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MATHEMATICIAN? FEELING OLD? A DISCUSSION

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Abstract You aren't the first, and you won't be the last, so don't fight it—accept and embrace the inexorable march of time. *Praemonitus praemunitus?*

"How old would you be if you didn't know how old you are?"
Leroy Robert Paige (American baseball pitcher)

1 Opening Thoughts

It is not easy, even under favourable conditions, to keep abreast of new concepts and developments in a field that grows deeper exponentially and whose frontiers are ever expanding; it asks much of us mathematicians, mentally and emotionally. American combinatorial expert Richard P. Stanley acknowledged in interview that, with age, he was no longer able to concentrate for the same length of period, and with as much intensity, compared to more youthful days. This is something that comes to all eventually, of course, *en route* to which the manner and scope of problems taken on evolve naturally to accommodate the passing of time before a stage is reached where the Law of Diminishing Returns kicks in and mathematical history/philosophy (or even administrative duties) might begin to appeal more than technical mathematics itself![1] Folklore has it that Hungarian-American mathematician (physicist and computer scientist) John von Neumann used to say that a mathematician is finished at thirty (recall the oft cited, but misleading, words of Englishman G.H. Hardy that mathematics is a "young man's game", with which Albert Einstein broadly agreed), a threshold he decided to increase as he got older to thirty-five, forty, forty-five, and beyond! I suppose someone as gifted as him had licence to do that. On a more humorous note still, the inimitable Italian-American mathematician (and philosopher) Gian-Carlo Rota shared his counselling on the ageing process along with other gems of wisdom, so to speak (characteristically quirky, funny and serious in varying degrees, as one would expect from him), in a 1996 colloquium talk. At the event—termed, in his honour, the 'Rotafest'—he introduced them as follows to an expectant audience: "The advice we give others is the advice that we ourselves need. Since it is too late for me to learn these lessons, I will discharge my unfulfilled duty by dishing them out to you." The final one was called, solemnly, 'Be Prepared for Old Age', which doesn't allow for the gentle time evolution of a mathematician and brutally admits not of any liminal stage before it is reached:

> "My late friend Stan Ulam used to remark that his life was sharply divided into two halves. In the first half, he was always the youngest person in the group; in the second half, he was always the oldest. There was no transitional period.
>
> I now realize how right he was. The etiquette of old age does not seem to have been written up, and we have to learn it the hard way. It depends on a basic realization, which takes time to adjust to. You must realize that after reaching a certain age you are no longer viewed as a person. You become an institution, and you are treated

[1] The allure of undergraduate teaching alone, at the expense of research, is (I have read) also a sure sign of the washed up renegade who has given up on him(her)self and in effect deserted the subject of mathematics.

> the way institutions are treated. You are expected to behave like a piece of period furniture, an architectural landmark, or an incunabulum.
>
> It matters little whether you keep publishing or not. If your papers are no good, they will say, "What did you expect? He is a fixture!"; and if an occasional paper of yours is found to be interesting, they will say, "What did you expect? He has been working at this all his life!" The only sensible response is to enjoy playing your newly found role as an institution." [13, p. 25].

"What did you expect?"—here we have a most convenient example of a false dilemma, known as Morton's Fork, which though amusing in this setting in fact carries a serious point, being that age prejudice seems to be one of the few remaining bastions of tolerated bigotry; it happens in mathematics, as well as in other academic subjects and many aspects of society, where a callous and mystifying derogation of the old occurs with impunity.

Returning to Hardy momentarily, the book from which the above short phrase is taken—*A Mathematician's Apology*—is well known for its melancholic tone as he picked over his life and work. It is an unusual and heartfelt essay wherein early on Hardy felt the need to clarify that

> "I write about mathematics because, like any other mathematician who has passed sixty, I have no longer the freshness of mind, the energy, or the patience to carry on effectively with my proper job." [8, p. 63],

and while he did at least acknowledge that some "great work" had been done by men of age, he viewed the abandonment of mathematics by those beyond their best, one might say, as of little consequence for either the field or the individual (we expand on his tendentious words, as the start of a backdrop to this essay, in an appendix), writing later

> "Mathematics is not a contemplative but a creative subject; no one can draw much consolation from it when he has lost the power or the desire to create; and that is apt to happen to a mathematician rather soon. It is a pity, but in that case he does not matter a great deal anyhow, and it would be silly to bother about him." (p. 143),

and, proud of his own collaborations with "exceptional mathematicians" J.E. Littlewood and (the genius) S. Ramanujan "on something like equal terms",

> "It is to them that I owe an unusually late maturity: I was at my best at a little past forty, when I was a professor at Oxford. Since then I have suffered from that steady deterioration which is the common fate of elderly men and particularly of elderly mathematicians. A mathematician may still be competent enough at sixty, but it is useless to expect him to have original ideas." (p. 148).

There are a host of results, constructs and connections found by researchers post forty, fifty, and indeed sixty (far too numerous to list), so Hardy's ruminations are overly downbeat (and generally agreed to have been largely caused by his state of mind at the time). It is a plain fact, however, that mathematical research is an exacting and arduous activity which—even as a vocational calling—does not get any easier beyond a certain point as the years elapse. As we approach it, facing our unavoidable professional destiny can be hard and unsettling as we become aware that having been put out to pasture, in a manner of speaking, the discipline will be occupied by, and flourish under the guardianship of, upcoming generations in our absence, leaving us to cope with the finality of our research contributions and wider engagement while being thankful for their existence in our lives and the purpose they brought—a bittersweet emotion (that may cup grief, sadness, repose and gratitude combined), and conveyed almost uniquely by the word *hiraeth* within the Welsh language.

