ON THE JACOBSTHAL, HORADAM AND GEOMETRIC MEAN SEQUENCES

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Abstract. This paper, in considering aspects of the geometric mean sequence, offers new results connecting Jacobsthal and Horadam numbers which are established and then proved independently.

1. Introduction

Consider the (scaled) geometric mean sequence $\{g_n(a,b;c)\}_{n=0}^{\infty} = \{g_n(a, b;c)\}_0^{\infty}$ defined, given $g_0 = a$, $g_1 = b$, through the recurrence

$$g_{n+1} = c\sqrt{g_n g_{n-1}}, \qquad n \geq 1, \tag{1.1}$$

where $c \in \mathbb{Z}^+$ is a scaling constant. Shiu and Yerger [7], on imposing $a = b = 1$, introduced the $c = 2$ instance of (1.1) as the recursion generating the geometric Fibonacci sequence $\{g_n(1,1;2)\}_0^{\infty}$, giving its growth rate along with that of an equivalent harmonic Fibonacci sequence (together with growth rates of integer valued versions). In this paper we assume $c = 1$ and begin by finding the growth rate of the sequence

$$\begin{aligned}\{g_n(a,b;1)\}_0^{\infty} \quad &= \quad \{a, b, (ab)^{\frac{1}{2}}, (ab^3)^{\frac{1}{4}}, (a^3b^5)^{\frac{1}{8}}, \\ &\qquad (a^5b^{11})^{\frac{1}{16}}, (a^{11}b^{21})^{\frac{1}{32}}, \ldots\}\end{aligned} \tag{1.2}$$

using three alternative approaches, two of which are routine with the other based on a connection between this sequence and the Jacobsthal sequence [6, Sequence No. A001045] discernible in the powers of a, b in (1.2). Other results follow accordingly, with Jacobsthal numbers expressed in terms of families of parameterized Horadam numbers in two particular identities that are established in different ways. Structurally, the type of recurrence (1.1) is familiar to us as part of the celebrated arithmetic-geometric mean named after Gauss.

2. Results and Analysis

2.1. **Background.** We first outline the essentials of two sequences (Jacobsthal and Horadam) requisite for our analysis. We then give results for the growth rate of the sequence $\{g_n(a,b;1)\}_0^{\infty}$ before developing some so called Jacobsthal-Horadam identities which are then proven independently.

The Horadam Sequence. The Horadam sequence $\{w_n\}_0^\infty = \{w_n(w_0, w_1; p, q)\}_0^\infty$ is defined, for given w_0, w_1, by the order two linear recurrence

$$w_{n+1} = pw_n - qw_{n-1}, \qquad n \geq 1. \tag{2.1}$$

Equation (2.1) dates back to the seminal work of A. F. Horadam in the 1960s, and is the recognized form for this particular recurrence [3]. For non-degenerate ($p^2 \neq 4q$) characteristic roots $\alpha(p, q) = (p + \sqrt{p^2 - 4q})/2$, $\beta(p, q) = (p - \sqrt{p^2 - 4q})/2$, it is easy to construct, for $n \geq 0$, a closed (Binet) form

$$w_n(a, b; p, q) = w_n(\alpha(p, q), \beta(p, q), a, b) = \frac{(b - a\beta)\alpha^n - (b - a\alpha)\beta^n}{\alpha - \beta} \tag{2.2}$$

for the general $(n + 1)$th Horadam term.

The Jacobsthal Sequence. The Jacobsthal sequence $\{J_n\}_0^\infty = \{J_0, J_1, J_2, J_3, J_4, \ldots\} = \{0, 1, 1, 3, 5, \ldots\}$ can (given $J_0 = 0$, $J_1 = 1$) be generated by the order two recursion

