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Multi-weighted Complex Structure on Fractional Order Coupled Neural Networks with Linear Coupling Delay: A Robust Synchronization Problem

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Abstract

This sequel is concerned with the analysis of robust synchronization for a multi-weighted complex structure on fractional-order coupled neural networks (MWCFCNNs) with linear coupling delays via state feedback controller. Firstly, by means of fractional order comparison principle, suitable Lyapunov method, Kronecker product technique, some famous inequality techniques about fractional order calculus and the basis of interval parameter method, two improved robust asymptotical synchronization analysis, both algebraic method and LMI method, respectively are established via state feedback controller. Secondly, when the parameter uncertainties are ignored, several synchronization criterion are also given to ensure the global asymptotical synchronization of considered MWCFCNNs. Moreover, two type of special cases for global asymptotical synchronization MWCFCNNs with and without linear coupling delays, respectively are investigated. Ultimately, the accuracy and feasibility of obtained synchronization criteria are supported by the given two numerical computer simulations.

Keywords Robust synchronization · Fractional order · Coupled neural networks · Kronecker product · Linear coupling delays

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1 Introduction

Nowadays, differential equations and dynamical networks have received great attention from many researchers as a result of their potential application in different fields which include biology [1–3], physics [4,5], engineering [6,7], mathematics [8,9], information technology and so forth [10,11]. Especially, synchronization of complex networks dynamical behaviors has become a heat research and it plays an immense role to mathematical modeling of the real world objects like social networks, global economic markets, disease network modeling, food web, power grid, WWW and so on, and some excellent results have been paid in more as of late, see e.g [12–17] and references in that. In [18] Lin et al. applied a decomposing matrix method to analyze the delayed complex networks under asymmetric coupling. Rui et al. [19] investigated the pinning synchronization of delayed complex networks by Taylor expansion. In [20], Yi et al. has delivered intermittent control with non-period based exponential synchronization problem of complex networks with time delays under non-linear coupling.

As is known to every one of, every network in real-world objects can be modeled by multiple weighted complex network dynamics, for instance, transportation networks, complex biological neural networks, public traffic networks, communication networks and so on. For example, we are contacting with friends via specific channels together with Gmail, Whatsapp, Facebook, Instagram, letters, mobile phone and so on, and every way of contact strategy depicts for various coupling. In this circumstance, social media networks can be modeled by more than one or multiple weights. Therefore, the investigation of complex dynamical networks with multi weights is necessary and intriguing issues. In recent years, synchronization analysis has always been a hot research topic in identical network systems, especially complex networks, neural networks and Boolean control networks [17,21–24], and many applications have been found in different areas. Nowadays, the investigation of the synchronization analysis of multi-weighted complex dynamical networks have been received much attention, for example [25–28]. In [27], Shui-Han et al. presented a finite-time synchronization criteria for multiple weighted complex dynamical with coupling delays and switching topology by using Dini derivative method and linear feedback control strategy. In [28], the authors developed the adaptive control strategies to achieve the H_∞ synchronization for multiple weighted complex dynamical networks via Lyapunov method and some famous inequality techniques.

Nowadays, the research on fractional-order delayed dynamical systems brought about numerous fruitful achievements due to the fact a few scholars and researchers were contributed to this area [29–31]. In the application perspective, fractional order calculus is applied in many fields, for instance epidemic models [32], control theory [33], biological models [34] and so on. On the other hand, the dynamical investigation of networks models with time delays have been gained more and more attention, recently, a variety of time delays have been considered in the study of various networks models [35–38]. In reality, many real-world systems need to be described with the aid of fractional order models because of the fact dynamics of fractional-order models are more correct than integer-order models. In plentiful applications, time delays are inevitable in realistic system designs, for example, complex networks, neural networks, echo cancelation, local loop equalization, multi path propagation in mobile communication, array signal processing, congestion analysis and control in high-speed networks and long transmission line in pneumatic systems. In recent years, fractional order complex dynamical networks (FOCNNs) with time delays has turned into a hot research topic because it has been utilized in different areas like metabolic systems,

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communication networks, global economic markets, and so on, and lots of remarkable outcomes about FOCNNs have been devoted in recent literature, as an instance [13,39–41]. In [42], we have to investigate the synchronization in finite time criteria for FOCNNs by hybrid control approach. In [43], the authors have demonstrated the pinning synchronization criteria for FOCNNs by fractional Proportional-Integral control. By utilizing the LaSalle invariance principle, the issues of outer synchronization criteria for FOCNNs was studied in [44].

Coupled neural networks (CNN), which is an extension of complex networks have attracted growing attention among numerous fields, together with secure communications, nonlinear optimization problems, image processing, and parallel computation, and it is the most powerful tool to analyze the passivity [45–47] and synchronization [22,48–50] of coupled neural networks. For example, the authors in [51] analyzed the pinning control for leader-following bipartite synchronization of CNN by M-matrix method and reciprocally convex approach. In [52], Shanrong et al. has deliberated pinning passivity analysis for different dimensional based CNN by some inequality techniques. The authors in [53] presented the complex structure on exponential synchronization criteria for delayed CNNs with stochastic perturbations by using the combination of combining the Lyapunov method with Kirchhoff's matrix-tree theorem and impulsive control method. Unfortunately, there are few results targeted on synchronization problem of fractional order coupled neural networks (FOCNNs), see Refs. [54,55]. For example, by using the well-known fractional order comparison theorem for a single delay, multi-quasi synchronization of FOCNNs with single weights was studied by novel pinning impulsive control strategies and also discussed the effect of coupling delays and pinning control matrix in [54]. In [55], Zhang et al. dealt with the issues of Riemann–Liouville sense synchronization stability criteria of single weighted complex structure on FOCNNs under linear coupling delays by applying LMI method and Lyapunov approach. Besides, in the natural implementation of the network model, the parameter uncertain factors are inevitable and it leads to breaking the synchronization performance of complex dynamical networks. Recently, the author have taken the uncertain parameter into the account of FOCNNs and some sufficient conditions have been established for pinning synchronization and robust pinning synchronization by using Kronecker product and Lyapunov functions [56].

In the meantime, it has been discovered that neural network with multi-weights reveals the more complex structure and unpredictable behaviors than a network with a single weight, which can substantially increase the applications of a neural network. For instance in [57], the authors gave some exponential synchronization criteria for integer order CNNs with multi weights by means of aperiodically pinning intermittent control method. In [58], by using some inequality scaling skills and Lyapunov–Krasovskii functionals, the author investigated about the finite time synchronization and finite time passivity criteria for multiple delayed CNNs with reaction-diffusion terms and coupling delays. Kronecker product technique and fractional order multiple delayed comparison principle are adopted to deal with the robust synchronization of single delayed FOCNNs with uncertain parameters by pinning control in [56]. Motivated by the above discussion, we try to explore firstly the robust synchronization of multi-weighted complex structure on fractional order coupled neural networks under linear coupling delays. However, the handling of multiple-weights complex structure, coupling delays and uncertain parameters are main challenge in this proposed research fields, there are no works not yet addressed in the same fields.

The crucial novelty of this work is highlighted in the following aspects:

1. Multi-weights, linear coupling delay term, and parameter uncertainty, are taking into consideration, robust asymptotical synchronization analysis for a class of FOCNNs with multiple delays are introduced.

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2. By means of Kronecker product technique and robust analysis scaling skills, a new brand of novel sufficient conditions with respect to FOCNNs are derived in form of both LMI and algebraic method, respectively by the state feedback controller.
3. As some special cases of proposed results, we also investigate the asymptotical synchronization for multi-weighted FOCNNs without parameter uncertainties, and some enhanced synchronization criteria have been derived for the problem of fractional order complex dynamical networks and fractional order neural network results.
4. Moreover, the present results in this paper are valid for single weighted FOCNNs and integer order coupled neural networks both single weight and multiple weights, respectively.
5. The conditions of the global asymptotic synchronization are deduced in term of LMI, and check the feasibility of obtaining results by using the LMI MATLAB control toolbox.

The rest of this proposed work is well organized as follows. In Sect. 2, basic definition and preliminaries are given including the problem statement will be addressed. A valid state feedback control scheme is designed and new conditions for robust synchronization are demonstrated in Sect. 3. Section 4 demonstrates our FOCNNs with multiple weight results with two computer simulations. At last, Sect. 5 ends with conclusions.

2 Preliminaries and Problem Statement

Notations In this article, $\mathbb{N}$ represents the space of natural numbers from 1 to n, $\mathbb{R}^n$ represents the space of n-D Euclidean space, respectively, and $\mathbb{R}^{n\times n}$ stands for a set of all $n \times n$ real matrices. For $z(t) = (z_1(t), \ldots, z_n(t))^T \in \mathbb{R}^n$, $\|z\| \in \mathbb{R}^n$ is denoted as arbitrary norm, which is described as:

$$\|z(t)\|_p = \sqrt[p]{\sum_{q=1}^{n} |z_q(t)|^p}, \quad p = 1, 2.$$

In this part, some basic knowledge of definitions, useful lemma's and problem statement will be given.

2.1 Basic Tools

Definition 2.1 [59] The Riemann–Liouville fractional integral order γ for a function z on interval $[t_0, T]$ is defined as

$$D_{t_0,t}^{-\gamma} z(t) = \frac{1}{\Gamma(\gamma)} \int_{t_0}^{t} (t-\chi)^{\gamma-1} z(\chi)\, \mathrm{d}\chi,$$

where $\gamma \in \mathbb{R}^+$.

Definition 2.2 [59] The Caputo type fractional-order derivative with order γ for a function z on interval $[t_0, T]$ is defined as

$$D_{t_0,t}^{\gamma} z(t) = \begin{cases} D_{t_0,t}^{-(n-\gamma)} \left(\frac{d^n}{dt^n} z(t)\right), & \text{if } \gamma \in (n-1, n) \\ \left(\frac{d^n}{dt^n} z(t)\right), & \text{if } \gamma = n. \end{cases}$$

where $\gamma \in \mathbb{R}^+,\ n \in \mathbb{Z}^+$.

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Definition 2.3 [59] The Mittag-Leffler function with two parameter is defined as

$$\mathbb{E}_{\gamma,\vartheta}(z) = \sum_{l=0}^{+\infty} \frac{z^l}{\Gamma(\gamma l + \vartheta)}$$

where $\gamma,\ \vartheta \in \mathbb{R}^+,\ z \in \mathbb{C}$.

Lemma 2.4 [60] *Let $y(t) \in \mathbb{R}^n$ be continuously derivable function and the positive definite matrix $M \in \mathbb{R}^{n\times n}$, the following inequality holds:*

$$D_{t_0,t}^{\gamma} y^T(t)My(t) \leq 2y^T(t)M\{D_{t_0,t}^{\gamma} y(t)\},\ \gamma \in (0, 1).$$

Lemma 2.5 [61] *Let $\varepsilon \in \mathbb{R},\ \Upsilon,\ \Lambda,\ \Phi,\ \Psi$ be matrices with suitable dimensions. Then the properties of Kronecker product is given by:*

(1). $(\varepsilon\Phi) \otimes \Psi = \Phi \otimes (\varepsilon\Psi)$;
(2). $(\Phi + \Psi) \otimes \Upsilon = (\Phi \otimes \Upsilon) + (\Psi \otimes \Upsilon)$;
(3). $(\Phi \otimes \Psi)^T = (\Phi^T \otimes \Psi^T)$;
(4). $(\Phi \otimes \Psi)(\Upsilon \otimes \Lambda) = (\Phi\Upsilon \otimes \Psi\Lambda)$.

