n-2} (p-1)^i = [1-(p-1)^{n-1}]/(2-p). \tag{2.31}$$

Thus,

$$\begin{aligned} \mathbf{T}_n(p) &= \mathbf{U}(p)\mathbf{X}(p)\mathbf{U}^{-1}(p) \\ &= \begin{pmatrix} p-1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_n(p) & 0 \\ 0 & n-1 \end{pmatrix} \cdot \frac{1}{p-2} \begin{pmatrix} 1 & -1 \\ -1 & p-1 \end{pmatrix} \\ &= \frac{1}{p-2} \begin{pmatrix} T_1^*(p,n) & T_2^*(p,n) \\ T_3^*(p,n) & T_4^*(p,n) \end{pmatrix}, \end{aligned} \tag{2.32}$$

where

$$\begin{aligned} T_1^*(p,n) &= (p-1)f_n(p) - (n-1), \\ T_2^*(p,n) &= (p-1)[n-1-f_n(p)], \\ T_3^*(p,n) &= f_n(p) - (n-1), \\ T_4^*(p,n) &= (n-1)(p-1) - f_n(p). \end{aligned} \tag{2.33}$$

Because the structure of the product matrix $\mathbf{H}^{n-1}(p)\mathbf{S} = \mathbf{H}^{n-1}(p, q(p))\mathbf{S}$ (2.18) remains unchanged (along with its entries save for the dependency $q(p) = 1-p$ here), then clearly, in this instance,

$$\begin{aligned} \alpha_n(p) &= (1-p)p^{n-2}P_{n-2}((p-1)/p^2), \\ \beta_n(p) &= p^{n-1}P_{n-1}((p-1)/p^2), \qquad n \geq 2, \end{aligned} \tag{2.34}$$

from the resultant matrix equation $\mathbf{F}_n(p) = \mathbf{H}^{n-1}(p)\mathbf{S} + \mathbf{T}_n(p)\mathbf{K}$ (equation (2.7), with $q = q(p)$), which also gives (using (2.33) and (2.31))

$$\begin{aligned}\gamma_n(p) &= T_1^*(p,n)/(p-2)\\ &= [(p-1)f_n(p) - (n-1)]/(p-2)\\ &= \frac{1}{p-2}\left((p-1)\frac{[1-(p-1)^{n-1}]}{2-p} - (n-1)\right)\\ &= \frac{1}{2-p}\left(n - \frac{[1-(p-1)^n]}{2-p}\right) \qquad (2.35)\end{aligned}$$

after simplification. It is worth observing that equations (2.34) are, of course, available directly from (2.20) but, on the other hand, that (2.21) does not deliver (2.35) (since $p+q=1$), which latter demands the separate formulation given.

Remark 2.2. In the same spirit as Section 2.2, we confirm that $T_3^*(p,n) = T_1^*(p,n-1)$ for completeness here, so as to secure the general procedure we have developed; from (2.33) this is merely equivalent to showing that $f_n(p) - (p-1)f_{n-1}(p) = 1$, being immediate by (2.31).

We end this subsection by remarking that we are in a position to recover those precise results from Case I in the form expressed. Note that, in terms of q, then from (2.35)

$$\gamma_n(q) = \gamma_n(p(q)) = \frac{1}{1+q}\left(n - \frac{[1-(-q)^n]}{1+q}\right) = N_c(q,n) \qquad (2.36)$$

of (1.4). Writing similarly (from (2.34))

$$\begin{aligned}\alpha_n(q) &= \alpha_n(p(q)) = q(1-q)^{n-2}P_{n-2}(-q/(1-q)^2),\\ \beta_n(q) &= \beta_n(p(q)) = (1-q)^{n-1}P_{n-1}(-q/(1-q)^2), \qquad n \geq 2, \qquad (2.37)\end{aligned}$$

we can reproduce $N_a(q,n)$ and $N_b(q,n)$ but this requires some work and for the sake of conciseness is devolved to the Appendix for any interested reader.

2.4. **The Case $p+q=1$, $p=2$ and $q=-1$: Case II.** In this instance we impose values for p and q, and formulate independently $\alpha_n(2,-1)$, $\beta_n(2,-1)$ and $\gamma_n(2,-1)$ in line with those results for Case II given earlier.

Here $\mathbf{H}(p,q)$ (2.6) gives simply

$$\mathbf{H}(2,-1) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \qquad (2.38)$$

for which is it easy to show (by induction, for example) that

$$\mathbf{H}^n(2,-1) = \begin{pmatrix} n+1 & -n \\ n & -(n-1) \end{pmatrix}. \qquad (2.39)$$

Thus, (2.9) delivers

$$\mathbf{T}_n(2,-1) = \sum_{i=0}^{n-2} \mathbf{H}^i(2,-1)$$
$$= \begin{pmatrix} \sum_{i=0}^{n-2}(i+1) & -\sum_{i=0}^{n-2} i \\ \sum_{i=0}^{n-2} i & \sum_{i=0}^{n-2}(1-i) \end{pmatrix}$$
$$= \frac{1}{2}(n-1)\begin{pmatrix} n & -(n-2) \\ n-2 & -(n-4) \end{pmatrix}. \tag{2.40}$$

This time (2.7) reads $\mathbf{F}_n(2,-1) = \mathbf{H}^{n-1}(2,-1)\mathbf{S}+\mathbf{T}_n(2,-1)\mathbf{K}$ which yields (omitting the details) $\gamma_n(2,-1) = (n-1)n/2 = D_c(n)$ of (1.6), together with[1]

$$\alpha_n(2,-1) = -2^{n-2}P_{n-2}(1/4),$$
$$\beta_n(2,-1) = 2^{n-1}P_{n-1}(1/4), \qquad n \geq 2, \tag{2.41}$$

or, finally,

$$\alpha_n(2,-1) = -(n-1) = D_a(n),$$
$$\beta_n(2,-1) = n = D_b(n), \tag{2.42}$$

as expected, since it is known [2, p. 103] that $P_n(1/4) = (n+1)/2^n$ $(n \geq 0)$.