2 On Matters of Legacy and Credit

Fear of becoming invisible the moment one stops research is a factor that drives people on to persist with it, for anyone whose productivity ends had better make sure that excellent work has gone before if they are to be remembered in any real and lasting way (assuming they are bothered—some aren't). On this latter point then, in certain instances, stature and rank can indeed be preserved and there are precedents for this. American statistician Herbert E. Robbins

said the following of Einstein and Isaac Newton—two of the most celebrated colossuses of mathematics and science—when quizzed:

> "He seems not to have had an inspiration in physics during the last thirty or so years of his life—at least none that could be compared with the great ones he had from 1905 [his *annus mirabilis*] to 1920. Here was a man who had perhaps the greatest intellect that God ever created and, in the last years of his life, nothing much came of it. It wasn't because Einstein was frivolous or dissipating his energy; he had done what he could and there came an end. As I watched him at the Institute [for Advanced Study, Princeton, U.S.A.] one year, he never complained about it and no one mentioned it, but everyone knew that he was essentially finished as a scientist. And Newton? The same thing. From about age thirty on, he did absolutely nothing in science. He had a career as Master of the Mint, he carried on a great deal of activity with friends—controversies over who invented the calculus, and so on—but the last half of his life was totally sterile from a scientific viewpoint." [11, p. 290].

This, however, was a long time ago, and they were exceptional individuals who made huge theoretical strides forward to change the way we think about the world around us (the vast majority of people do not). Not surprisingly, Hardy had something to say about both the subjective and objective worth of his own professional work, and he did so set against the likes of solicitors, stockbrokers or bookmakers (though it still hinted of unapologetic elitism, and a self-satisfied hostility to utility, declared by like minds such as Gauss and Jacobi to name but two forerunners and giants of their eras):

> "Is there any sense in which I can claim that my life has been less futile than theirs? It seems to me . . . that there is only one possible answer: yes, perhaps, but, if so, for one reason only.
>
> I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world. I have helped to train other mathematicians, but mathematicians of the same kind as myself, and their work has been, so far at any rate as I have helped them to it, as useless as my own. Judged by all practical standards, the value of my mathematical life is nil; and outside mathematics it is trivial anyhow. I have just one chance of escaping a verdict of complete triviality, that I may be judged to have created something worth creating. And that I have created something is undeniable: the question is about its value." [8, pp. 150–151].

One of life's certainties for the masses is that they are forgotten soon enough—we academics can buck the trend, but it happens rarely.

Mathematicians feel keenly, more than most it seems, the need to assert a right of possession to what they find in research, especially as repute and legacy can rest on such things. Often though, a moment of intimate comprehension is built on the work of others, so the whole issue of academic proprietorship and approbation is rather muddy at times. Recognition may not be accorded where it should or, even worse, it might be wrongly directed elsewhere. Descartes, for instance—to whom was ascribed the creation of analytic geometry—put together the earlier ideas of others as a new methodology in the form of a workable engine of proof, discovery and invention. Eric T. Bell (Scottish-born mathematician and science fiction writer who lived in the United States for most of his life) wrote (in his text *Men of Mathematics*, first printed in 1937)

> "Similar remarks apply to most of the other advances of modern mathematics. A new concept may be "in the air" for generations until some one man—occasionally two or three together—sees clearly the essential detail that his predecessors missed, and the new thing comes into being." [4, p. 7],

by way of example citing Einstein as the author of relativity, and not Minkowski to whose name it is sometimes attached. It happens, as history has shown, that attribution of a weighty innovation or sweeping breakthrough is not always justly placed, and the person who first applied a fresh technique more powerfully than its originator may get more than his or her professional due from peers and those thereafter—the notion of, for example, calculus had been around (under

the care of Archimedes) long before Newton and Leibniz took control of its exposure.[2] Bell made a couple of relevant points which are worth bearing in mind, reminding us, firstly, that

> "It seems rather meaningless to say that So-and-so might have done this or that if circumstances had been other than they were. Any one of us no doubt could jump over the moon if we and the physical universe were different from what we and it are, but the truth is we do not make the jump." (p. 7),

and then of something we all know to be the case:

> "Priority after all gradually loses its irritating importance as we recede in time from the men to whom it was a hotly contested cause of verbal battles while they and their partisans lived." (p. 8).

Rota commented on this topic as well, with typical candour:

> "The history of mathematics is replete with injustice. There is a tendency to exhibit toward the past a forgetful, oversimplifying, hero-worshiping attitude that we have come to identify with mass behavior. Great advances in science are pinned on a few extraordinary white-maned individuals. By the magic powers of genius denied to ordinary mortals (thus safely getting us off the hook), they alone are made responsible for Progress.
>
> The public abhors detail. Revealing that behind every great man one can find a beehive of lesser-known individuals who paved his way and obtained most of the results for which he is known is a crime of *lèse majesté*." [14, p. 211],

repeating the general view in his Foreword to a book by Bergeron *et al.*:

> "Advances in mathematics occur in one of two ways.
>
> The first occurs by the solution of some outstanding problem, ... Such solutions are justly acclaimed by the mathematical community. The solution of every famous mathematical problem is the result of joint effort of a great many mathematicians. It always comes as an unexpected application of theories that were previously developed without a specific purpose, theories whose effectiveness was at first thought to be highly questionable.
>
> Mathematicians realized long ago that it is hopeless to get the lay public to understand the miracle of unexpected effectiveness of theory. The public, misled by two hundred years of Romantic fantasies, clamors for some "genius" whose brain power cracks open the secrets of nature. It is therefore a common public relations gimmick to give the entire credit for the solution of famous problems to the one mathematician who is responsible for the last step.
>
> It would probably be counterproductive to let it be known that behind every "genius" there lurks a [colony] of research mathematicians who gradually built up to the "final" step in seemingly pointless research papers. And it would be fatal to let it be known that the showcase problems of mathematics are of little or no interest for the progress of mathematics. We all know that they are dead ends, curiosities, good only as confirmation of the effectiveness of theory." [15, p. v].