$$J_{n+1} = J_n + 2J_{n-1}, \qquad n \geq 1, \tag{2.3}$$

or (given $J_0 = 0$) by the order one recursion

$$J_{n+1} = 2J_n + (-1)^n, \qquad n \geq 0, \tag{2.4}$$

and has a well known $(n + 1)$th term closed form

$$J_n = [2^n - (-1)^n]/3, \qquad n \geq 0, \tag{2.5}$$

which follows from (2.2) (with $a = J_0 = 0$, $b = J_1 = 1$, and $\alpha(1, -2) = 2$, $\beta(1, -2) = -1$) since, by (2.1) and (2.3),

$$J_n = w_n(0, 1; 1, -2). \tag{2.6}$$

Note that the fully characterized sequence $\{w_n(0, 1; 1, -2)\}_0^\infty$ is the $p = 1$, $q = -2$ case of the so called fundamental sequence $\{w_n(0, 1; p, q)\}_0^\infty$, which is a part specialized Horadam sequence much studied in the relevant literature [3].

2.2. **Growth Rate of $\{g_n(a, b; 1)\}_0^\infty$.** We begin by showing that the growth rate of the sequence $\{g_n(a, b; 1)\}_0^\infty$ is 1. That is to say the following.

Theorem 2.1. *The sequence $\{g_n(a, b; 1)\}_0^\infty$ grows according to*

$$\lim_{n\to\infty} \{g_{n+1}(a, b; 1)/g_n(a, b; 1)\} = 1.$$

As alluded to in the Introduction, we approach the proof of Theorem 2.1 in three ways.

Method I. This is elementary, and parallels that seen in [7].

Proof. Writing $L = \lim_{n\to\infty}\{g_{n+1}/g_n\}$, then $L^2 = \lim^2_{n\to\infty}\{g_{n+1}/g_n\} = \lim_{n\to\infty}\{(g_{n+1}/g_n)^2\} = \lim_{n\to\infty}\{g_n g_{n-1}/g_n^2\}$ (by (1.1)) $= \lim_{n\to\infty}\{g_{n-1}/g_n\} = 1/L$, so that $L^3 = 1$ of which $L = 1$ is a solution. □

Method II. This is rather more interesting, since it relies on a closed form for the sequence term $g_n(a, b; 1)$ in which Jacobsthal numbers make an appearance.

Lemma 2.1. *For $n \geq 1$, $g_n(a, b; 1) = (a^{J_{n-1}} b^{J_n})^{2^{-(n-1)}}$.*

Proof. Denote $\ln(g_n)$ by t_n and, assuming $a, b > 0$, consider (1.1) which can be written

$$t_{n+1} = (t_n + t_{n-1})/2. \tag{L.1}$$

Applying this recurrence once gives $t_{n+1} = \frac{1}{2}[\frac{1}{2}(t_{n-1} + t_{n-2}) + t_{n-1}] = \frac{1}{4}(3t_{n-1} + t_{n-2}) = \frac{1}{2^2}(J_3 t_{n-1} + J_2 t_{n-2})$. Repeating the procedure, we see that $t_{n+1} = \frac{1}{4}[3 \cdot \frac{1}{2}(t_{n-2} + t_{n-3}) + t_{n-2}] = \frac{1}{8}(5t_{n-2} + 3t_{n-3}) = \frac{1}{2^3}(J_4 t_{n-2} + J_3 t_{n-3})$. After iteration $k \geq 1$, therefore,[1]

$$t_{n+1} = \frac{1}{2^{k+1}}[J_{k+2} t_{n-k} + J_{k+1} t_{n-(k+1)}], \tag{L.2}$$

which is exhausted at $k = n - 1$ and yields

$$t_{n+1} = (J_{n+1} t_1 + J_n t_0)/2^n, \tag{L.3}$$

or $g_{n+1}(a, b; 1) = (a^{J_n} b^{J_{n+1}})^{2^{-n}}$ (to which the restriction $a, b > 0$ does not actually apply). □

The proof of Theorem 2.1 is now immediate.