Remark 2.3. The keen reader may like to check—as a straightforward but pleasing algebraic exercise—that the identities (2.24) and (2.25) hold when, resp., $\alpha_n(p(q),q) = N_a(q,n) = [q+(-q)^n]/(1+q)$ and $\beta_n(p(q),q) = N_b(q,n) = [1-(-q)^n]/(1+q)$ of Case I; the Case II forms $\alpha_n(2,-1) = D_a(n) = 1-n$ and $\beta_n(2,-1) = D_b(n) = n$ are trivial ones to verify.

3. Summary

The original power product recurrence of M. W. Bunder has been analyzed here in a more generalized form (that is, with scalar multiplier), and some recent results both recovered and extended. Closed forms for the associated sequence terms have been derived for a range of recursion parameter conditions, in which the role of Catalan polynomials is a central one. It is worth emphasizing that while results in [4] have been reformulated in Sections 2.3 and 2.4 of this paper, those of Section 2.2 were inaccessible from the line of enquiry taken in that precursory work due to the imposition of the 'balanced-power' constraint $p+q=1$ underpinning it. We also mention an examination of the $p=q=1/2$ (so called geometric mean) version of (1.1) in [5] with $c=1$—this work identifies connections between Jacobsthal and Horadam numbers (and the condition on c is also relaxed in an appendix).

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[1]Also consistent with (2.34) for $p = 2$.

There is nothing in principle to prevent our approach being applied to the deeper third order recurrence $z_n = c(z_{n-1})^p(z_{n-2})^q(z_{n-3})^r$ ($n \geq 3$ given z_0, z_1, z_2). There is, however, a practical problem in that the matrix $\mathbf{H} = \mathbf{H}(p,q,r)$ capturing the system power variables would—as the analogue to $\mathbf{H}(p,q)$ (2.6)—be a 3-square matrix in this case and at present we know of no delineation of such a matrix which, when raised to an arbitrary power, has polynomial entries so as to give a compact realization of those functional exponents of the general term of the sequence $\{z_n(z_0, z_1, z_2, p, q, r; c)\}_0^\infty$. This is left as an open problem.

Appendix

Consider $\alpha_n(q) = q(1-q)^{n-2}P_{n-2}(-q/(1-q)^2)$ of (2.37). To show that this representation tallies with $N_a(q,n)$ of (1.4), we argue inductively.

Proof. It is trivially true for $n = 2, 3$ (where (2.37) and (1.4) give $\alpha_2(q) = N_a(q,2) = q$ and $\alpha_3(q) = N_a(q,3) = q(1-q)$), so we assume the result holds for some $n = k, k-1$ ($k \geq 3$), which is to say $\alpha_k(q) = q(1-q)^{k-2}P_{k-2}(-q/(1-q)^2) = [q+(-q)^k]/(1+q) = N_a(q,k)$ and $\alpha_{k-1}(q) = q(1-q)^{k-3}P_{k-3}(-q/(1-q)^2) = [q+(-q)^{k-1}]/(1+q) = N_a(q,k-1)$. Thus,

$$
\begin{aligned}
\alpha_{k+1}(q) &= q(1-q)^{k-1}P_{k-1}(-q/(1-q)^2) \\
&= q(1-q)^{k-1}\{P_{k-2}(-q/(1-q)^2) \\
&\qquad\qquad + [q/(1-q)^2]P_{k-3}(-q/(1-q)^2)\} \\
&= (1-q)\cdot q(1-q)^{k-2}P_{k-2}(-q/(1-q)^2) \\
&\qquad\qquad + q\cdot q(1-q)^{k-3}P_{k-3}(-q/(1-q)^2) \\
&= (1-q)\cdot\frac{[q+(-q)^k]}{1+q} + q\cdot\frac{[q+(-q)^{k-1}]}{1+q} \qquad \text{(P.1)}
\end{aligned}
$$

employing (2.22) and the inductive hypothesis. Continuing,

$$
\begin{aligned}
&= \{(1-q)[q+(-q)^k] + q[q+(-q)^{k-1}]\}/(1+q) \\
&\;\vdots \\
&= [q+(-q)^{k+1}]/(1+q) \\
&= N_a(q,k+1) \qquad \text{(P.2)}
\end{aligned}
$$

after a little algebra, as required (upholding the inductive step). □

A similar process establishes that $\beta_n(q) = (1-q)^{n-1}P_{n-1}(-q/(1-q)^2)$ of (2.37) corresponds to $N_b(q,n)$ of (1.4) (reader exercise).

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