Happily, overwhelming numbers of us do not hesitate to acknowledge the work of another on which we draw as a fundamental matter of principle, with English academic James Joseph Sylvester—in Part III (pp. 418–479) of a paper published as Article No. 74 in Volume II of his *Collected Papers*[3]—capturing beautifully the way mathematicians use such prior labours to their own advantage in searching for a solution, proof, theory, methodological improvement, *etc*. Speaking of Charles Hermite's work on fifth degree polynomial roots, whose characterisation Sylvester (some years later) simplified in its algebraic complexity, he wrote

> "The arrow from my hand may have been the first to hit the mark, but it was his hand which had previously shaped, bent, and strung the bow." [18, p. 418];

[2] The so called 'calculus controversy', or *prioritätsstreit* ('priority dispute'), was a bitter argument on ownership between Newton and Leibniz that began simmering in 1699 and broke out in full force some years later.

[3] This last part having been added to the original paper delivered before the Royal Society of London.

a lovely image, expressed exquisitely ("*Nanos gigantum humeris insidentes*", one recalls).

As in most walks of life, someone can slip through the proverbial net to gain an unwarranted prominence in the public eye—this can happen by chance or design. In his notes 'To Any Prospective Reader' at the start of another tome [3] Bell remarked (p. v) "Once we venture beyond the rudiments, we may agree that those who cultivate mathematics have more interesting things to say than those who merely venerate." He echoed the much referenced sentiment expressed even more scornfully by Hardy (whose short treatise [8], in which his now familiar words appeared, was published during the same year as Bell's book [3]), noting that Roger Bacon—known as a celebrated 13th century medieval English philosopher—"all but fully awake in science, was still as fast asleep in the mathematics which he eulogized as were any of his European opponents . . ." (p. 91). Bell opined that Bacon's "rudimentary knowledge of mathematics, as represented in his published writings, scarcely guarantees his frequently quoted testimonial—"mathematics is the gate and key of the sciences," etc.", and went even further: "Curiously, the most flattering testimonials for mathematics have been (and still are) the enthusiastic utterances of men who knew very little about the subject." (Footnote 8 to Chapter 4, p. 596).[4]

3 On Upsides and Downsides

Robbins submitted that an increasing expectation to sustain and (better still) improve one's work raises complicated questions that involve things such as choice, incentive, effort, inspiration, taste, opportunity and energy. He was also aware of being taught to continually evaluate his attainments—gauging their worth alongside those from an ongoing personal timeline, and against others as an indicator of relative standing—creating private anxieties and tensions not easily assuaged. Some thirty odd years later, this measuring mindset remains very much around, and is more acute still. It has been drilled into us, and we live with it daily as a behaviour trigger absorbed through the subliminal and overt messages we receive which promote a rather mindless 'hamster wheel' strategy to the modern day enterprise that is research (whose rhythm in the U.K. beats to the tune of the omnipresent Research Excellence Framework, that self-invented quasi-periodic H.E. research 'assessment' exercise which is divisive and discriminatory in equal part). That said, we are, with the gathering of experience, in a position to develop our own style of work, and to appreciate more the boundless corpus of a bewilderingly diverse discipline—exploring its undulating geography of contours, fine-tuning our mathematical instincts, unveiling its secrets, and gradually interpreting what we learn in our own ways. This gives us a mental freedom that is potentially priceless and of real merit in research (and teaching in ways) as it affords the luxury of shifting, as the mood takes, between the two longstanding and dichotomous Platonistic and formalistic philosophical stances on mathematical endeavour so that we become more rounded in our beliefs, at peace with ourselves, and less judgemental of the positions of colleagues (and in turn swayed little by those around us, able to place value on one's own work for the sake of it); these are all positives that may arrive as the years go by, refined by our decisions to work collaboratively or in solitude, and framed by an overarching pluralistic tolerance coupled with an untouchable irreverence to authority that would each serve us efficaciously.

There is more good news, as well—a suggestion that our beloved subject engages us in a worthwhile pursuit that helps us to stick around in reasonable shape even beyond retirement. John E. Littlewood—Hardy's long time collaborator and friend—wrote, in an entertaining piece (republished posthumously, the year following his death),

> "Mathematics is very hard work, and dons tend to be above the average in health and vigor. Below a certain threshold a man cracks up; but above it, hard mental work *makes* for health and vigor (also—on much historical evidence throughout the ages—for longevity)." [10, p. 116].

Sylvester—in a late 1860s Presidential Address to the Exeter British Association—extolled the virtues of mathematics in raising its adherents ". . . to higher and higher states of conscious intellectual being." He added "This accounts, I believe, for the extraordinary longevity of all the greatest masters of the Analytical art, the Dii Majores of the mathematical Pantheon." Listing

[4] He also wrote "To dispose here of another myth, Leonardo da Vinci's . . . published jottings on mathematics are trivial, even puerile, and show no mathematical talent whatever."

those worthy of mention (in his eyes, they were Leibnitz, Euler, Lagrange, Laplace, Gauss, Plato, Newton, Archimedes and Pythagoras), he finished with the following reflection:

> "The mathematician lives long and lives young; the wings of his soul do not early drop off, nor do its pores become clogged with the earthly particles blown from the dusty highways of vulgar life." [19, p. 658].