Lemma 2.6 [62] *Consider the fractional order differential inequality with time delays as follows:*

$$\begin{cases} D^{\gamma} H(t) \leq -\zeta H(t) + \varsigma H(t-\mu_1) + \eta H(t-\mu_2),\ 0 < \gamma \leq 1, \\ H(\chi) = \hat{h}(\chi),\ \chi \in [-\hat{\mu} = -\max\{\mu_1, \mu_2\}, 0]. \end{cases} \tag{1}$$

If all the eigenvalues of $\hat{H}$ satisfy $|arg\big(\lambda\big)| > \frac{\pi}{2}$, and the characteristic equation $det\big(\Delta(s)\big)$ has no purely imaginary roots for any $\mu_1,\ \mu_2 > 0$, then the zero solution of system (1) is Lyapunov asymptotically stable, where

$$\hat{H} = \begin{pmatrix} \varsigma_{11} + \eta_{11} - \zeta_{11} & \varsigma_{12} + \eta_{12} - \zeta_{12} & \cdots & \varsigma_{1n} + \eta_{1n} - \zeta_{1n} \\ \varsigma_{21} + \eta_{21} - \zeta_{21} & \varsigma_{22} + \eta_{22} - \zeta_{22} & \cdots & \varsigma_{2n} + \eta_{2n} - \zeta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varsigma_{n1} + \eta_{n1} - \zeta_{n1} & \varsigma_{n2} + \eta_{n2} - \zeta_{n2} & \cdots & \varsigma_{nn} + \eta_{nn} - \zeta_{nn} \end{pmatrix} \tag{2}$$

$$\Delta(s) = \begin{pmatrix} s^{\gamma} - e^{-\mu_1}\varsigma_{11} - e^{-\mu_2}\eta_{11} + \zeta_{11} & \cdots & -e^{-\mu_1}\varsigma_{1n} - e^{-\mu_2}\eta_{1n} + \zeta_{1n} \\ -e^{-\mu_1}\varsigma_{21} - e^{-\mu_2}\eta_{21} + \zeta_{21} & \cdots & -e^{-\mu_1}\varsigma_{2n} - e^{-\mu_2}\eta_{2n} + \zeta_{2n} \\ \vdots & \vdots & \vdots \\ -e^{-\mu_1}\varsigma_{n1} - e^{-\mu_2}\eta_{n1} + \zeta_{n1} & \cdots & s^{\gamma} - e^{-\mu_1}\varsigma_{nn} - e^{-\mu_2}\eta_{nn} + \zeta_{nn} \end{pmatrix}. \tag{3}$$

Lemma 2.7 [62] *If the characteristic equation i.e.,$s^{\gamma} - \varsigma e^{-\mu_1} - \eta e^{-\mu_2} + \zeta$, of (1) has no pure imaginary roots for any $\mu_1,\ \mu_2 > 0$, and $\varsigma + \eta - \zeta > 0$, then the zero solution of system (1) is Lyapunov asymptotically stable.*

Lemma 2.8 [62] *Consider the fractional order differential inequality with multiple time delays as follows:*

$$\begin{cases} D^{\gamma} H(t) \leq -\zeta H(t) + \varsigma H(t-\mu_1) + \eta H(t-\mu_2),\ 0 < \gamma \leq 1, \\ H(\chi) = \hat{h}(\chi),\ \chi \in [-\hat{\mu} = -\max\{\mu_1, \mu_2\}, 0]. \end{cases} \tag{4}$$

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and the linear fractional order differential inequality with multiple time delays

$$\begin{cases} D^{\gamma} E(t) \leq -\zeta E(t) + \varsigma E(t-\mu_1) + \eta E(t-\mu_2), \ 0 < \gamma \leq 1, \\ E(\chi) = \hat{h}(\chi), \ \chi \in [-\hat{\mu} = -\max\{\mu_1, \mu_2\}, 0], \end{cases} \tag{5}$$

where $H(t)$ *and* $E(t)$ *are continuous and non negative in* $[0, +\infty)$, *and* $\hat{h}(t) \geq 0, \ t \in [-\max\{\mu_1, \mu_2\}, 0]$. *If* $\zeta, \ \varsigma$ *and* $\eta > 0$, *then* $H(t) \leq E(t) \ \forall \ t \in [0, +\infty)$.

Lemma 2.9 [63] *For any vectors* $\varepsilon_1, \ \varepsilon_2 \in \mathbb{R}^n$, *one constant* $\mu > 0$ *and any positive definite matrix* $0 < M \in \mathbb{R}^{n \times n}$, *the following relationship holds:*

$$2\varepsilon_1^T \varepsilon_2 \leq \mu \varepsilon_1^T M \varepsilon_1 + \mu^{-1} \varepsilon_2^T M^{-1} \varepsilon_2.$$

Lemma 2.10 [64] *If* $y(t)$ *is the continuously derivable function, the following relationship true almost everywhere:*

$$\mathcal{D}^{\gamma} |y(t)| \leq \mathrm{sgn}(y(t)) \mathcal{D}^{\gamma} y(t), \ 0 < \gamma < 1.$$

2.2 Problem Statement

Consider the following multi-weighted complex structure on fractional-order coupled neural networks (MWCFCNNs) with linear coupling delay and uncertain parameter described by:

$$\begin{aligned} D^{\gamma} z_k(t) &= -P z_k(t) + Q g\big(z_k(t)\big) + R h\big(z_k(t-\mu_1)\big) \\ &\quad + \sum_{l=1}^{N} \alpha_1 V_{kl}^1 \Lambda_1 z_l(t) + \sum_{l=1}^{N} \alpha_2 V_{kl}^2 \Lambda_2 z_l(t) \\ &\quad + \cdots + \sum_{l=1}^{N} \alpha_x V_{kl}^x \Lambda_x z_l(t) + \sum_{l=1}^{N} \beta_1 W_{kl}^1 \Upsilon_1 z_l(t-\mu_2) \\ &\quad + \sum_{l=1}^{N} \beta_2 W_{kl}^2 \Upsilon_2 z_l(t-\mu_2) \\ &\quad + \cdots + \sum_{l=1}^{N} \beta_x W_{kl}^x \Upsilon_x z_l(t-\mu_2), \end{aligned} \tag{6}$$

where $k = 1, 2, \ldots, N$, N is the total number of nodes in the networks, $z_k(t) = \big(z_{k1}(t), \ldots, z_{kn}(t)\big)^T$ stands for the state of the k-th neuron at time t; $P = diag\{p_1, \ldots, p_n\}$ with $p_i > 0, \ (i \in \mathbb{N})$ signifies the weight of self feedback connection; $Q = (q_{kl})_{n\times n}$ and $R = (r_{kl})_{n\times n}$ are the connection strengths of the l-th neuron on k-th neuron; $\mu_1 > 0$ and $\mu_2 > 0$ stands for the positive and constant delays; $g(z_k(t)) = \big(g_1(z_{k1}(t)), \ldots, g_n(z_{kn}(t))\big)^T$ and $h(z_k(t-\mu_1)) = \big(h_1(z_{k1}(t-\mu_1)), \ldots, h_n(z_{kn}(t-\mu_1))\big)^T$ represents the activation function of the neurons at time t and $t-\mu_1$, respectively; $0 < \alpha_j, \ 0 < \beta_j, \ (j = 1, 2, \ldots, x)$ denotes the coupling strengths of the jth coupling form; $\Lambda_j = diag\{\Lambda_{j1}, \ldots, \Lambda_{jn}\} > 0$ and $\Upsilon_j = diag\{\Upsilon_{j1}, \ldots, \Upsilon_{jn}\} > 0, \ (j = 1, 2, \ldots, x)$ are inner linking strengths of the jth coupling form, respectively; $V^j = \big(V_{kl}^j\big)_{N\times N}$ and $W^j = \big(W_{kl}^j\big)_{N\times N}$ are the coupling configuration matrix of the jth coupling form, in which V_{kl}^j and W_{kl}^j are described by the

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following form:

$$\begin{cases} V_{kk}^{j} = -\sum_{l=1,k\neq l}^{N} V_{kl}^{j}, & k=1,2,\ldots,N,\ j=1,2,\ldots,x \\ V_{kl}^{j}\ (k\neq l) > 0, & \text{if node } k \text{ and } l \text{ are linked of the } j\text{th coupling form} \\ V_{kl}^{j}\ (k\neq l) = 0, & \text{otherwise,} \end{cases}$$

$$\begin{cases} W_{kk}^{j} = -\sum_{l=1,k\neq l}^{N} W_{kl}^{j}, & k=1,2,\ldots,N,\ j=1,2,\ldots,x \\ W_{kl}^{j}\ (k\neq l) > 0, & \text{if node } k \text{ and } l \text{ are linked of the } j\text{th coupling form} \\ W_{kl}^{j}\ (k\neq l) = 0, & \text{otherwise.} \end{cases}$$

In practical systems, many uncertain factors occur and it also affects the synchronization performance of complex dynamical networks. In this work, the parameters $\alpha_j,\ \beta_j,\ Q,\ R,\ \Lambda_j,\ \Upsilon_j,\ V^j,\ W^j$ change in some given precision, which is intervalized as following ranges:

$$\alpha_I := \Big\{0 < \underline{\alpha}_j \le \alpha_j \le \overline{\alpha}_j,\ j=1,2,\ldots,x,\ \ \forall\, \alpha_j \in \alpha_I\Big\};$$

$$\beta_I := \Big\{0 < \underline{\beta}_j \le \beta_j \le \overline{\beta}_j,\ j=1,2,\ldots,x,\ \ \forall\, \beta_j \in \beta_I\Big\};$$

$$P_I := \Big\{P = diag(p_k) : \underline{P} \le P \le \overline{P},\ 0 < \underline{p}_k \le p_k \le \overline{p}_k,\ k=1,2,\ldots n,\ \forall\, P \in P_I\Big\};$$

$$Q_I := \Big\{Q = (q_{kl})_{n\times n} : \underline{Q} \le Q \le \overline{Q},\ 0 < \underline{q}_{kl} \le q_{kl} \le \overline{q}_{kl},\ k=1,2,\ldots,n,$$
$$l=1,2,\ldots,n\ \forall\, Q \in Q_I\Big\};$$

$$R_I := \Big\{R = (r_{kl})_{n\times n} : \underline{R} \le R \le \overline{R},\ 0 < \underline{r}_{kl} \le r_{kl} \le \overline{r}_{kl},\ k=1,2,\ldots,n,$$
$$l=1,2,\ldots,n\ \forall\, R \in R_I\Big\};$$

$$\Lambda_I := \Big\{\Lambda_j = diag(\lambda_k^j) : \underline{\Lambda}_j \le \Lambda_j \le \overline{\Lambda}_j,\ 0 < \underline{\lambda}_k^j \le \lambda_k^j \le \overline{\lambda}_k^j,\ j=1,2,\ldots,x,$$
$$k=1,2,\ldots n,\ \forall\, \Lambda_j \in \Lambda_I\Big\};$$

$$\Upsilon_I := \Big\{\Upsilon_j = diag(\upsilon_k^j) : \underline{\Upsilon}_j \le \Upsilon_j \le \overline{\Upsilon}_j,\ 0 < \underline{\upsilon}_k^j \le \upsilon_k^j \le \overline{\upsilon}_k^j,\ j=1,2,\ldots,x,$$
$$k=1,2,\ldots n,\ \forall\, \Upsilon_j \in \Upsilon_I\Big\};$$

$$V_I := \Big\{V^j = (V_{kl}^j)_{N\times N} : \underline{V}^j \le V^j \le \overline{V}^j,\ 0 < \underline{V}_{kl}^j \le V_{kl}^j \le \overline{V}_{kl}^j,\ k\neq l,$$
$$j=1,2,\ldots,x,\ k=1,2,\ldots,N,\ l=1,2,\ldots,N,\ V^j \in V_I\Big\};$$

$$W_I := \Big\{W^j = (W_{kl}^j)_{N\times N} : \underline{W}^j \le W^j \le \overline{W}^j,\ 0 < \underline{W}_{kl}^j \le W_{kl}^j \le \overline{W}_{kl}^j,\ k\neq l,$$
$$j=1,2,\ldots,x,\ k=1,2,\ldots,N,\ l=1,2,\ldots,N,\ W^j \in W_I\Big\}; \qquad (7)$$