Proof. Consider

$$\begin{aligned}
\lim_{n\to\infty}\left\{\frac{g_{n+1}}{g_n}\right\} &= \lim_{n\to\infty}\left\{\frac{a^{J_n/2^n} b^{J_{n+1}/2^n}}{a^{J_{n-1}/2^{n-1}} b^{J_n/2^{n-1}}}\right\} \\
&= \lim_{n\to\infty}\left\{a^{\frac{J_n}{2^n} - \frac{J_{n-1}}{2^{n-1}}} b^{\frac{J_{n+1}}{2^n} - \frac{J_n}{2^{n-1}}}\right\} \\
&= \lim_{n\to\infty}\left\{a^{(-1)^{n-1}/2^n} b^{(-1)^n/2^n}\right\}, \qquad \text{(T.1)}
\end{aligned}$$

on using (2.4). In other words,

$$\lim_{n\to\infty}\left\{\frac{g_{n+1}}{g_n}\right\} = \lim_{n\to\infty}\left\{(b/a)^{(-1/2)^n}\right\}, \tag{T.2}$$

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[1] To establish formally that (L.2) is correct we need to show that the next iterative step is self-consistent: we write $t_{n+1} = [J_{k+2} t_{n-k} + J_{k+1} t_{n-(k+1)}]/2^{k+1} = [J_{k+2}(t_{n-(k+1)} + t_{n-(k+2)})/2 + J_{k+1} t_{n-(k+1)}]/2^{k+1} = [(J_{k+2}/2 + J_{k+1}) t_{n-(k+1)} + J_{k+2} t_{n-(k+2)}/2]/2^{k+1} = [(J_{k+3}/2) t_{n-(k+1)} + J_{k+2} t_{n-(k+2)}/2]/2^{k+1}$ (using (2.3)) $= [J_{k+3} t_{n-(k+1)} + J_{k+2} t_{n-(k+2)}]/2^{k+2}$, as required.

and since $\lim_{n\to\infty}\{(-1/2)^n\} = 0$ then $\lim_{n\to\infty}\{g_{n+1}/g_n\} = (b/a)^0 = 1$ trivially (interestingly, the sequence $\{(b/a)^{(-1/2)^n}\}_0^\infty = \{b/a, (a/b)^{1/2}, (b/a)^{1/4}, (a/b)^{1/8}, (b/a)^{1/16}, \ldots\}$ has terms in which a, b take the same exponentially decreasing power as they oscillate between numerator and denominator). □

Method III. This is also very straightforward, being a reformulation of the ratio g_{n+1}/g_n in (T.2) from which Theorem 2.1 follows.

Proof. From (1.1) $g_{n+1}/g_n = \sqrt{g_{n-1}/g_n} = (g_n/g_{n-1})^{-1/2} = (r_n)^{-1/2}$ writing the ratio g_n/g_{n-1} as r_n. Thus, $g_{n+1}/g_n = (r_n)^{-1/2} = (r_{n-1})^{(-1/2)^2} = (r_{n-2})^{(-1/2)^3} = \cdots = (r_1)^{(-1/2)^n} = (g_1/g_0)^{(-1/2)^n} = (b/a)^{(-1/2)^n}$. □

Remark 2.1. We observe that if the scaling variable c is retained in (1.1) then the growth rate of the sequence $\{g_n(a,b;c)\}_0^\infty$ is, by Method I or III, easily seen to be $c^{2/3}$.

2.3. **Jacobsthal-Horadam Identities.** We now develop a couple of identities that link terms of the Jacobsthal sequence with those of particular Horadam sequences—these are then generalized, with proofs given.

The power product recurrence

$$z_{n+1} = (z_n)^p (z_{n-1})^q, \qquad n \geq 1, \tag{2.7}$$

with initial values $z_0 = a$, $z_1 = b$, is known to produce a sequence $\{z_n(a,b,p,q)\}_0^\infty$ for which

$$z_n(a,b,p,q) = a^{w_n(1,0;p,-q)} b^{w_n(0,1;p,-q)}, \qquad n \geq 0. \tag{2.8}$$