I like this quote, which is echoed by Davis and Hersh who, in their respected text *The Mathematical Experience*, pointed out that

> "The Greek philosophers thought of mathematics as a bridge between theology and the perceptible, physical world, and this view was stressed and developed by the Neoplatonists. The quadrivium: arithmetic, music, geometry, astronomy, already known to Protagoras (d. 411 B.C.), was thought to lead the mind upward through mathematics to the heavenly sphere where the eternal movements were the perceptible form of the world soul." [5, pp. 54–55].

One can, not surprisingly, find similar feelings in others which are no coincidence, and I suspect a good many of us mathematicians are taken out of ourselves in glorious and uplifting ways through research, for after all it is those challenges we take on, rather than what survives them in print as carefully crafted and distilled residues, that is surely the essence of what it means to be a mathematician—respect, praise, honour, accolades, and the like, are an earthly bonus (not an entitlement) and should be regarded as such. Intellectual demands and zealous enquiries can effect a fierce fray for supremacy over our subject, where stark and sudden bursts of conscious perception visited upon us might grant fleeting glimpses of a higher truth, so maybe there is a lesson here—to treasure the processes of mathematical inventiveness, inspiration and artistry for what they are, and at least as much as any bodily fruit they might bear, before the cerebral powers and strengths of our youth begin to dwindle. We should always acknowledge with humility that in the inestimable mathematical collective only a meagre fraction of work remains of interest, while most is ignored and dismissed, or else overwritten and subsumed, never again to see the light of day beyond the short period during which, like every toiling author responsible, it may live and breathe—this is scant reward for a lifetime spent mostly in the pitfalls and blind alleys of research, and is just the way of things. But then something of it must remain, at least, according to Bell who offers a small branch of hope to "the labors of thousands of men"—that their efforts may endure in some way (for anything non-zero is infinitely bigger than zero itself); one would very much like to think so:

> "... like the tiny creatures whose empty frames survive in massive coral reefs that can wreck a battleship, these hordes of all but anonymous mathematicians have left something in the structure of mathematics more durable than their own brief and commonplace lives." [3, p. ix].

There is solace to be taken in this, long after we have ceased to be robust and resolute in our mathematical activities, and later as we begin to slip from existence. I am often drawn to the words of Hungarian-born American Paul R. Halmos—a prolific writer about research and teaching, and populariser of mathematics across many decades—who over half a century ago described a spirit and ethos that defines the mathematical community, writing

> "The mathematical fraternity is a little like a self-perpetuating priesthood. The mathematicians of today train the mathematicians of tomorrow and, in effect, decide whom to admit ... Most people do not find it easy to join—mathematical talent and genius are apparently exactly as rare as talent and genius in painting and music—but anyone can join, everyone is welcome. The rules are nowhere explicitly formulated, but they are intuitively felt by everyone in the profession. Mistakes are forgiven and so is obscure exposition—the indispensable requisite is mathematical insight. Sloppy thinking, verbosity without content, and polemic have no role, and—this to me is one of the most wonderful aspects of mathematics—they are much easier to spot than in the non-mathematical fields of human endeavor (much easier than, for instance, in literature among the arts, in art criticism among the humanities, and in your favorite abomination among the social sciences)." [6, p. 381].

They remain largely intact today, fortunately, despite having inevitably succumbed to the competitive edges that have intruded on all aspects of academic life[5] and mould our customs and conventions. Nevertheless, anyone blessed enough to be inducted into the hallowed Fellowship of Mathematical Research can, when vitality subsides and age catches up and overtakes, count themselves one of a privileged few and rejoice in their acts of membership and participation.

Undeniably, such is the incessant pull of mathematics—even though we might often be hampered by insecurities and the gnawing whispers of self-doubt—lots of us feel we are never quite finished with it; it's simply what we do. Halmos summed this up at the very end of his 'automathography' (where he looked back at, and chronicled, his own working life in detail and with commendable sincerity):

> "I thought, I taught, I wrote, and I talked mathematics for fifty years, and I am glad I did. I wanted to be a mathematician. I still do." [7, p. 403].

I know what he's saying here. We may anchor enormous significance to our mathematical rituals, habits and sacraments, no less dedicated to them in faith than devotees of a religion as we seek perfection in forms we are able to cognise and decode. More pragmatically, there are other considerations to take into account as part of the big picture. As one becomes older one realises that, in the words of Sir Michael F. Atiyah,

> "Many would-be mathematicians also have talents and interests in other directions and they may have a difficult choice to make between embarking on a mathematical career and pursuing something else. The great Gauss is reputed to have wavered between mathematics and philology, Pascal deserted mathematics at an early age for theology, while Descartes and Leibniz are also famous as philosophers. ... You should not regard mathematics as a closed world, and the interaction between mathematics and other disciplines is healthy both for the individual and for society." [2, p. 1000].