Let $\tilde{z}(t) = \frac{1}{N}\sum_{k=1}^{N} z_k(t)$. Then, one gets

$$D^{\gamma}\tilde{z}(t) = \frac{1}{N}\sum_{k=1}^{N} D^{\gamma} z_k(t)$$
$$= \frac{1}{N}\sum_{k=1}^{N}\Big[-Pz_k(t) + Qg\big(z_k(t)\big) + Rh\big(z_k(t-\mu_1)\big)$$

$$+\sum_{l=1}^{N}\alpha_1 V_{kl}^1\Lambda_1 z_l(t)+\sum_{l=1}^{N}\alpha_2 V_{kl}^2\Lambda_2 z_l(t)$$

$$+\cdots+\sum_{l=1}^{N}\alpha_x V_{kl}^x\Lambda_x z_l(t)+\sum_{l=1}^{N}\beta_1 W_{kl}^1\Upsilon_1 z_l(t-\mu_2)+\sum_{l=1}^{N}\beta_2 W_{kl}^2\Upsilon_2 z_l(t-\mu_2)$$

$$+\cdots+\sum_{l=1}^{N}\beta_x W_{kl}^x\Upsilon_x z_l(t-\mu_2)\Big]$$

$$=-\frac{P}{N}\sum_{k=1}^{N}z_k(t)+\frac{1}{N}\sum_{k=1}^{N}Qg\big(z_k(t)\big)+\frac{1}{N}\sum_{k=1}^{N}Rh\big(z_k(t-\mu_1)\big)$$

$$+\frac{1}{N}\sum_{l=1}^{N}\alpha_1\Big(\sum_{k=1}^{N}V_{kl}^1\Big)\Lambda_1 z_l(t)$$

$$+\frac{1}{N}\sum_{l=1}^{N}\alpha_2\Big(\sum_{k=1}^{N}V_{kl}^2\Big)\Lambda_2 z_l(t)+\cdots+\sum_{l=1}^{N}\alpha_x\Big(\sum_{k=1}^{N}V_{kl}^x\Big)\Lambda_x z_l(t)$$

$$+\sum_{l=1}^{N}\beta_1\Big(\sum_{k=1}^{N}W_{kl}^1\Big)\Upsilon_1 z_l(t-\mu_2)$$

$$+\sum_{l=1}^{N}\beta_2\Big(\sum_{k=1}^{N}W_{kl}^2\Big)\Upsilon_2 z_l(t-\mu_2)+\cdots+\sum_{l=1}^{N}\beta_x\Big(\sum_{k=1}^{N}W_{kl}^x\Big)\Upsilon_x z_l(t-\mu_2)$$

$$=-\frac{P}{N}\sum_{k=1}^{N}z_k(t)+\frac{1}{N}\sum_{k=1}^{N}Qg\big(z_k(t)\big)+\frac{1}{N}\sum_{k=1}^{N}Rh\big(z_l(t-\mu_1)\big) \tag{8}$$

It should be noted that $\frac{1}{N}\sum_{j=1}^{x}\sum_{l=1}^{N}\alpha_j\Big(\sum_{k=1}^{N}V_{kl}^j\Big)\Lambda_j z_l(t)=\sum_{j=1}^{x}\sum_{l=1}^{N}\beta_j\Big(\sum_{k=1}^{N}W_{kl}^j\Big)$ $\Upsilon_j z_l(t-\mu_2)=0$ by mean of Definition V^j and W^j, that is $\sum_{k=1}^{N}V_{kl}^j=\sum_{k=1}^{N}W_{kl}^j=0,\ j=1,2,\ldots,x,\ l=1,2,\ldots,N.$

For the system (6), we design the following linear feedback controller:

$$\delta_k(t)=-F_k\Big(z_k(t)-\frac{1}{N}\sum_{k=1}^{N}z_k(t)\Big),\ k=1,2,\ldots,N. \tag{9}$$

Then, we have

$$D^{\gamma}z_k(t)=-Pz_k(t)+Qg\big(z_k(t)\big)+Rh\big(z_k(t-\mu_1)\big)+\sum_{l=1}^{N}\alpha_1 V_{kl}^1\Lambda_1 z_l(t)$$

$$+\sum_{l=1}^{N}\alpha_2 V_{kl}^2\Lambda_2 z_l(t)$$

$$+\cdots+\sum_{l=1}^{N}\alpha_x V_{kl}^x\Lambda_x z_l(t)+\sum_{l=1}^{N}\beta_1 W_{kl}^1\Upsilon_1 z_l(t-\mu_2)$$

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$$+\sum_{l=1}^{N}\beta_2 W_{kl}^2\Upsilon_2 z_l(t-\mu_2)$$

$$+\cdots+\sum_{l=1}^{N}\beta_x W_{kl}^x\Upsilon_x z_l(t-\mu_2)-F_k\Big(z_k(t)-\frac{1}{N}\sum_{k=1}^{N}z_k(t)\Big), \qquad (10)$$

The error vector $y_k(t)=z_k(t)-\frac{1}{N}\sum_{k=1}^{N}z_k(t)$ is given by:

$$\begin{aligned}
D^{\gamma}y_k(t) &= -Py_k(t)+Qg\big(z_k(t)\big)-\frac{1}{N}\sum_{k=1}^{N}Qg\big(z_k(t)\big)+Rh\big(z_k(t-\mu_1)\big)\\
&\quad-\frac{1}{N}\sum_{k=1}^{N}Rh\big(z_k(t-\mu_1)\big)\\
&\quad+\sum_{j=1}^{x}\alpha_j\sum_{l=1}^{N}V_{kl}^{j}\Lambda_j y_l(t)+\sum_{j=1}^{x}\beta_j\sum_{l=1}^{N}W_{kl}^{j}\Upsilon_j y_l(t-\mu_2)-F_k y_k(t)\\
&= -Py_k(t)+Q\tilde{g}\big(y_k(t)\big)+R\tilde{h}\big(y_k(t-\mu_1)\big)+\sum_{j=1}^{x}\alpha_j\sum_{l=1}^{N}V_{kl}^{j}\Lambda_j y_l(t)\\
&\quad+\sum_{j=1}^{x}\beta_j\sum_{l=1}^{N}W_{kl}^{j}\Upsilon_j y_l(t-\mu_2)-F_k y_k(t).
\end{aligned} \qquad (11)$$

where $\tilde{g}\big(y_k(t)\big)=Qg\big(z_k(t)\big)-\frac{1}{N}\sum_{k=1}^{N}g\big(z_k(t)\big),\ \tilde{h}\big(y_k(t-\mu_1)\big)=h\big(z_k(t-\mu_1)\big)-\frac{1}{N}\sum_{k=1}^{N}Rh\big(z_k(t-\mu_1)\big)$.

Remark 2.11 To the best of author's knowledge, many real-world objects can be depicted by multiple coupling strengths of complex dynamical behaviors. Unfortunately, there are no results paid to be investigated on fractional-order complex dynamical behaviors, especially neural networks systems. Consequently, it's far very necessary and important to further investigate the synchronization analysis of FOCNNs with multiple weights.

In this article, the following definition and assumption condition will be needed.

Definition 2.12 The complex structure on MWCFCNNs with linear coupling delay under uncertainty (11) is asymptotically synchronized if

$$\lim_{t\to+\infty}\Big\|z_k(t)-\frac{1}{N}\sum_{k=1}^{N}z_k(t)\Big\|=0,\ k=1,2,\ldots,N. \qquad (12)$$

Assumption $[\mathcal{A}_1]$: The non linear activation function $g_k(\cdot),\ h_k(\cdot)$ satisfies the Lipschitz continuous if there exists a constants $\psi_k>0,\ \phi_k>0$ such that

$$\big|g_k(\chi_1)-g_k(\chi_2)\big|\le\psi_k\big|\chi_1-\chi_2\big|,\ k=1,2,\ldots,n,\ \chi_1,\ \chi_2\in\mathbb{R},$$
$$\big|h_k(\chi_1)-h_k(\chi_2)\big|\le\phi_k\big|\chi_1-\chi_2\big|,\ k=1,2,\ldots,n,\ \chi_1,\ \chi_2\in\mathbb{R},$$

where $\big|(\cdot)\big|$ is the absolute value.

3 Main Results

For the sake of convenience, we define

$$\begin{aligned}
\hat{q}_{kl} &= \max\{|\underline{q}_{kl}|, |\overline{q}_{kl}|\},\ l = 1, 2, \ldots, n,\ k = 1, 2, \ldots, n \\
\hat{r}_{kl} &= \max\{|\underline{r}_{kl}|, |\overline{r}_{kl}|\},\ l = 1, 2, \ldots, n,\ k = 1, 2, \ldots, n \\
\hat{V}^j_{kk} &= \sum_{l=1, l\neq k}^{N} \overline{V}^j_{lk},\ \hat{V}^j_{kl}\ (k \neq l) = \overline{V}^j_{kl}, \\
&\qquad j = 1, 2, \ldots, x,\ k = 1, 2, \ldots, N,\ l = 1, 2, \ldots, N \\
\hat{W}^j_{kk} &= \sum_{l=1, l\neq k}^{N} \overline{W}^j_{lk},\ \hat{W}^j_{kl}\ (k \neq l) = \overline{W}^j_{kl}, \\
&\qquad j = 1, 2, \ldots, x,\ k = 1, 2, \ldots, N,\ l = 1, 2, \ldots, N.
\end{aligned}$$

Theorem 3.1 *Under Assumption $[\mathcal{A}_1]$, the MWCFCNNs (6) is robust asymptotically synchronized under the controller (9) if the following condition holds:*

$$\begin{aligned}
&\max\left\{\phi_i \sum_{l=1}^{n} \hat{r}_{li},\ i = 1, 2, \ldots, n\right\} \\
&\quad + \max\left\{\sum_{j=1}^{x} \overline{\beta}_j \overline{\upsilon}^j_i \Big(\sum_{l=1}^{N} \hat{W}^j_{lk}\Big),\ k = 1, 2, \ldots, N,\ i = 1, 2, \ldots, n\right\} \\
&\quad < \min\left\{F_k + \underline{p}_i - \psi_i \sum_{l=1}^{n} \hat{q}_{li}\right. \\
&\quad \left. - \sum_{j=1}^{x} \overline{\alpha}_j \overline{\lambda}^j_i \Big(\sum_{l=1}^{N} \hat{V}^j_{lk}\Big),\ k = 1, 2, \ldots, N,\ i = 1, 2, \ldots, n\right\} \sin\left(\frac{\gamma\pi}{2}\right).
\end{aligned}$$