It was first put forward by Bunder in 1975 [1], having recently been proved inductively and generalized [2], and reproved again from first principles elsewhere [4]. Since our recursion (1.1) (with $c = 1$) is the $p = q = \frac{1}{2}$ case of (2.7) we can infer immediately from (2.8) and Lemma 2.1 that

$$\begin{aligned} g_n(a,b;1) &= (a^{J_{n-1}} b^{J_n})^{2^{-(n-1)}} \\ &= z_n(a,b,1/2,1/2) \\ &= a^{w_n(1,0;1/2,-1/2)} b^{w_n(0,1;1/2,-1/2)}, \end{aligned} \tag{2.9}$$

delivering

$$J_{n-1} = 2^{n-1} w_n(1,0;1/2,-1/2), \qquad n \geq 1, \tag{2.10}$$

and

$$J_n = 2^{n-1} w_n(0,1;1/2,-1/2), \qquad n \geq 0, \tag{2.11}$$

which we believe are new relations in that they express Jacobsthal numbers in terms of Horadam numbers other than $w_n(0,1;1,-2)$ of (2.6).

Remark 2.2. We remark, for completeness, that values $p = q = 1$ in

(2.7) yield, by (2.8), $z_n(a,b,1,1) = a^{w_n(1,0;1,-1)}b^{w_n(0,1;1,-1)} = a^{F_{n-1}}b^{F_n}$ $(n \geq 1)$, where $\{F_0, F_1, F_2, F_3, F_4, \ldots\} = \{F_n\}_0^\infty = \{0, 1, 1, 2, 3, \ldots\}$ is the Fibonacci sequence. This special case recurrence $z_{n+1} = z_n z_{n-1}$ (with arbitrary starting values) was examined in [8], and the sequence therefrom—although quite different from that in [7]—also called a geometric Fibonacci sequence for which a number of results were produced (theorems and a range of identities) for geometric Fibonacci numbers.

Remark 2.3. Noting that terms of the sequence $\{g_n(2,3;1)\}_0^\infty$ seemed to converge, Maynard [5] considered the more general recurrence of Bunder (2.7) and examined convergence properties of the resulting sequence. As a corollary he gave $\lim_{n\to\infty}\{g_n(a,b;1)\} = (ab^2)^{1/3}$ (powers of a and b in (1.2) form resp. sequences $\{1, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \frac{11}{32}, \frac{21}{64}, \ldots\}$ and $\{0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \frac{21}{32}, \frac{43}{64}, \ldots\}$, with rapid convergence to $\frac{1}{3}$ and $\frac{2}{3}$ by inspection), concluding that the motivating limit was precisely $18^{1/3} = \lim_{n\to\infty}\{g_n(2,3;1)\}$ and from which Theorem 2.1 is immediate since $(ab^2)^{1/3}$ is independent of n. We also see, in consequence, that $J_{n-1}/2^{n-1} \sim \frac{1}{3}$ and $J_n/2^{n-1} \sim \frac{2}{3}$ for some large n; thus, $\lim_{n\to\infty}\{J_n/J_{n-1}\} = 2$, which can be obtained from any of (2.3)-(2.5) by elementary methods.

Remark 2.4. In view of (2.6) we have $2^{n-1}w_n(1,0;1/2,-1/2) = w_{n-1}(0,1;1,-2)$ $(n \geq 1)$ and $2^{n-1}w_n(0,1;1/2,-1/2) = w_n(0,1;1,-2)$ $(n \geq 0)$ from, resp., (2.10) and (2.11); others that similarly relate terms of different Horadam sequences are available directly from (2.3) and (2.4), as the reader is invited to discover.