Thomas Paine—English-born American political activist/theorist/critic, philosopher, and busy revolutionary—wrote, in his well known text *The Age of Reason* which promoted reason and free thought (it was published in three parts either side of the dawn of the 19th century, the author arguing against institutionalised religion in general and Christian doctrine in particular),

> "To be happy in old age it is necessary that we accustom ourselves to objects that can accompany the mind all the way through life, and that we take the rest as good in their day. The mere man of pleasure is miserable in old age; and the mere drudge in business is but little better: whereas, natural philosophy, mathematical and mechanical science, are a continual source of tranquil pleasure, and in spite of the gloomy dogmas of priests, and of superstition, the study of those things is the study of the true theology; it teaches man to know and to admire the Creator, for the principles of science are in the creation, and are unchangeable, and of divine origin." [12, p. 128],

adding "...; for when we cease to have an object we become like an invalid in [a] hospital waiting for death." Although *The Age of Reason* is a treatise on faith and belief, Paine captured the essence of the exciting life that a true mathematician, for one, has before him, fixed firm by an undying inquisitiveness. Einstein said, when being interviewed for *Life* Magazine, "Curiosity has its own reason for existence."[6] In Chapter 5 of her book *Men Who Made a New Physics: Physicists and the Quantum Theory* (University of Chicago Press (1987; first published in 1965)), Barbara Lovett Cline wrote of Einstein's approach, as a private instructor, to the education of two boys who were struggling at school. Objecting to the prevailing "education machine", as

[5] Some of which have always been a part of it. The 1982 Fields Medal winner William P. Thurston wrote

> "In addition to our inner motivation and our informal social motivation for doing mathematics, we are driven by considerations of economics and status. Mathematicians, like other academics, do a lot of judging and being judged. Starting with grades, and continuing through letters of recommendation, hiring decisions, promotion decisions, referees reports, invitations to speak, prizes, ... we are involved in many ratings, in a fiercely competitive system." [20, p. 171],

in a much read 1994 essay where he deliberated on the nature of proof and progress in mathematics.

[6] William Miller—one of the magazine's editors—called on Einstein (with two guests) at his home in Princeton a few months before his death, writing up a three page personal memoir that was published in the May 2nd 1955 issue; the quote appears towards the end of the article (on p. 64), which is titled 'Old Man's Advice to Youth: 'Never Lose a Holy Curiosity".

he termed it (a phrase pertinent today still), she recalled his words: "Curiosity" [he said] was a "delicate little plant [which], aside from stimulation, stands mainly in need of freedom." (p. 64).

Another point. Anybody wanting to carve out a long and successful career that is invigorating and pleasurable probably needs a stable home life and should strive to both foster and maintain a healthy balance between mathematical drive and other things outwith the subject (academic or otherwise) that take one away from limiting internalised patterns of conduct and familiar cerebral sound chambers, particularly when they provide mathematical regeneration and new means of self-expression ('time away' does not signal a shortage of research ideas, either, as is sometimes the lazy *non sequitur* charge levelled). Common sense, yes, but worth remembering, and I am obliged to mention a sombre tone adopted by Robbins—either serving as a warning by him, or else surfacing as regretfulness on his part—which should resonate with most of us:

> "Younger mathematicians have [an increased] desire to become known and make a reputation. This weakens with age, either through frustration if they don't succeed or through satiation if they do. And even if one does make it to the top, was it worthwhile? Is it worth continuing to strive for more? The really successful mathematician, if he's honest, must assess his life in terms of having foregone meaningful relations with others—wife, children, colleagues, friends, etc. As one becomes older, he becomes less likely to want to pay the price for new successes. The theorems that I've proved aren't going to be much good or as comforting to me as would be close friends when I'm old and perhaps infirm." [11, p. 290].[7]

I can relate to this, for sure. If you don't push yourself you may not consummate your potential and achieve anything much at all in your own eyes, but operating relentlessly at or near to one's limit level, or else being drawn into an obsessive workaholic culture to the exclusion of other interests—which for some feeds a desperate craving for permanence (the 'mirage of immortality' syndrome[8]) that will almost certainly be doomed to take the form of but a minor footprint left in the academic landscape—can come at considerable cost to yourself and those nearest to you; all may be damaged by concerns that seem terribly important, but ultimately may not be. Sudden and debilitating burnout can lie in wait for those who cannot, or will not, avoid surfeit and excess, but evolving fatigue and jadedness are quite natural and can be mitigated by taking appropriate steps to compensate for declining energy, memory and computational ability—the most reliable recipe, and a common one, is to combine hard won accumulated knowledge and cunning in techniques with the vibrancy and gusto of a younger collaborator.

4 More Thoughts

I allow myself a short digression. The young are beginning to dominate society somewhat, and the older one gets the more it is noticeable. Inside a recently coined notion of 'youthocracy'—an essentially social media driven phenomenon that has created oversensitivity, callowness and

[7] Interestingly, on thinking about days of yore, Halmos felt able to write

> "[One] way in which I wish I had then followed the advice I give now has to do with the old adage about all work and no play. I don't believe in it; I think all work and no play is the only way to get anything done."

Surely delivered for dramatic effect (as he was wont to do on occasions), he immediately back-tracked and modified the statement:

> "What I am saying is that the work of a scholar is not torture that would be insupportable without distraction, and that, for most of us, two consuming passions are one too many.
>
> . . . We don't need hobbies. I might go so far as to say that we shouldn't have any. Most of us ordinary mortals don't have enough psychic energy to split between two passions; we cannot be good at both mathematics and cello, we cannot be satisfyingly creative in both mathematics and cabinet making, we cannot be effective in both mathematics and politics." [7, p. 53];

ironically, he violated his own rule at one point by becoming involved in a "temporary but frightfully time-consuming attention [given over] to political ideals."