Proof For error system (11), we consider the following Lyapunov functional:

$$H(t) = \sum_{k=1}^{N} \|y_k(t)\| \tag{13}$$

Then, applying the Caputo-fractional derivative for Lyapunov functional (13), and by means of Lemma 2.10, Assumption $[\mathcal{A}_1]$, one has

$$\begin{aligned}
D^\gamma H(t) &= D^\gamma\left[\sum_{k=1}^{N} \|y_k(t)\|\right] \\
&= D^\gamma\left[\sum_{k=1}^{N}\sum_{i=1}^{n} |y_{ki}(t)|\right] \\
&\leq \sum_{k=1}^{N}\sum_{i=1}^{n} \operatorname{sgn} y_{ki}(t) D^\gamma y_{ki}(t)
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{k=1}^{N} \operatorname{sgn} y_k(t) D^{\gamma} y_k(t) \\
&= \sum_{k=1}^{N} \operatorname{sgn} y_k(t) \Big[- P y_k(t) + Q \tilde{g}\big(y_k(t)\big) \\
&\quad + R \tilde{h}\big(y_k(t-\mu_1)\big) + \sum_{j=1}^{x} \alpha_j \sum_{l=1}^{N} V_{kl}^{j} \Lambda_j y_l(t) \\
&\quad + \sum_{j=1}^{x} \beta_j \sum_{l=1}^{N} W_{kl}^{j} \Upsilon_j y_l(t-\mu_2) - F_k y_k(t) \Big] \\
&\le - \sum_{k=1}^{N} \sum_{i=1}^{n} p_i |y_{ki}(t)| - \sum_{k=1}^{N} \sum_{i=1}^{n} F_k |y_{ki}(t)| + \sum_{k=1}^{N} \sum_{i=1}^{n} \sum_{l=1}^{n} |q_{il}| |g_l\big(y_{ki}(t)\big)| \\
&\quad + \sum_{k=1}^{N} \sum_{i=1}^{n} \sum_{l=1}^{n} |r_{il}| |h_l\big(y_{ki}(t-\mu_1)\big)| + \sum_{j=1}^{x} \alpha_j \sum_{i=1}^{n} \lambda_i^j \Big[\sum_{k=1}^{N} \sum_{l=1}^{N} |V_{kl}^{j}| |y_{li}(t)| \Big] \\
&\quad + \sum_{j=1}^{x} \beta_j \sum_{i=1}^{n} \upsilon_i^j \Big[\sum_{k=1}^{N} \sum_{l=1}^{N} |W_{kl}^{j}| |y_{li}(t-\mu_2)| \Big] \\
&\le - \sum_{k=1}^{N} \sum_{i=1}^{n} \underline{p}_i |y_{ki}(t)| - \sum_{k=1}^{N} \sum_{i=1}^{n} F_k |y_{ki}(t)| + \sum_{k=1}^{N} \sum_{i=1}^{n} \sum_{l=1}^{n} \hat{q}_{il} \psi_l |y_{kl}(t)| \\
&\quad + \sum_{k=1}^{N} \sum_{i=1}^{n} \sum_{l=1}^{n} \hat{r}_{il} \phi_l |y_{kl}(t-\mu_1)| + \sum_{j=1}^{x} \overline{\alpha}_j \sum_{i=1}^{n} \overline{\lambda}_i^j \Big[\sum_{k=1}^{N} \sum_{l=1}^{N} \hat{V}_{kl}^{j} |y_{li}(t)| \Big] \\
&\quad + \sum_{j=1}^{x} \overline{\beta}_j \sum_{i=1}^{n} \overline{\upsilon}_i^j \Big[\sum_{k=1}^{N} \sum_{l=1}^{N} \hat{W}_{kl}^{j} |y_{li}(t-\mu_2)| \Big] \\
&\le - \sum_{k=1}^{N} \sum_{i=1}^{n} \Big[F_k + \underline{p}_i - \psi_i \sum_{l=1}^{n} \hat{q}_{li} - \sum_{j=1}^{x} \overline{\alpha}_j \overline{\lambda}_i^j \Big(\sum_{l=1}^{N} \hat{V}_{lk}^{j} \Big) \Big] |y_{ki}(t)| \\
&\quad + \sum_{k=1}^{N} \sum_{i=1}^{n} \Big[\phi_i \sum_{l=1}^{n} \hat{r}_{li} \Big] |y_{ki}(t-\mu_1)| \\
&\quad + \sum_{k=1}^{N} \sum_{i=1}^{n} \Big[\sum_{j=1}^{x} \overline{\beta}_j \overline{\upsilon}_i^j \Big(\sum_{l=1}^{N} \hat{W}_{lk}^{j} \Big) \Big] |y_{ki}(t-\mu_2)| \\
&\le -\zeta \sum_{k=1}^{N} \sum_{i=1}^{n} |y_{ki}(t)| + \varsigma \sum_{k=1}^{N} \sum_{i=1}^{n} |y_{ki}(t-\mu_1)| + \eta \sum_{k=1}^{N} \sum_{i=1}^{n} |y_{ki}(t-\mu_2)| \\
&= -\zeta H(t) + \varsigma H(t-\mu_1) + \eta H(t-\mu_2) \qquad (14)
\end{aligned}
$$

where

$$\zeta = \min \Big\{ F_k + \underline{p}_i - \psi_i \sum_{l=1}^{n} \hat{q}_{li}$$

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$$- \sum_{j=1}^{x} \overline{\alpha}_j \overline{\lambda}_i^j \Big(\sum_{l=1}^{N} \hat{V}_{lk}^j \Big), \ k = 1, 2, \dots, N, \ i = 1, 2, \dots, n \Big\} > 0,$$

$$\varsigma = \max \Big\{ \phi_i \sum_{l=1}^{n} \hat{r}_{li}, \ i = 1, 2, \dots, n \Big\} > 0,$$

$$\eta = \max \Big\{ \sum_{j=1}^{x} \overline{\beta}_j \overline{\upsilon}_i^j \Big(\sum_{l=1}^{N} \hat{W}_{lk}^j \Big), \ k = 1, 2, \dots, N, \ i = 1, 2, \dots, n \Big\} > 0.$$

Consider the following linear fractional order system with multiple time delays

$$\begin{cases} D^{\gamma} E(t) = -\zeta E(t) + \varsigma E(t - \mu_1) + \eta E(t - \mu_2) \\ E(\chi) = \hat{h}(\chi), \ \chi \in [-\hat{\mu}, 0] \end{cases} \tag{15}$$

and assume $E(t) \geq 0$ $\big(E(t) \in \mathbb{R}\big)$. A Laplace transform of (15) is

$$\begin{aligned} s^{\gamma} E(s) - s^{\gamma-1} E(0) &= -\zeta E(s) + \varsigma \int_0^{+\infty} e^{-st} E(t - \mu_1)\, \mathrm{d}t + \eta \int_0^{+\infty} e^{-st} E(t - \mu_2)\, \mathrm{d}t \\ &= -\zeta E(s) + \varsigma \Big[\int_{-\mu_1}^{+\infty} e^{-s(\kappa+\mu_1)} E(\kappa)\, \mathrm{d}\kappa \Big] \\ &\quad + \eta \Big[\int_{-\mu_2}^{+\infty} e^{-s(\kappa+\mu_2)} E(\kappa)\, \mathrm{d}\kappa \Big] \\ &= -\zeta E(s) + \varsigma e^{-s\mu_1} \int_{-\mu_1}^{0} e^{-s\kappa} E(\kappa)\, \mathrm{d}\kappa + \eta e^{-s\mu_2} \int_{-\mu_2}^{0} e^{-s\kappa} E(\kappa)\, \mathrm{d}\kappa \\ &\quad + \varsigma e^{-s\mu_1} \int_0^{+\infty} e^{-s\kappa} E(\kappa)\, \mathrm{d}\kappa + \eta e^{-s\mu_2} \int_0^{+\infty} e^{-s\kappa} E(\kappa)\, \mathrm{d}\kappa \\ &= -\zeta E(s) + \varsigma e^{-s\mu_1} E(s) + \eta e^{-s\mu_2} E(s) \\ &\quad + \varsigma e^{-s\mu_1} \int_{-\mu_1}^{0} e^{-s\kappa} E(\kappa)\, \mathrm{d}\kappa \\ &\quad + \eta e^{-s\mu_2} \int_{-\mu_2}^{0} e^{-s\kappa} E(\kappa)\, \mathrm{d}\kappa, \end{aligned} \tag{16}$$

where $E(s)$ is Laplace transform of $E(t)$. An equivalence of (16) is

$$\Delta(s) E(s) = d_1(s) \tag{17}$$

where

$$\Delta(s) = \big(s^{\gamma} + \zeta - \varsigma e^{-s\mu_1} - \eta e^{-s\mu_2} \big)$$

$$d_1(s) = s^{\gamma-1} E(0) + \varsigma e^{-s\mu_1} \int_{-\mu_1}^{0} e^{-s\kappa} E(\kappa)\, \mathrm{d}\kappa + \eta e^{-s\mu_2} \int_{-\mu_2}^{0} e^{-s\kappa} E(\kappa)\, \mathrm{d}\kappa$$

Now, we will prove that there is no pure imaginary roots for characteristic equation of $det(\Delta(s)) = 0$ for any $\mu_1, \ \mu_2 \geq 0$. Suppose that there exists a pure imaginary roots for any $\mu_1, \ \mu_2 \geq 0$, that is

$$s = \varrho i = |\varrho| \Big[\cos \Big(\frac{\pi}{2} \Big) + i \sin \Big(\frac{\pm\pi}{2} \Big) \Big],$$

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where ϱ is a real number. If

$$\varrho < 0,\ s = \varrho i = |\varrho|\Big[\cos\Big(\frac{\pi}{2}\Big) - i\sin\Big(\frac{\pi}{2}\Big)\Big], \quad \text{while if}$$

$$\varrho > 0,\ s = \varrho i = |\varrho|\Big[\cos\Big(\frac{\pi}{2}\Big) + i\sin\Big(\frac{\pi}{2}\Big)\Big]. \tag{18}$$

Substitute them into $det(\Delta(s)) = 0$, one has

$$|\varrho|^{\gamma}\Big[\cos\Big(\frac{\gamma\pi}{2}\Big) + i\sin\Big(\frac{\pm\gamma\pi}{2}\Big)\Big] + \zeta - \varsigma\big[\cos\big(\varrho\mu_1\big) - i\sin(\varrho\mu_1)\big]$$
$$-\eta\big[\cos\big(\varrho\mu_2\big) - i\sin\big(\varrho\mu_2\big)\big] = 0 \tag{19}$$

Separating real and imaginary parts of (19), one has

$$|\varrho|^{\gamma}\cos\Big(\frac{\gamma\pi}{2}\Big) + \zeta - \varsigma\eta\cos\big(\varrho\mu_1\big) - \eta\cos\big(\varrho\mu_2\big) = 0 \tag{20}$$

and

$$|\varrho|^{\gamma}\sin\Big(\pm\frac{\gamma\pi}{2}\Big) + \varsigma\sin\big(\varrho\mu_1\big) + \eta\sin\big(\varrho\mu_2\big) = 0. \tag{21}$$

From (20) and (21), it can get

$$|\varrho|^{2\gamma} + 2\zeta|\varrho|^{\gamma}\cos\Big(\frac{\gamma\pi}{2}\Big) + \zeta^2 - 2\varsigma\cos\varrho\big(\mu_1 - \mu_2\big) - \big(\varsigma^2 + \eta^2\big) = 0.$$