Relations (2.10),(2.11) emerge naturally as a result of what we know about Bunder's sequence $\{z_n(a,b,p,q)\}_0^\infty$. It is readily seen, however, that they are merely instances of general ones. Setting $w_0 = a = 1$, $w_1 = b = 0$, (2.2) gives $w_n(1,0;p,q) = w_n(\alpha(p,q), \beta(p,q), 1, 0) = \alpha\beta(\beta^{n-1} - \alpha^{n-1})/(\alpha-\beta)$, and choosing further, for arbitrary γ, $p(\gamma) = \gamma$, $q(\gamma) = -2\gamma^2$ (for which $\alpha(\gamma, -2\gamma^2) = 2\gamma$, $\beta(\gamma, -2\gamma^2) = -\gamma$), then $w_n(1,0;\gamma,-2\gamma^2) = (2\gamma^n/3)[2^{n-1} - (-1)^{n-1}] = 2\gamma^n J_{n-1}$ by (2.5) and we have a generalized form of (2.10), parameterized by γ, which recovers it when $\gamma = 1/2$:

Identity I. For $n \geq 1$,

$$J_{n-1} = \frac{1}{2\gamma^n} w_n(1,0;\gamma,-2\gamma^2).$$

In a similar fashion, (2.2) gives $w_n(0,1;p,q) = w_n(\alpha(p,q), \beta(p,q), 0, 1) = (\alpha^n - \beta^n)/(\alpha-\beta)$ (the general term of the fundamental sequence mentioned earlier), with $w_n(0,1;\gamma,-2\gamma^2) = (\gamma^{n-1}/3)[2^n - (-1)^n] = \gamma^{n-1}J_n$ and so a general form of (2.11) which latter is also reproduced for the same value $\gamma = 1/2$:

Identity II. For $n \geq 0$,

$$J_n = \frac{1}{\gamma^{n-1}} w_n(0, 1; \gamma, -2\gamma^2).$$

We now offer proofs of both Identities I and II, necessarily in reverse order beginning with Identity II.

Proofs of Identity II: Proof 1. This proof uses a matrix approach.

Proof. Writing the recurrence (2.1) as

$$\text{(II.1)} \qquad \begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_{n-1} \\ w_{n-2} \end{pmatrix}$$

in matrix form leads iteratively to the matrix power equation

$$\text{(II.2)} \qquad \begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} w_1 \\ w_0 \end{pmatrix},$$

which holds for $n \geq 1$. Thus,

$$\text{(II.3)} \qquad w_n(a, b; p, q) = (1, 0) \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} b \\ a \end{pmatrix}$$

and, in particular,

$$\text{(II.4)} \qquad w_n(0, 1; p, q) = (1, 0) \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let us define matrices

$$\text{(II.5)} \qquad \mathbf{F}(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}, \qquad \mathbf{G}(\gamma) = \begin{pmatrix} \gamma & 2\gamma^2 \\ 1 & 0 \end{pmatrix}.$$

Then, observing the decomposition

$$\text{(II.6)} \qquad \frac{1}{\gamma}\mathbf{F}(\gamma)\mathbf{G}(\gamma)\mathbf{F}^{-1}(\gamma) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix},$$

it follows, using (2.6) as a starting point, that

$$\begin{aligned} J_n &= w_n(0, 1; 1, -2) \\ &= (1, 0) \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= (1, 0)[\gamma^{-1}\mathbf{F}(\gamma)\mathbf{G}(\gamma)\mathbf{F}^{-1}(\gamma)]^{n-1}(1, 0)^T \\ \text{(II.7)} \qquad &= \frac{1}{\gamma^{n-1}}(1, 0)\mathbf{F}(\gamma) \cdot \mathbf{G}^{n-1}(\gamma) \cdot \mathbf{F}^{-1}(\gamma)(1, 0)^T, \end{aligned}$$

having employed (II.4) and (II.6). Noting that $(1, 0)\mathbf{F}(\gamma) = (1, 0)$ and $\mathbf{F}^{-1}(\gamma)(1, 0)^T = (1, 0)^T$, (II.7) now reads

$$\text{(II.8)} \qquad J_n = \frac{1}{\gamma^{n-1}}(1, 0)\mathbf{G}^{n-1}(\gamma)(1, 0)^T = \frac{1}{\gamma^{n-1}} w_n(0, 1; \gamma, -2\gamma^2),$$

by (II.4). □

Proof 2. This proof takes a different route.