[8] We turn again to Sylvester, a troubled man during periods of life who was determined to stamp a mark on the field aided by unabashed self-congratulatory claims to boost his cause; a good example of this occurs in an article of 1887:

> "Perhaps I may without immodesty lay claim to the appellation of the Mathematical Adam, as I believe that I have given more names (passed into general circulation) to the creatures of the mathematical reason than all the other mathematicians of the age combined." [17, p. 152];

the title of this short piece (on various terminologies for prime number properties) says it all really.

panic in those of more tender years—lurks oversimplifications and superficiality that overspill into the modern day tertiary sector, pushing the urgent 'publish, publish, publish' banner held aloft by younger colleagues which, as the only recognisable 'meal ticket' for something designated as success, does not show enough deference to wisdom, knowledge and forbearance nurtured by elders as long term return investments in work and life. Research that is 'slow-release' or not necessarily tailored towards a specific (possibly restricted) goal is becoming devalued as the highly ambitious, super focused, and ultra conspicuous—programmed to work within short project timescales—gain prominence at the expense of those endowed with a different intellectual wealth who take a more relaxed and possibly deeper attitude towards research, ever mindful to uphold their fortunate position through a default sense of *noblesse oblige*. Over the last thirty years there has been a discernible shift in evocations of paradigmatic excellence where valued praxes—in which academics were held in esteem for their imagination and courage of thought, creative work was protected and valued above market or political use, and there was an inbuilt sensuality and wildness to genuine discovery—have been eroded by the attritional march of neoliberalism whose pervasiveness seems unstoppable. Those who know this will eventually fade away, leaving behind as they do ever smaller pockets of vacuum to be filled with one dimensional hypernormalisation where careerism and quasi-pathological egotism can run riot. The traditional university culture—foregrounding integrity, liberty, and a gentler collegiate stewardship which lauds what age and experience can offer—is already twisted by the creeping imposition of facile metrics embedded in the design of belligerent systems and protocols, and may soon be lost forever. Age? Who needs it? It has become almost inexplicably and absurdly *passé*.

It does, though, bring a sense of overview which can be of use in assessing, and responding to, transformation in the education sector. Consideration as to predicted effects of new network technologies and electronic developments on aspects of the mathematical community (language, methods, purpose, gatherings), aired by Stanway in a thoughtful 2006 piece [16], foreshadowed the pace of change those of us of a certain age are experiencing in the way mathematics is prosecuted and communicated—as needs and expectations in knowledge management exert themselves, as digital boundaries between teaching and research become more flexible and porous, and as some of the old ways of doing things fall away to reflect the ongoing march of scientific progress through educational models, tools, systems and stratagems. One particular facet that may undergo a radical transition is our intellectual culture, where the scholarly information environment could have territories dominated by content rather than distinct individual contribution. Authorship, Stanway points out, might well then become problematic for some in the long run, sitting alongside the matter of ownership in a domain whose future form would then perforce resemble a medieval one when author identity did not command the same level of significance as possession of hand crafted texts and manuscripts.[9]

At this juncture I open up the discussion a little further, presenting the last four verses (of ten) from Thomas Hardy's 'An Ancient to Ancients' published in his 1922 collection of poems *Late Lyrics and Earlier, With Many Other Verses* (I have obtained a first edition of the book (published by Macmillan and Co.) where the ode can be seen on pp. 282–284; I am grateful to James Stanton for bringing this work to my attention):[10]

"We who met sunrise sanguine-souled,
Gentlemen,
Are wearing weary. We are old;
These younger press; we feel our rout
Is imminent to Aïdes' den,—

[9]This is an interesting point. Notions of ownership and authorship, together with intellectual property rights, only gained agency in the move from a scribal culture to a typographical one driven by centralised production of physical documents in publishing houses and their mass distribution. The current electronic culture that is upon us, whilst exciting, carries with it a potential legacy from the dim and distant past—a pre-typographical scribal culture (albeit then characterised by localised production and limited distribution) where users of manuscripts were indifferent to authorship and its chronology.

[10]In this poem Hardy tames a rising inequity felt naturally by the old as they are forced to make way for the young, doing so through a fine composure in the narrator's voice which speaks to both groups in earnest tones. It is a poem of lamentation, subtle observation and hope, based on the inspiration of lifelong autodidactic learning and the defiance of determined, persistent and avid scholars (pooled from painters, novelists, poets, writers, philosophers, historians) whose fires of fervour remain stoked and who still have work to do before the heavy baton of cultural and scientific progress is passed on to an impatient next generation; unruly nostalgia—an interlude on the way to "forgottenness"—must be disciplined, and Hardy does it with touching sensitivity, cataloguing classical late succeeders before a climactic address to youth [I have, in writing this *précis*, appealed to an analysis of the verses by W. Kerrigan (*The Sewanee Rev.* **111**, 383–391 (2003))].

That evening's shades are stretching out,
Gentlemen!

And yet, though ours be failing frames,
Gentlemen,
So were some others' history names,
Who trode their track light-limbed and fast
As these youth, and not alien
From enterprise, to their long last,
Gentlemen.

Sophocles, Plato, Socrates,
Gentlemen,
Pythagoras, Thucydides,
Herodotus, and Homer,—yea,
Clement, Augustin, Origen,
Burnt brightlier towards their setting-day,
Gentlemen.

And ye, red-lipped and smooth-browed; list,
Gentlemen;
Much is there waits you we have missed;
Much lore we leave you worth the knowing,
Much, much has lain outside our ken:
Nay, rush not: time serves: we are going,
Gentlemen."

The mathematical creativity and youth correlation is something of a false statistic, and weak at that, with outliers in number (vouchsafed to us regularly over centuries) sensibly accommodated as positive White Swan Events in any case.[11] The very top mathematicians are few, always excite, and are quite often of relatively advancing years (leaving the rest to dream of admission to an elite band of 'gentlemen' such as those in Hardy's poem who sit in good company together).