When $(\varsigma + \eta)^2 < \zeta^2\sin^2(\pm\frac{\gamma\pi}{2})$, because $\varsigma,\ \eta \geq 0$, one has
$|\varrho|^{2\gamma} + 2\zeta|\varrho|^{\gamma}\cos(\frac{\gamma\pi}{2}) + \zeta^2 - 2\varsigma\eta\cos\varrho(\mu_1 - \mu_2) - (\varsigma^2 + \eta^2)$

$$= |\varrho|^{2\gamma} + 2\zeta|\varrho|^{\gamma}\cos\Big(\frac{\gamma\pi}{2}\Big) + \zeta^2 + 2\varsigma\eta\Big(1 - \cos\varrho\big(\mu_1 - \mu_2\big)\Big) - \big(\varsigma + \eta\big)^2$$
$$> \Big[|\varrho| + \zeta\cos\Big(\frac{\gamma\pi}{2}\Big)\Big]^2 + 2\varsigma\eta\big[1 - \cos\varrho(\mu_1 - \mu_2)\big]$$
$$\geq 0.$$

Based from condition of Theorem 3.1, we have $\varsigma + \eta < \zeta\sin(\frac{\gamma\pi}{2})$ which implies the characteristic equation $det\big(\Delta(s)\big) = 0$ has no purely imaginary roots for any $\mu_1,\ \mu_2 \geq 0$, which means the zero solution of system (15) is globally asymptotically stable. Then, by virtue of Lemma 2.8, we have $0 \leq H(t) \leq E(t)$, and based on above discussion, we can get $\sum_{k=1}^{N}\|y_k(t)\| \to 0$ and $\|y_k(t)\| \to 0$ as $t \to +\infty$. Therefore we declare that, MWCFCNNs (6) achieves robust asymptotically synchronization under the controller (9). □

The following kinds of MWCFCNNs are also very interesting issues. One without difficulty derives the following asymptotic synchronization criteria on MWCFCNNs (22) and MWCFCNNs (23) from the proof of Theorem 3.1 and based on the comparison result in Theorem 1 of Ref [65].

Case 1: If $W^j = 0$ $(j = 1, 2\ldots, x)$, let $\mu_1 = \mu$, MWCFCNNs (6) is turned into the following expression:

$$D^{\gamma}z_k(t) = -Pz_k(t) + Qg\big(z_k(t)\big) + Rh\big(z_k(t - \mu)\big)$$

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$$+\sum_{l=1}^{N}\alpha_1 V_{kl}^1\Lambda_1 z_l(t)+\sum_{l=1}^{N}\alpha_2 V_{kl}^2\Lambda_2 z_l(t)$$

$$+\cdots+\sum_{l=1}^{N}\alpha_x V_{kl}^x\Lambda_x z_l(t) \tag{22}$$

Corollary 3.2 *Under Assumption* $[\mathcal{A}_1]$*, the MWCFCNNs (22) is robust asymptotically synchronized under the controller (9) if the following condition holds:*

$$\max\left\{\phi_i\sum_{l=1}^{n}\hat{r}_{li},\ i=1,2,\ldots,n\right\}<\min\left\{F_k+\underline{p}_i-\psi_i\sum_{l=1}^{n}\hat{q}_{li}-\sum_{j=1}^{x}\overline{\alpha}_j\overline{\lambda}_i^j\left(\sum_{l=1}^{N}\hat{V}_{lk}^j\right),\right.$$

$$\left.k=1,2,\ldots,N,\ i=1,2,\ldots,n\right\}\sin\left(\frac{\gamma\pi}{2}\right).$$

Case 2: If $V^j=0$ $(j=1,2\ldots,x=0)$, MWCFCNNs (6) is turned into the following expression:

$$D^{\gamma}z_k(t)=-Pz_k(t)+Qg\big(z_k(t)\big)+Rh\big(z_k(t-\mu_1)\big)+\sum_{l=1}^{N}\beta_1 W_{kl}^1\Upsilon_1 z_l(t-\mu_2)$$

$$+\sum_{l=1}^{N}\beta_2 W_{kl}^2\Upsilon_2 z_l(t-\mu_2)+\cdots+\sum_{l=1}^{N}\beta_x W_{kl}^x\Upsilon_x z_l(t-\mu_2), \tag{23}$$

Corollary 3.3 *Under Assumption* $[\mathcal{A}_1]$*, the MWCFCNNs (23) is robust asymptotically synchronized under the controller (9) if the following condition holds:*

$$\max\left\{\phi_i\sum_{l=1}^{n}\hat{r}_{li},\ i=1,2,\ldots,n\right\}$$

$$+\max\left\{\sum_{j=1}^{x}\overline{\beta}_j\overline{\upsilon}_i^j\left(\sum_{l=1}^{N}\hat{W}_{lk}^j\right),\ k=1,2,\ldots,N,\ i=1,2,\ldots,n\right\}$$

$$<\min\left\{F_k+\underline{p}_i-\psi_i\sum_{l=1}^{n}\hat{q}_{li},\ k=1,2,\ldots,N,\ i=1,2,\ldots,n\right\}\sin\left(\frac{\gamma\pi}{2}\right).$$

Theorem 3.4 *Suppose that the Assumption* $[\mathcal{A}_1]$ *hold. If* $\zeta>0,\ \varsigma>0,\ \eta>0$ *be known constants and* $\varsigma+\eta<\zeta$*, then MWCFCNNs (6) is robust asymptotically synchronized under the controller (9), if there exists a positive diagonal matrix* $M\in\mathbb{R}^{n\times n}$ *such that following condition holds,*

1. $I_N\otimes\Big(-M\underline{P}-\overline{P}M+\Big(\delta_q+\delta_r+\sum_{j=1}^{x}\overline{\alpha}_j\delta_v^j+\sum_{j=1}^{x}\overline{\beta}_j\delta_w^j\Big)M^2+\Psi+\sum_{j=1}^{x}\overline{\alpha}_j\delta_\lambda^j I_n+\zeta M\Big)-\big(2F\otimes M\big)<0,$
2. $I_N\otimes(\Phi-\varsigma M)<0,$
3. $I_N\otimes\Big(\sum_{j=1}^{x}\overline{\beta}_j\delta_u^j I_n-\eta M\Big)<0,$

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where $\delta_q = \sum_{k=1}^{n}\sum_{l=1}^{n}\hat{q}_{kl}^2$, $\delta_r = \sum_{k=1}^{n}\sum_{l=1}^{n}\hat{r}_{kl}^2$, $\delta_v^j = \sum_{j=1}^{x}\sum_{k=1}^{N}\sum_{l=1}^{N}\hat{V}_{kl}^{j\,2}$, $\delta_w^j = \sum_{j=1}^{x}\sum_{k=1}^{N}\sum_{l=1}^{N}\hat{W}_{kl}^{j\,2}$, $\delta_\lambda^j = \sum_{j=1}^{x}\sum_{k=1}^{n}\overline{\lambda}_k^{j\,2}$, $\delta_u^j = \sum_{j=1}^{x}\sum_{k=1}^{n}\overline{\upsilon}_k^{j\,2}$, $\Psi = diag\{\psi_1^2,\ldots,\psi_n^2\}$, $\Phi = diag\{\phi_1^2,\ldots,\phi_n^2\}$ and $F = diag\{F_1,\ldots,F_N\}$.

Proof For error system (11), we take the following Lyapunov functional:

$$H(t) = \sum_{k=1}^{N} y_k^T(t) M y_k(t) = y^T(t)\big(I_N \otimes M\big)y(t) \tag{24}$$

Then, applying the Caputo-fractional derivative for Lyapunov functional (24) and by utilizing Lemma 2.4, we have

$$\begin{aligned}
D^\gamma H(t) &\le 2\sum_{k=1}^{N} y_k^T(t) M\{D^\gamma y_k(t)\}\\
&= 2\sum_{k=1}^{N} y_k^T(t) M\Big\{-Py_k(t) + Q\tilde{g}\big(y_k(t)\big) + R\tilde{h}\big(y_k(t-\mu_1)\big)\\
&\quad + \sum_{j=1}^{x}\alpha_j \sum_{l=1}^{N} V_{kl}^j \Lambda_j y_l(t)\\
&\quad + \sum_{j=1}^{x}\beta_j \sum_{l=1}^{N} W_{kl}^j \Upsilon_j y_l(t-\mu_2) - F_k y_k(t)\Big\}\\
&= -2\sum_{k=1}^{N} y_k^T(t) M P y_k(t) - 2\sum_{k=1}^{N} y_k^T(t) M F y_k(t) + 2\sum_{k=1}^{N} y_k^T(t) M Q\tilde{g}\big(y_k(t)\big)\\
&\quad + 2\sum_{k=1}^{N} y_k^T(t) M R\tilde{h}\big(y_k(t-\mu_1)\big) + 2\sum_{j=1}^{x}\alpha_j\Big(\sum_{k=1}^{N}\sum_{l=1}^{N} V_{kl}^j y_k^T(t) M \Lambda_j y_l(t)\Big)\\
&\quad + 2\sum_{j=1}^{x}\beta_j\Big(\sum_{k=1}^{N}\sum_{l=1}^{N} W_{kl}^j y_k^T(t) M \Upsilon_j y_l(t-\mu_2)\Big).
\end{aligned} \tag{25}$$

Based on Assumption $[\mathcal{A}_1]$, one gets

$$\begin{aligned}
2\sum_{k=1}^{N} y_k^T(t) M Q\tilde{g}\big(y_k(t)\big) &\le \sum_{k=1}^{N} y_k^T(t) M Q Q^T M y_k(t) + \sum_{k=1}^{N} y_k^T(t)\Psi y_k(t)\\
&\le y^T(t)\Big(I_N \otimes (\delta_q M^2 + \Psi)\Big)y(t).
\end{aligned} \tag{26}$$

Similarly,

$$\begin{aligned}
2\sum_{k=1}^{N} y_k^T(t) M R\tilde{h}\big(y_k(t-\mu_1)\big) &\le y^T(t)\Big(I_N \otimes \delta_r M^2\Big)y(t)\\
&\quad + y^T(t-\mu_1)\Big(I_N \otimes \Phi\Big)y(t-\mu_1)
\end{aligned} \tag{27}$$

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From Lemma 2.9, we can get

$$
\begin{aligned}
2\sum_{j=1}^{x}\alpha_j\Big(\sum_{k=1}^{N}\sum_{l=1}^{N}V_{kl}^{j}y_k^T(t)M\Lambda_j y_l(t)\Big) &= 2\sum_{j=1}^{x}\alpha_j y^T(t)\big(V^j\otimes M\big)\big(I_N\otimes\Lambda_j\big)y(t)\\
&\le \sum_{j=1}^{x}\alpha_j y^T(t)\big[\big(V^j\otimes M\big)\big(V^j\otimes M\big)^T\big]y(t)\\
&\quad+\sum_{j=1}^{x}\alpha_j y^T(t)\big[\big(I_N\otimes\Lambda_j\big)\big(I_N\otimes\Lambda_j\big)^T\big]y(t)\\
&= \sum_{j=1}^{x}\alpha_j y^T(t)\big(V^jV^{jT}\otimes M^2\big)y(t)\\
&\quad+\sum_{j=1}^{x}\alpha_j y^T(t)\big(I_N\otimes\Lambda_j^2\big)y(t)\\
&\le \sum_{j=1}^{x}\overline{\alpha}_j\delta_v^j y^T(t)\big(I_N\otimes M^2\big)y(t)\\
&\quad+\sum_{j=1}^{x}\overline{\alpha}_j\delta_\lambda^j y^T(t)y(t). \qquad (28)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
2\sum_{j=1}^{x}\beta_j\Big(\sum_{k=1}^{N}\sum_{l=1}^{N}W_{kl}^{j}y_k^T(t)M\Upsilon_j y_l(t-\mu_2)\Big) &\le \sum_{j=1}^{x}\overline{\beta}_j\delta_w^j y^T(t)\big(I_N\otimes M^2\big)y(t)\\
&\quad+\sum_{j=1}^{x}\overline{\beta}_j\delta_u^j y^T(t-\mu_2)y(t-\mu_2). \qquad (29)
\end{aligned}
$$