Proof. We define a sequence $\{T_n\}_0^\infty = \{\gamma^{n-1}J_n\}_0^\infty$, noting that $T_0 = J_0/\gamma = 0$ and $T_1 = J_1 = 1$. Then $T_{n+1} = \gamma^n J_{n+1} = \gamma^n(J_n + 2J_{n-1})$ (by (2.3)) $= \gamma(\gamma^{n-1}J_n) + 2\gamma^2(\gamma^{n-2}J_{n-1}) = \gamma T_n + 2\gamma^2 T_{n-1}$. This being a Horadam recurrence (2.1) for the sequence $\{T_n\}_0^\infty$ (with $p = \gamma$, $q = -2\gamma^2$), we have, for $n \geq 0$, $\gamma^{n-1}J_n = T_n = w_n(T_0, T_1; \gamma, -2\gamma^2) = w_n(0, 1; \gamma, -2\gamma^2)$ as required. □

The proofs of Identity I—while running along the same lines of argument as those just seen—have nuances that warrant their inclusion.

Proofs of Identity I: Proof 1.

Proof. Using Proof 1 of Identity II we can write down, from (II.8), that

$$\begin{aligned} J_{n-1} &= \frac{1}{\gamma^{n-2}}(1,0)\mathbf{G}^{n-2}(\gamma)(1,0)^T \\ &= \frac{1}{\gamma^{n-2}}(1,0)\mathbf{G}^{n-1}(\gamma)\cdot\mathbf{G}^{-1}(\gamma)(1,0)^T, \end{aligned} \tag{I.1}$$

and since

$$\mathbf{G}^{-1}(\gamma)(1,0)^T = \frac{1}{2\gamma^2}(0,1)^T \tag{I.2}$$

then (I.1) becomes

$$J_{n-1} = \frac{1}{2\gamma^n}(1,0)\mathbf{G}^{n-1}(\gamma)(0,1)^T = \frac{1}{2\gamma^n}w_n(1,0;\gamma,-2\gamma^2), \tag{I.3}$$

by (II.4). □

Proof 2.

Proof. Let us define a sequence $\{T_n\}_0^\infty$ for which T_0 is undefined and $\{T_n\}_1^\infty = \{2\gamma^n J_{n-1}\}_1^\infty$. We see that, for $n \geq 0$, $T_n = w_n(T_0, T_1; \gamma, -2\gamma^2)$ since $T_{n+1} = 2\gamma^{n+1}J_n = 2\gamma^{n+1}(J_{n-1} + 2J_{n-2})$ (by (2.3)) $= 2[\gamma(\gamma^n J_{n-1}) + 2\gamma^2(\gamma^{n-1}J_{n-2})] = 2[\gamma(\frac{1}{2}T_n) + 2\gamma^2(\frac{1}{2}T_{n-1})] = \gamma T_n + 2\gamma^2 T_{n-1}$. Now by definition $T_1 = 2\gamma J_0 = 0$ and $T_2 = 2\gamma^2 J_1 = 2\gamma^2$, but the Horadam recurrence (2.1) gives $T_2 = \gamma T_1 + 2\gamma^2 T_0 = 2\gamma^2 T_0$ and so $T_0 = 1$. Thus, for $n \geq 1$, $2\gamma^n J_{n-1} = T_n = w_n(1, 0; \gamma, -2\gamma^2)$. □

Remark 2.5. Equation (2.6) is, we note, but the $\gamma = 1$ version of Identity II, while Identity I correspondingly reads, for $n \geq 1$, $J_{n-1} = w_n(1, 0; 1, -2)/2$.

Remark 2.6. Using (2.5) it is simple to see that $J_{n-1} + J_n = 2^{n-1}$ ($n \geq 1$) which, employing Identities I and II, gives $w_n(1, 0; \gamma, -2\gamma^2) + 2\gamma w_n(0, 1; \gamma, -2\gamma^2) = (2\gamma)^n$ as a new identity; this is confirmed, for $n \geq$

0, by inspection thus: $\{w_n(1,0;\gamma,-2\gamma^2)\}_0^\infty + 2\gamma\{w_n(0,1;\gamma,-2\gamma^2)\}_0^\infty = \{1,0,2\gamma^2,2\gamma^3,6\gamma^4,10\gamma^5,\dots\} + 2\gamma\{0,1,\gamma,3\gamma^2,5\gamma^3,11\gamma^4,\dots\} = \{1,2\gamma,4\gamma^2, 8\gamma^3,16\gamma^4,32\gamma^5,\dots\} = \{(2\gamma)^0,(2\gamma)^1,(2\gamma)^2,(2\gamma)^3,(2\gamma)^4,(2\gamma)^5,\dots\}$.