Our destiny is forged in the main by the skill we show in playing those cards that fate deals us, but some hands are better than others and of course may outweigh any gains age on its own may bring—that is the luck of the draw, as we say. As much as anything, it could be argued that a prime factor in making it as a professional mathematician is simply the willingness to immerse oneself in the subject over and above all else—whoever we are and whatever our station—keeping alive a flame of warmth and affection for it as sustenance to fuel the hard work that always lies ahead. Halmos had some interesting (if slightly idiosyncratic) words on this thought, writing

> "To be a mathematician you must love mathematics more than family, religion, money, comfort, pleasure, glory. I do not mean that you must love it to the exclusion of family, religion, and the rest, and I do not mean that if you do love it, you'll never have any doubts, you'll never be discouraged, you'll never be ready to chuck it all ...
>
> "Mathematician" is, to be sure, an undefined term, and it is possible that some people called mathematicians nowadays (or ever) do not (or did not) love mathematics all that much. ... I am not saying that the love of mathematics is more important than the love of other things. What I am saying is that to the extent that one's loves can be ordered, the greatest love of a mathematician (the way I would like to use the term) is mathematics. I have known many mathematicians, great and small, and I feel sure that what I am saying is true about them. ...
>
> ... the love of mathematics is a hypothesis about which the conclusion doesn't follow. If you want to be a mathematician, look into your soul and ask yourself how much

[11] A metaphorical 'Black Swan Event'—from which 'Grey/White Swan Events' are designated—is one that comes as a surprise, has a major effect, and is often inappropriately rationalised after the fact with the benefit of hindsight—it was brought into common public usage by Nassim Nicholas Taleb in his 2001 book *Fooled By Randomness* which explained the idea in the context of financial markets.

> you want to be one. If the wish isn't very deep and very great, if it is not, in fact, maximal, if you have another desire that takes precedence, or even more than one, then you should not try to be a mathematician. The "should" is not a moral one; it is a pragmatic one. I think that you would probably not succeed in your attempt, and, in any event, you would probably feel frustrated and unhappy." [7, pp. 400–401];

I think he's articulated something quite profound here if truth be told—Mistress Mathematics will arouse passion, pain and loyalty, but the favours she bestows over time are worth it. One's research (and teaching) can be likened to a ticket that affords regular entry to the sanctum of mathematics. The riches found there act as the fruits and flowers of a lotus tree which, once tasted, can never be left alone. Its mythological powers—as a narcotic that caused a longing to remain in the presence of fellow lotus eaters—are mirrored by the hold that mathematics has on its community of research workers where prosaic realism and utilitarianism are subordinate to pure discovery and uninhibited freedom effected with strident vigour; once this intellectual delicacy has been sampled it is craved evermore—uniting those similarly blessed with the affliction, providing certitude to our industry, quenching an undying thirst for order, beauty and veracity. As the distinguished and influential mathematician David Hilbert said (of mathematicians), "Wir müssen wissen. Wir werden wissen" ["We must know. We shall know"].[12] Littlewood wrote that a desire for truth is not the driving force behind our creations—whether we produce seminal work of the highest importance or else belong to a vast army of people who are gifted but fall short of genius—but rather that "A *sine qua non* is an intense conscious curiosity about the subject, with a craving to exercise the mind in it, quite like physical hunger." [10, p. 113].

The slip roads off the main mathematical thoroughfares signpost quiet towns called "Obsolete" and "Outworn", and you'd be best living with the idea that your path of travel will lead you in the end to both so that you at least give yourself time to take in the scenery and find some pleasure in it on your busy professional journeying. You may as well, especially if what the American Alfred W. Adler wrote is true:

> "... almost no one is capable of doing significant mathematics. There are no acceptably good mathematicians. Each generation has its few great mathematicians, and mathematics would not even notice the absence of the others. They are useful as teachers, and their research harms no one, but it is of no importance at all. A mathematician is great or he is nothing. ...
>
> ... [And] there is never any doubt about who is and who is not a creative mathematician, so all that is required is to keep track of the activities of these few men." [1, p. 39].

Now there's a thought (Davis and Hersh [5] described his article as a statement of views that were "romanticized, manic-depressive, and apocalyptic", painting a "dismal picture"), and hardly an encouragement, which—chiming with (G.H.) Hardy—he supplemented with (p. 40)

> "The mathematical life of a mathematician is short. Work rarely improves after the age of twenty-five or thirty. If little has been accomplished by then, little will ever be accomplished. If greatness has been attained, good work may continue to appear, but the level of accomplishment will fall with each decade. Perhaps this is due to an early failure of the nerve for excellence. It is easy to believe that life is long and one's gifts are vast—easy at the beginning, that is. But the limits of life grow more evident; it becomes clear that great work can be done rarely, if at all. ... [Creativity] requires work of truly immense concentration. Such consuming commitment can rarely be continued into middle and old age, and mathematicians after a time do minor work."

[12] The words ("unsere losung" ["our slogan"]), which appear as the epitaph on Hilbert's gravestone, were given in response to the Latin maxim *"Ignoramus et ignorabimus"* ["We do not know and we shall not know"] which represents the idea that scientific knowledge is limited (by the ultimate nature of matter and the enigma of consciousness) and was promulgated by the 19th century German physiologist Emil du Bois-Reymond. Hilbert spoke them at the conclusion of his retirement address to the yearly meeting of the Society of German Natural Scientists and Physicians (the Gesellschaft der Deutschen Naturforscher und Ärzte) in Königsberg on September 8th, 1930. Shortly afterwards, he read on German radio a four-minute version of the finale of his speech. The excerpt stands as a dramatic anthem that—offered at a certain point in the history of the discipline by one of the greatest minds—has inspired many mathematicians; it climaxed a development of thought evidenced in Hilbert's earlier work, and reacted to broad and deep trends in philosophy and mathematical culture at the time.