Combining (25)–(29), we have

$$
\begin{aligned}
D^\gamma H(t) &\le y^T(t)\big(I_N\otimes(-M\underline{P}-\overline{P}M)\big)y(t)+y^T(t)\Big(I_N\otimes\big((\delta_q+\delta_r)M^2+\Psi\big)\Big)y(t)\\
&\quad+y^T(t-\mu_1)\Big(I_N\otimes\Phi\Big)y(t-\mu_1)+y^T(t)\Big(I_N\otimes\sum_{j=1}^{x}\overline{\alpha}_j\delta_v^jM^2\Big)y(t)\\
&\quad+y^T(t)\Big(I_N\otimes\sum_{j=1}^{x}\overline{\alpha}_j\delta_\lambda^jI_n\Big)y(t)+y^T(t)\Big(I_N\otimes\sum_{j=1}^{x}\overline{\beta}_j\delta_w^jM^2\Big)y(t)\\
&\quad+y^T(t-\mu_2)\Big(I_N\otimes\sum_{j=1}^{x}\overline{\beta}_j\delta_u^jI_n\Big)y(t-\mu_2)-y^T(t)\Big(2F\otimes M\Big)y(t)\\
&\le y^T(t)\Big[-\big(2F\otimes M\big)+I_N\otimes\Big(-M\underline{P}-\overline{P}M\\
&\quad+\Big(\delta_q+\delta_r+\sum_{j=1}^{x}\overline{\alpha}_j\delta_v^j+\sum_{j=1}^{x}\overline{\beta}_j\delta_w^j\Big)M^2+\Psi
\end{aligned}
$$

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$$
\begin{aligned}
&+\sum_{j=1}^{x}\overline{\alpha}_j\delta_\lambda^j I_n+\zeta M\Big)\Big]y(t)-\zeta y^T(t)\Big[I_N\otimes M\Big]y(t)\\
&+y^T(t-\mu_1)\Big[I_N\otimes(\Phi-\varsigma M)\Big]y(t-\mu_1)\\
&+\varsigma y^T(t-\mu_1)\Big[I_N\otimes M\Big]y(t-\mu_1)\\
&+y^T(t-\mu_2)\Big[I_N\otimes\Big(\sum_{j=1}^{x}\overline{\beta}_j\delta_u^j I_n-\eta M\Big)\Big]y(t-\mu_2)\\
&+\eta y^T(t-\mu_2)\Big[I_N\otimes M\Big]y(t-\mu_2).
\end{aligned}
\tag{30}
$$

According to the conditions $1-3$ of Theorem 3.4, we can obtain

$$
\begin{aligned}
D^\gamma H(t)\le&-\zeta y^T(t)\Big[I_N\otimes M\Big]y(t)+\varsigma y^T(t-\mu_1)\Big[I_N\otimes M\Big]y(t-\mu_1)\\
&+\eta y^T(t-\mu_2)\Big[I_N\otimes M\Big]y(t-\mu_2)\\
=&-\zeta H(t)+\varsigma H(t-\mu_1)+\eta H(t-\mu_2)
\end{aligned}
\tag{31}
$$

Then, similar to the proof of Theorem 3.1, we can obtain $H(t)\to 0$ as $t\to+\infty$. i.e.,$\sum_{k=1}^{N}y_k^T(t)My_k(t)=\sum_{k=1}^{N}\sum_{i=1}^{n}m_i y_{ki}^2(t)\to 0$ as $t\to+\infty$. Hence, we conclude that MWCFCNNs (6) realizes robust asymptotically synchronization under the state feedback controller (9). The proof is ended. □

Corollary 3.5 *Suppose that the Assumption $[\mathcal{A}_1]$ hold. If $\zeta>0,\ \varsigma>0$ be known constants and $\varsigma<\zeta$, then MWCFCNNs (22) is robust asymptotically synchronized under the controller (9), if there exists a positive diagonal matrix $M\in\mathbb{R}^{n\times n}$ such that following condition holds,*

1. $I_N\otimes\Big(-M\underline{P}-\overline{P}M+\Big(\delta_q+\delta_r+\sum_{j=1}^{x}\overline{\alpha}_j\delta_v^j\Big)M^2+\Psi+\sum_{j=1}^{x}\overline{\alpha}_j\delta_\lambda^j I_n+\zeta M\Big)-\Big(2F\otimes M\Big)<0,$
2. $I_N\otimes(\Phi-\varsigma M)<0,$

where $\delta_q,\ \delta_r,\ \delta_v^j,\ \delta_\lambda^j,\ \Phi,\ \Psi$ and F are already defined in Theorem 3.4.

Corollary 3.6 *Suppose that the Assumption $[\mathcal{A}_1]$ hold. If $\tilde{\zeta}>0,\ \varsigma,\ \eta>0$ be known constants and $\varsigma+\eta<\tilde{\zeta}$, then MWCFCNNs (23) is robust asymptotically synchronized under the controller (9), if there exists a positive diagonal matrix $M\in\mathbb{R}^{n\times n}$ such that following condition holds,*

1. $I_N\otimes\Big(-M\underline{P}-\overline{P}M+\Big(\delta_q+\delta_r+\sum_{j=1}^{x}\overline{\beta}_j\delta_w^j\Big)M^2+\Psi+\tilde{\zeta}M\Big)-\Big(2F\otimes M\Big)<0,$
2. $I_N\otimes(\Phi-\varsigma M)<0,$
3. $I_N\otimes\Big(\sum_{j=1}^{x}\overline{\beta}_j\delta_u^j I_n-\eta M\Big)<0,$

where $\delta_q,\ \delta_r,\ \delta_w^j,\ \delta_u^j,\ \Phi,\ \Psi$ and F are already defined in Theorem 3.4.
For $j=1,2,\ldots x$, when $\underline{\alpha}_j=\overline{\alpha}_j=\alpha_j$, $\underline{\beta}_j=\overline{\beta}_j=\beta_j$, $\underline{P}=\overline{P}=P$, $\underline{Q}=\overline{Q}=Q$, $\underline{R}=\overline{R}=R$, $\underline{\Lambda}_j=\overline{\Lambda}_j=\Lambda_j$, $\underline{\Upsilon}_j=\overline{\Upsilon}_j=\Upsilon$, $\underline{V}^j=\overline{V}^j=V^j$, $\underline{W}^j=\overline{W}^j=W^j$, i.e., MWCFCNNs (6) is without uncertainty parameter, then we have the following interesting results.

Theorem 3.7 *Under Assumption* $[\mathcal{A}_1]$, *the MWCFCNNs* (6) *is asymptotically synchronized under the controller* (9) *if the following condition holds:*

$$\max\left\{\phi_i \sum_{l=1}^{n} |r_{li}|,\ i = 1, 2, \ldots, n\right\}$$

$$+\max\left\{\sum_{j=1}^{x} \beta_j \Upsilon_i^j \left(\sum_{l=1}^{N} |W_{lk}^j|\right),\ k = 1, 2, \ldots, N,\ i = 1, 2, \ldots, n\right\}$$

$$< \min\left\{F_k + p_i - \psi_i \sum_{l=1}^{n} |q_{li}|\right.$$

$$\left. - \sum_{j=1}^{x} \alpha_j \lambda_i^j \left(\sum_{l=1}^{N} |V_{lk}^j|\right),\ k = 1, 2, \ldots, N,\ i = 1, 2, \ldots, n\right\} \sin\left(\frac{\gamma\pi}{2}\right).$$

Corollary 3.8 *Under Assumption* $[\mathcal{A}_1]$, *the MWCFCNNs* (22) *is asymptotically synchronized under the controller* (9) *if the following condition holds:*

$$\max\left\{\phi_i \sum_{l=1}^{n} |r_{li}|,\ i = 1, 2, \ldots, n\right\} < \min\left\{F_k + p_i - \psi_i \sum_{l=1}^{n} |q_{li}| - \sum_{j=1}^{x} \alpha_j \lambda_i^j \left(\sum_{l=1}^{N} |V_{lk}^j|\right),\right.$$

$$\left. k = 1, 2, \ldots, N,\ i = 1, 2, \ldots, n\right\} \sin\left(\frac{\gamma\pi}{2}\right).$$

Corollary 3.9 *Under Assumption* $[\mathcal{A}_1]$, *the MWCFCNNs* (23) *is asymptotically synchronized under the controller* (9) *if the following condition holds:*

$$\max\left\{\phi_i \sum_{l=1}^{n} |r_{li}|,\ i = 1, 2, \ldots, n\right\}$$

$$+\max\left\{\sum_{j=1}^{x} \beta_j \Upsilon_i^j \left(\sum_{l=1}^{N} |W_{lk}^j|\right),\ k = 1, 2, \ldots, N,\ i = 1, 2, \ldots, n\right\}$$

$$< \min\left\{F_k + p_i\right.$$

$$\left. - \psi_i \sum_{l=1}^{n} |q_{li}|,\ k = 1, 2, \ldots, N,\ i = 1, 2, \ldots, n\right\} \sin\left(\frac{\gamma\pi}{2}\right).$$

Theorem 3.10 *Suppose that the Assumption* $[\mathcal{A}_1]$ *hold. If* $\zeta_1 > 0$, $\varsigma_1 > 0$, $\eta_1 > 0$ *be known constants and* $\varsigma_1 + \eta_1 < \zeta_1$, *then MWCFCNNs* (6) *is asymptotically synchronized under the controller* (9), *if there exists a positive diagonal matrix* $M \in \mathbb{R}^{n\times n}$ *and constants* ξ_j $(j = 1, 2, \ldots, x)$ *such that following condition holds,*

1. $I_N \otimes \Delta - (2F \otimes M) + \sum_{j=1}^{x} \alpha_j \left(2V^j \otimes M\Lambda_j\right) + \sum_{j=1}^{x} \xi_j^{-1} \beta_j \left(W^j \otimes M\Upsilon_j\right)\left(W^j \otimes \Upsilon_j M\right) < 0,$
2. $I_N \otimes (\Phi - \varsigma_1 M) < 0,$
3. $I_N \otimes \left(\sum_{j=1}^{x} \xi_j \beta_j I_n - \eta_1 M\right) < 0,$

where $\Delta = -2MP + MQQ^T M + MRR^T M + \zeta_1 M + \Psi$, F, Ψ and Φ are already defined in Theorem 3.4.