Some Final Remarks. Leaving c unassigned in (1.1) then some experimental computations indicate that the power of c within $g_n(a,b;c)$ is $S_{n-1}/2^{n-2}$, where $\{S_n\}_0^\infty = \{S_0,S_1,S_2,S_3,S_4,\dots\} = \{0,1,3,9,23,\dots\}$ is the sequence for which

$$S_n = [(3n+1)2^n - (-1)^n]/9, \qquad n \geq 0, \tag{2.12}$$

is the closed form of its $(n+1)$th term [6, Sequence No. A045883]. Accordingly, we establish formally this empiric observation for completeness in the Appendix (adapting the routine methodology of the study [7]) as part of a new and succinct direct proof of a generalized form of Lemma 2.1 that accommodates the scaling variable c. In view of the similarity between (2.12) and (2.5) in structure it is, perhaps, not surprising that the power of c in the corresponding sequence growth rate ratio $g_{n+1}(a,b;c)/g_n(a,b;c)$ is dependent upon a Jacobsthal number, and this allows immediate confirmation of Remark 2.1 (see again the Appendix).

3. Summary

Geometric mean sequences, however defined, have been studied in the past, but not to any great extent. This paper considers the notion of the growth rate of what we regard as the standard geometric mean sequence, and from that develops new identities which express Jacobsthal numbers in terms of parameterized families of Horadam numbers. As an extension of this article, future work will examine properties of the scaled version of Bunder's recurrence (2.7).

Appendix

Consider the recurrence equation (1.1) with c held as a free variable. We state and prove the following (which reduces to Lemma 2.1 for $c = 1$):

Theorem A.1. For $n \geq 1$,

$$g_n(a,b;c) = (a^{J_{n-1}} b^{J_n} c^{2S_{n-1}})^{2^{-(n-1)}}.$$

Proof. Let $h_n = \ln(g_n)$.[2] Then (1.1) may be written

$$2h_{n+1} - h_n - h_{n-1} = 2\ln(c), \tag{P.1}$$

[2]As in the proof of Lemma 2.1, we should assume $a,b,c > 0$ at the outset; this constraint, however, does not apply to Theorem A.1 itself.

the characteristic equation of which is $0 = 2x^2 - x - 1 = (2x+1)(x-1)$ and gives a homogeneous solution of form $A(-\frac{1}{2})^n + B$ (where A, B are arbitrary constants). A particular solution to (P.1), chosen to be $Cn\ln(c)$, has $C = \frac{2}{3}$ when substituted into it, so that the general solution is

$$h_n(c) = A(-1/2)^n + B + (2/3)n\ln(c). \tag{P.2}$$

Applying the initial values $h_0 = \ln(a)$ and $h_1 = \ln(b)$ yields simultaneous equations

$$\begin{aligned} \ln(a) &= A + B, \\ \ln(b^2/c^{4/3}) &= -A + 2B, \end{aligned} \tag{P.3}$$

for A, B, with solutions $A = \ln(a^{2/3}c^{4/9}/b^{2/3})$ and $B = \ln(a^{1/3}b^{2/3}/c^{4/9})$, and in turn a full solution