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Proof For error system (12), we choose the same Lyapunov functional in (22), then one has

$$
\begin{aligned}
D^{\gamma}H(t) &\leq 2\sum_{k=1}^{N} y_k^T(t)M\{D^{\gamma}y_k(t)\}\\
&= 2\sum_{k=1}^{N} y_k^T(t)M\bigg\{-Py_k(t)+Q\tilde{g}\big(y_k(t)\big)+R\tilde{h}\big(y_k(t-\mu_1)\big)\\
&\quad+\sum_{j=1}^{x}\alpha_j\sum_{l=1}^{N}V_{kl}^j\Lambda_j y_l(t)\\
&\quad+\sum_{j=1}^{x}\beta_j\sum_{l=1}^{N}W_{kl}^j\Upsilon_j y_l(t-\mu_2)-F_k y_k(t)\bigg\}\\
&\leq -y^T(t)\Big[I_N\otimes(2MP)\Big]y(t)+y^T(t)\Big[I_N\otimes\big(MQQ^TM+MRR^TM+\Psi\big)\Big]y(t)\\
&\quad+y^T(t-\mu_1)\Big[I_N\otimes\Phi\Big]y(t-\mu_1)+2\sum_{j=1}^{x}\alpha_j y^T(t)\Big[V^j\otimes M\Lambda_j\Big]y(t)\\
&\quad+2\sum_{j=1}^{x}\beta_j y^T(t)\Big[W^j\otimes M\Upsilon_j\Big]y(t-\mu_2)-y^T(t)\Big[2F\otimes M\Big]y(t). \qquad (32)
\end{aligned}
$$

It is easy to compute

$$
\begin{aligned}
2\sum_{j=1}^{x}\beta_j y^T(t)\Big[W^j\otimes M\Upsilon_j\Big]y(t-\mu_2) &\leq \sum_{j=1}^{x}\xi_j^{-1}\beta_j y^T(t)\Big[W^j\otimes M\Upsilon_j\Big]\Big[W^j\otimes\Upsilon_j M\Big]y(t)\\
&\quad+\sum_{j=1}^{x}\xi_j\beta_j y^T(t-\mu_2)\big(I_N\otimes I_n\big)y(t-\mu_2). \qquad (33)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
D^{\gamma}H(t) \leq y^T(t)\Bigg[&\Big(I_N\otimes\big(-2MP+MQQ^TM+MRR^TM+\varsigma_1 M+\Psi\big)\Big)\\
&+2\sum_{j=1}^{x}\alpha_j\big(V^j\otimes M\Lambda_j\big)\\
&+\sum_{j=1}^{x}\xi_j^{-1}\beta_j\Big(W^j\otimes M\Upsilon_j\Big)\Big(W^j\otimes\Upsilon_j M\Big)-\big(2F\otimes M\big)\Bigg]y(t)\\
&-\varsigma_1 y^T(t)[I_N\otimes M]y(t)\\
&+y^T(t-\mu_1)\Big[I_N\otimes\big(\Phi-\zeta_1 M\big)\Big]y(t-\mu_1)\\
&+\zeta_1 y^T(t-\mu_1)\Big[I_N\otimes M\Big]y(t-\mu_1)\\
&+y^T(t-\mu_2)\Big[I_N\otimes\Big(\sum_{j=1}^{x}\xi_j\beta_j I_n-\eta_1 M\Big)\Big]y(t-\mu_2)\\
&+\eta_1 y^T(t-\mu_2)\Big[I_N\otimes M\Big]y(t-\mu_2).
\end{aligned}
$$

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According to the conditions 1–3 of Theorem 3.10, we can obtain

$$\begin{aligned} D^{\gamma} H(t) &\le -\zeta_1 y^T(t)\Big[I_N \otimes M\Big]y(t) + \varsigma_1 y^T(t-\mu_1)\Big[I_N \otimes M\Big]y(t-\mu_1) \\ &\quad + \eta_1 y^T(t-\mu_2)\Big[I_N \otimes M\Big]y(t-\mu_2) \\ &= -\zeta_1 H(t) + \varsigma_1 H(t-\mu_1) + \eta_1 H(t-\mu_2). \end{aligned} \tag{34}$$

Then, similar to the proof of Theorem 3.1, the MWCFCNNs error system (11) will be asymptotically stable, i.e., the MWCFCNNs system (6) is globally synchronized via controller (9). The proof is completed. □

Corollary 3.11 *Suppose that the Assumption* $[\mathcal{A}_1]$ *hold. If* $\zeta_1 > 0$, $\varsigma_1 > 0 > 0$ *be known constants and* $\varsigma_1 < \zeta_1$, *then MWCFCNNs (22) is asymptotically synchronized under the controller (9), if there exists a positive diagonal matrix* $M \in \mathbb{R}^{n\times n}$ *such that following condition holds,*

1. $I_N \otimes \Delta - (2F \otimes M) + \sum_{j=1}^{x} \alpha_j \big(2V^j \otimes M\Lambda_j\big) < 0,$
2. $I_N \otimes (\Phi - \varsigma_1 M) < 0,$

where $\Delta = -2MP + MQQ^TM + MRR^TM + \zeta_1 M + \Psi$, F, Ψ and Φ are already defined in Theorem 3.4.

Corollary 3.12 *Suppose that the Assumption* $[\mathcal{A}_1]$ *hold. If* $\tilde{\zeta}_1 > 0$, $\varsigma_1 > 0$, $\eta_1 > 0$ *be known constants and* $\varsigma_1 + \eta_1 < \tilde{\zeta}_1$, *then MWCFCNNs (23) is asymptotically synchronized under the controller (9), if there exists a positive diagonal matrix* $M \in \mathbb{R}^{n\times n}$ *and constants* ξ_j $(j = 1, 2, \ldots, x)$ *such that following condition holds,*

1. $I_N \otimes \Delta - 2F \otimes M + \sum_{j=1}^{x} \xi_j^{-1}\beta_j \Big(W^j \otimes M\Upsilon_j\Big)\Big(W^j \otimes \Upsilon_j M\Big) < 0,$
2. $I_N \otimes (\Phi - \varsigma_1 M) < 0,$
3. $I_N \otimes \Big(\sum_{j=1}^{x} \xi_j \beta_j I_n - \eta_1 M\Big) < 0,$

where $\Delta = -2MP + MQQ^TM + MRR^TM + \tilde{\zeta}_1 M + \Psi$, F, Ψ and Φ are already defined in Theorem 3.4.

Remark 3.13 If weights of FOCNNs is assumed to be single weight, then the model (6) which turns into FOCNNs with single weight. Then the proposed results are also holds to guarantee the asymptotical synchronization criteria for FOCNNs with and without parameter uncertainties, these results not yet considered in the existing works. When $\gamma = 1$, one obtains the integer order case.

4 Numerical Simulations

In this section, two numerical simulations are given to demonstrate the accuracy of the required synchronization results in this paper.

Example 4.1 Consider a multi-weighted complex structure on fractional-order coupled neural networks with linear coupling delay and parameter uncertainty described by

$$D^{0.997} z_k(t) = -Pz_k(t) + Qg\big(z_k(t)\big) + Rh\big(z_k(t-0.1)\big)$$

$$
\begin{aligned}
&+\sum_{l=1}^{4}\alpha_1 V_{kl}^{1}\Lambda_1 z_l(t)+\sum_{l=1}^{4}\alpha_2 V_{kl}^{2}\Lambda_2 z_l(t)\\
&+\sum_{l=1}^{4}\beta_1 W_{kl}^{1}\Upsilon_1 z_l(t-0.2)+\sum_{l=1}^{4}\beta_2 W_{kl}^{2}\Upsilon_2 z_l(t-0.2)\\
&-F_k\Big(z_k(t)-\frac{1}{4}\sum_{l=1}^{4}z_l(t)\Big), \qquad (35)
\end{aligned}
$$

where $k=1,2,3,4$, $g_k(\tau)=\tanh(\tau)$, $h_k(\tau)=\sinh(\tau)$ $(k=1,2)$, $F_k=0.1k$, $k=1,2,3,4$. The parameters in P, Q, R, α_j, β_j, V^j, W^j $(j=1,2)$ in (35) change in some given precision, which is intervalized as below:

$$
\begin{aligned}
\alpha_I &:= \Big\{0.005j \le \alpha_j \le 0.05j,\ j=1,2,\ \ \forall\,\alpha_j \in \alpha_I\Big\};\\
\beta_I &:= \Big\{0.004j \le \beta_j \le 0.04j,\ j=1,2,\ \ \forall\,\beta_j \in \beta_I\Big\};\\
P_I &:= \Big\{P=diag(p_k): \underline{P}\le P\le \overline{P},\ \frac{8}{0.4k+0.4}+0.04\le p_k\le \frac{8}{0.4k+0.4}+0.4,\\
&\qquad k=1,2,\ \forall\,P\in P_I\Big\};\\
Q_I &:= \Big\{Q=(q_{kl})_{n\times n}: \underline{Q}\le Q\le \overline{Q},\ \frac{1}{2k+3l}+0.02\le q_{kl}\le \frac{1}{2k+3l}+0.2,\\
&\qquad k=1,2,\ l=1,2,\ \forall\,Q\in Q_I\Big\};\\
R_I &:= \Big\{R=(r_{kl})_{n\times n}: \underline{R}\le R\le \overline{R},\ \frac{1}{k+2l}+0.04\le r_{kl}\le \frac{1}{k+2l}+0.4,\\
&\qquad k=1,2,\ l=1,2,\ \forall\,R\in R_I\Big\};\\
\Lambda_I &:= \Big\{\Lambda_j=diag(\lambda_k^j): \underline{\Lambda}_j\le \Lambda_j\le \overline{\Lambda}_j,\ \frac{j}{k+1}+0.03\le \lambda_k^j\le \frac{j}{k+1}+0.11,\ j=1,2,\\
&\qquad k=1,2,\ \forall\,\Lambda_j\in \Lambda_I\Big\};\\
\Upsilon_I &:= \Big\{\Upsilon_j=diag(\upsilon_k^j): \underline{\Upsilon}_j\le \Upsilon_j\le \overline{\Upsilon}_j,\ \frac{j}{k+1}+0.05\le \Upsilon_j\le \frac{j}{k+1}+0.5,\ j=1,2,\\
&\qquad k=1,2\ \forall\,\Upsilon_j\in \Upsilon_I\Big\};\\
V_I &:= \Big\{V^j=(V_{kl}^j)_{4\times 4}: \underline{V}^j\le V^j\le \overline{V}^j,\ \frac{j}{2k+2l}+0.01\le V_{kl}^j\le \frac{j}{2k+2l}+0.1,\ k\ne l,\\
&\qquad j=1,2,\ k=1,2,3,4,\ l=1,2,3,4,\ V^j\in V_I\Big\};\\
W_I &:= \Big\{W^j=(W_{kl}^j)_{4\times 4}: \underline{W}^j\le W^j\le \overline{W}^j,\ \frac{j}{k+l}+0.03\le W_{kl}^j\le \frac{j}{k+l}+0.3,\ k\ne l,\\
&\qquad j=1,2,\ k=1,2,3,4,\ l=1,2,3,4,\ W^j\in W_I\Big\};
\end{aligned}
$$

The activation function satisfies with Assumption $[\mathcal{A}_1]$ with $\phi_k=2,\ \psi_k=1$ $(k=1,2)$. By employing Theorem 3.1, one can obtains

$$
6.12=\min\Big\{F_k+\underline{p}_i-\psi_i\sum_{l=1}^{n}\hat{q}_{li}
$$

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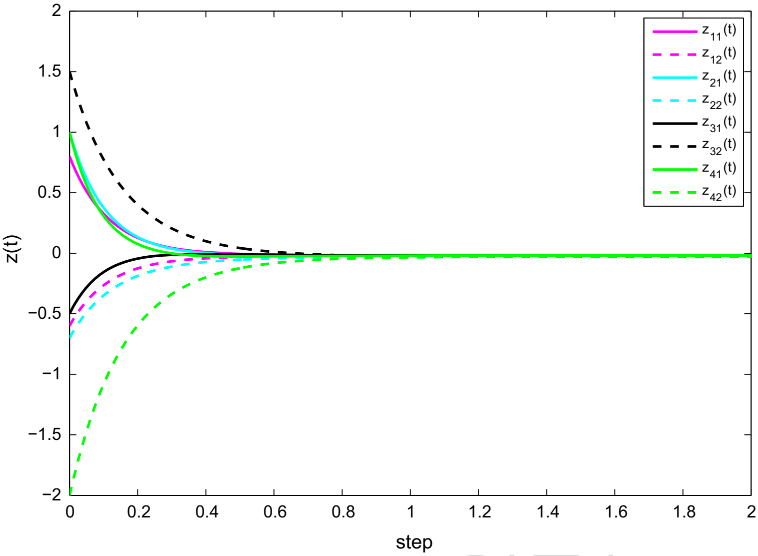

Fig. 1 $z_{k1}(t),\ z_{k2}(t),\ k = 1, 2, 3, 4$

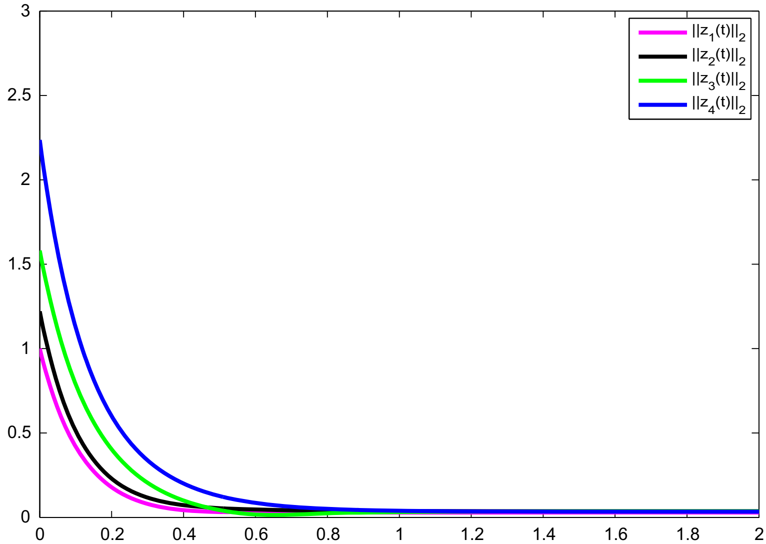

Fig. 2 $\|z_k(t)\|_2,\ k = 1, 2, 3, 4$

562 $$- \sum_{j=1}^{x} \overline{\alpha}_j \overline{\lambda}_i^j \Big(\sum_{l=1}^{N} \hat{V}_{lk}^j \Big),\ k = 1, 2, \ldots, N,\ i = 1, 2, \ldots, n \Big\} > 0,$$

563 $$2.76 = \max \Big\{ \phi_i \sum_{l=1}^{n} \hat{r}_{li},\ i = 1, 2, \ldots, n \Big\} > 0,$$

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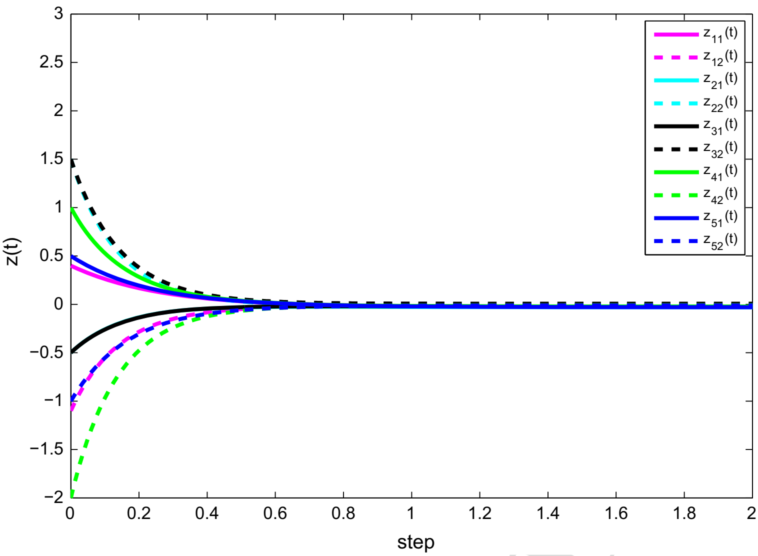

Fig. 3 $z_{k1}(t),\ z_{k2}(t),\ k = 1, 2, 3, 4$

$$1.87 = \max\left\{\sum_{j=1}^{x} \overline{\beta}_j \overline{\upsilon}_i^j \Big(\sum_{l=1}^{N} \hat{W}_{lk}^j\Big),\ k = 1, 2, \ldots, N,\ i = 1, 2, \ldots, n\right\} > 0.$$

Therefore, it follows from that the system (35) realize robust asymptotical synchronization from Theorem 3.1. The computer simulations are depicted in Figs. 1 and 2

Example 4.2 For the MWCFCNNs with linear coupling delay:

$$\begin{aligned} D^{0.99} z_k(t) = & -P z_k(t) + Q g\big(z_k(t)\big) + R h\big(z_k(t-0.05)\big) \\ & + \sum_{l=1}^{5} \alpha_1 V_{kl}^1 \Lambda_1 z_l(t) + \sum_{l=1}^{5} \alpha_2 V_{kl}^2 \Lambda_2 z_l(t) \\ & + \sum_{l=1}^{5} \beta_1 W_{kl}^1 \Upsilon_1 z_l(t-0.05) + \sum_{l=1}^{5} \beta_2 W_{kl}^2 \Upsilon_2 z_l(t-0.05) \\ & - F_k\Big(z_k(t) - \frac{1}{5}\sum_{l=1}^{4} z_l(t)\Big) \end{aligned} \tag{36}$$

where $k = 1, 2, 3, 4, 5$, $g_k(\tau) = h_k(\tau) = \tanh(\tau)\ (k = 1, 2)$, $F_k = 0.1,\ k = 1, 2, 3, 4, 5$, $\alpha_1 = 0.6,\ \alpha_2 = 0.5,\ \beta_1 = 0.7,\ \beta_2 = 0.5$, the matrices are chosen as respectively

$$P = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix},\quad Q = \begin{bmatrix} 1.2 & -0.3 \\ -1 & 1.2 \end{bmatrix},\quad R = \begin{bmatrix} 0.7 & 0.8 \\ 0.6 & -1 \end{bmatrix},\quad \Lambda_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix},$$

$$\Lambda_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad \Upsilon_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},\quad \Upsilon_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

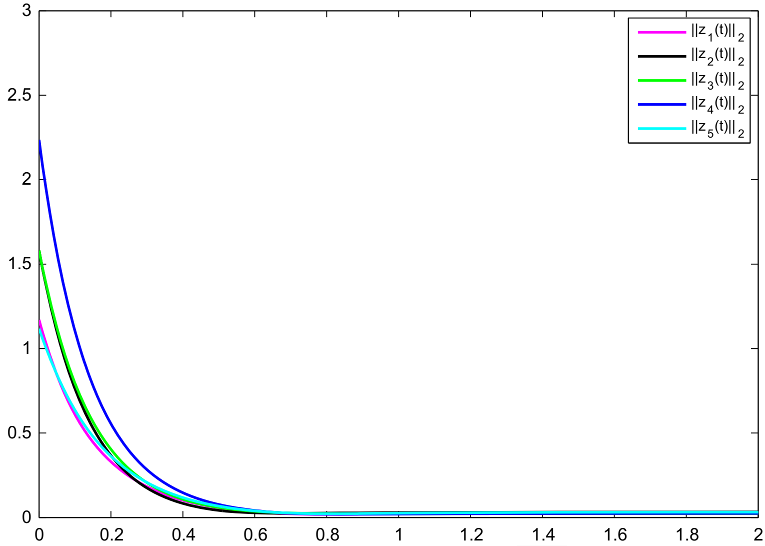

Fig. 4 $\|z_k(t)\|_2,\ k = 1, 2, 3, 4$

The topology structure of (36) is defined by

$$V^1 = \begin{bmatrix} -0.55 & 0 & 0.2 & 0.1 & 0.25 \\ 0 & -0.9 & 0.5 & 0 & 0.4 \\ 0.2 & 0.5 & -1.1 & 0.2 & 0.2 \\ 0.1 & 0 & 0.2 & -0.5 & 0.2 \\ 0.25 & 0.4 & 0.2 & 0.2 & -1.05 \end{bmatrix}, \quad V^2 = \begin{bmatrix} -0.5 & 0 & 0.1 & 0 & 0.4 \\ 0 & -0.7 & 0.4 & 0 & 0.3 \\ 0.1 & 0.4 & -1 & 0.3 & 0.2 \\ 0 & 0 & 0.3 & -0.8 & 0.5 \\ 0.4 & 0.3 & 0.2 & 0.5 & -1.4 \end{bmatrix}$$

$$W^1 = \begin{bmatrix} -2.2 & 0 & 0.7 & 0.7 & 0.8 \\ 0 & -1.3 & 0.3 & 0.3 & 0.7 \\ 0.7 & 0.3 & -1.2 & 0 & 0.2 \\ 0.7 & 0.3 & 0 & -1.2 & 0.2 \\ 0.8 & 0.7 & 0.2 & 0.2 & -1.9 \end{bmatrix}, \quad W^2 = \begin{bmatrix} -0.5 & 0.4 & 0 & 0.1 & 0 \\ 0.4 & -0.9 & 0.2 & 0 & 0.3 \\ 0 & 0.2 & -0.4 & 0 & 0.2 \\ 0.1 & 0 & 0 & -0.3 & 0.2 \\ 0 & 0.3 & 0.2 & 0.2 & -0.7 \end{bmatrix}$$

The activation function satisfies with Assumption $[\mathcal{A}_1]$ with $\phi_k = 1,\ \psi_k = 5\ (k = 1, 2)$. Let us choose $\xi_1 = 4,\ \xi_2 = 3.5,\ \zeta = 1,\ \varsigma = 0.5,\ \eta = 0.2$. By means of MATLAB toolbox to solve the conditions of LMIs in Theorem 3.10 and the feasible solution is given by

$$P = \begin{bmatrix} 13.3450 & 0 \\ 0 & 8.5568 \end{bmatrix}.$$

Therefore the MWCFCNNs (36) is globally synchronized according to Theorem 3.10. The computer simulations are presented in Figs. 3 and 4, which confirms the validity of proposed results.

5 Conclusions

This sequel mainly deals with the robust asymptotical synchronization for coupling delayed FOCNNs with multi weights. On the one hand, by a key role of fractional order comparison

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principle, robust analysis skills, Lyapunov method, and Kronecker product technique, several robust asymptotical synchronization and synchronization results are established by the linear feedback controller. On the other hand, two kinds of special cases of multi-weighted complex structure on FOCNNs with and without linear coupling delays are concerned. Then based on proposed models, several synchronization results, both algebraic method and LMI method, respectively are demonstrated. Finally, we provide three computer simulations to illustrate the correctness of the proposed main results. The proposed approach herein is possible for the investigation and application of some other fractional order memristor neural networks including adaptive synchronization of fractional-order Cohen-Grossberg memristor based coupled neural networks and pinning synchronization of fractional-order memristor based coupled complex neural networks. This will occur in the near future.

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