$$\begin{aligned} h_n(a,b,c) &= \ln\left(\frac{a^{\frac{2}{3}(-\frac{1}{2})^n}c^{\frac{4}{9}(-\frac{1}{2})^n}}{b^{\frac{2}{3}(-\frac{1}{2})^n}} \cdot \frac{a^{\frac{1}{3}}b^{\frac{2}{3}}}{c^{\frac{4}{9}}} \cdot c^{\frac{2n}{3}}\right) \\ &= \ln\left(a^{\Omega_a(n)}b^{\Omega_b(n)}c^{\Omega_c(n)}\right), \end{aligned} \tag{P.4}$$

where, after a little algebra using (2.5) and (2.12), the exponent functions simplify as

$$\begin{aligned} \Omega_a(n) &= [2(-1/2)^n + 1]/3 &=& J_{n-1}/2^{n-1}, \\ \Omega_b(n) &= 2[1 - (-1/2)^n]/3 &=& J_n/2^{n-1}, \\ \Omega_c(n) &= 4[(-1/2)^n - 1]/9 + (2n)/3 &=& S_{n-1}/2^{n-2}; \end{aligned} \tag{P.5}$$

thus, $g_n(a,b;c) = a^{\Omega_a(n)}b^{\Omega_b(n)}c^{\Omega_c(n)} = (a^{J_{n-1}}b^{J_n}c^{2S_{n-1}})^{2^{-(n-1)}}$. □

As an aside, we see that with reference to growth rate the power $G_c(n)$ of c within the growth rate ratio $g_{n+1}(a,b;c)/g_n(a,b;c)$ may now be determined as, for $n \geq 1$, the simple difference $G_c(n) = \Omega_c(n+1) - \Omega_c(n) = S_n/2^{n-1} - S_{n-1}/2^{n-2} = (S_n - 2S_{n-1})/2^{n-1} = \{[(3n+1)2^n - (-1)^n]/9 - 2[(3n-2)2^{n-1} + (-1)^n]/9\}/2^{n-1}$ (using (2.12)) $= [2^n - (-1)^n]/(3 \cdot 2^{n-1}) = J_n/2^{n-1}$ by (2.5); this final form holds for $n \geq 0$, for which $\lim_{n\to\infty}\{G_c(n)\} = 2/3$ trivially and, as stated in Remark 2.1, the growth rate of the sequence $\{g_n(a,b;c)\}_0^\infty$ is $\lim_{n\to\infty}\{g_{n+1}(a,b;c)/g_n(a,b;c)\} = c^{2/3}$.

Finally, the geometric Fibonacci sequence of Shiu and Yerger corresponds to $\{g_{n-1}(1,1;2)\}_1^\infty$ here. With (by (P.5)) $\Omega_c(n-1) = 4[(-1/2)^{n-1} - 1]/9 + 2(n-1)/3 = [2n - (8/3)(-1/2)^n - 10/3]/3$, we recover their exponent of 2 [7, p. 84] and so the general term $2^{\Omega_c(n-1)} = g_{n-1}(1,1;2)$ of their sequence.

References

[1] M. W. Bunder, *Products and powers*, The Fibonacci Quarterly, **13** (1975), 279.

[2] P. J. Larcombe and O. D. Bagdasar, *On a result of Bunder involving Horadam sequences: a proof and generalization*, The Fibonacci Quarterly, **51** (2013), 174–176.

[3] P. J. Larcombe, O. D. Bagdasar, and E. J. Fennessey, *Horadam sequences: a survey*, Bulletin of the I.C.A., **67** (2013), 49–72.
[4] P. J. Larcombe, O. D. Bagdasar, and E. J. Fennessey, *On a result of Bunder involving Horadam sequences: a new proof*, The Fibonacci Quarterly, **52** (2014), 175–177.
[5] P. Maynard, *Geometric-mean sequences*, Mathematical Gazette, **89** (2005), 270–275.
[6] O.E.I.S. Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.
[7] A. J. Shiu and C. R. Yerger, *Geometric and harmonic variations of the Fibonacci sequence*, Mathematical Spectrum, **41** (2009), 81–86.
[8] W. R. Spickerman, R. L. Creech, and R. N. Joyner, *An introduction to geometric Fibonacci sequences*, Journal of the Elisha Mitchell Scientific Society, **109** (1993), 193–196.

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