This is harsh stuff from Adler (particularly since he appears to have had a very low profile as an academic researcher (if he had one at all), probably falling into Bell's category of disproportionately loud voices), but he was, as we have seen, clearly not alone in thinking along these lines. Was he striking a lively chord of cautionary rationalism with Thomas Hardy who wrote, in an opening 'Apology' to his said *Late Lyrics*, "Heine observed nearly a hundred years ago that the soul has her eternal rights; that she will not be darkened by statutes, nor lullabied by the music of bells. And what is to-day, in allusions to the present author's pages, alleged to be "pessimism" is, in truth, only such "questionings" in the exploration of reality, and is the first step towards the soul's betterment, and the body's also." (pp. vii–viii)?[13] Who can tell?

At the end of the day, each of us knows deep down that we should ideally bow out gracefully, which is probably when, to paraphrase Sylvester, we do not consider that our mathematical erudition is sufficiently extensive, nor the vigour of our mental constitution adequate, to keep us conversant with the continually advancing tide of mathematical progress.[14] Staving off the clouds of introspective dolefulness after our departure from the active scene of research is a different matter, though Einstein—that detached, amused and non-conformist individual—had a few succouring words of fortification for us all in a March 1950 letter to P. Moos (the Moos family were connected to the Einsteins, Albert's grandfather Abraham having married German Jew Helene Moos in 1839) where he wrote "I am content in my later years. I have kept my good humor and take neither myself nor the next person seriously." Laying great store on one's chosen field (that is, holding close an indomitable zest for learning and new knowledge) while keeping one's ego in check and maintaining a sense of perspective, as he did, is a great assistance in remaining happy and conserving properly centred emotions. Longevity can help the ride, figuratively speaking, Einstein having observed something (in a different letter addressed to Queen Elizabeth of Belgium at the start of 1954) that we can actually apply to our academic travails as the years roll by:

> "In one's youth every person and every event appear to be unique. With age one becomes much more aware that similar events recur. Later on one is less often delighted or surprised, but also less disappointed than in earlier years."[15]

In some ways membership of the academy—because of its allure to outsiders (those wanting to 'belong' to university life and carve out a purposeful career), but also because of the way it shapes behaviours and belief systems of its insiders (by requiring compliance and defining what success looks like)—has been said to be a kind of cult. But cults, through the abuses they all too often inflict (coercion, bullying, obedience, submission, conformity, and the like), are known to be dangerous to the well being of members, and this is certainly true of H.E. for many staff at present. I was recently reading the thoughts of Cardiff University's Michael Marinetto in the *Times Higher Education*,[16] where he asserts that being "outside the whale" of professional dogma—that is, being *in* but not *of* academe—is an alternative existence which has real appeal and virtue for self-preservation, being founded according to him on three principles espoused by sociologist C. Wright Mills (described as the mid-20th century "closet Marxist" from Waco, Texas, and an intellectual outcast in the urbane Ivy League setting of Columbia University): Firstly, professional focus should be on mastering authentic scholarship, and not on impressing peers by outcompeting or outproducing them as rivals. Secondly, we should value depth of thinking and scholarship over instrumental performativity measures. Thirdly, we should remember that academic writing elevates the methodical practice of relevant skills above the pursuit of rewards or goals. Working on your own terms, and being at ease with it, is no mean feat, and we can learn from those who are invested practically in the university while maintaining a spiritual and emotional detachment from it—they are indeed a fortunate minority who, mentally living outside the metaphorical bloated whale and, as Marinetto aptly puts it, away from its "bottomless

[13] Heinrich Heine was a German poet, writer, literary critic, and political radical. Thomas~~Heinrich Heine~~ Hardy makes no excuse for the fact that (p. vii) ". . . some grave, positive, stark, delineations are interspersed among those of the passive, lighter, and traditional sort presumably nearer to stereotyped tastes." In his search for insights into, and understanding of, humanity he was compelled to visit all sides of it and report his findings in poetry; Adler seems to have wished to do likewise narratively in his vivid, and in places severe, smaller scale examination of mathematics and the archetypal mathematician.

[14] He gave these as his reasons for leaving John Hopkins University (Baltimore, Maryland), in his late 60s, in a January 1884 letter to German Christian Felix Klein.

[15] These two letter extracts appear (resp.) on p. 190 & p. 191 of A. Calaprice's edited collection of citations in *The Quotable Einstein* (Princeton University Press (1996)); I have been unable to access the original documents from which they are taken.

[16] 'How to Survive in the Era of Academic Overproduction', September 2019.

appetite for publications", are "free to dive for pearls"; this may come to some earlier rather than later, or not at all, as the case may be, but one likes to think it will at some point, and age may be a determining factor in being able to fashion and curate such an internalised safe space (that is to say, raise a defensive psychological shield robust enough to thwart those pressures which are endemic to the tertiary sector). Anyone in this position is liberated, a good case in point being the much respected Israeli-American mathematician Doron Zeilberger—forthright, opinionated, provocative, humerous and entertaining—who, on being interviewed by Ron Aharoni in 2015 (a (lightly edited) record is available from Zeilberger's personal homepage), bemoaned the unsatisfactory nature of refereeing/reviewing (and the standard of feedback) in the context of article/grant submissions, stating

> "..., *asymptotically*, like in my case, once one is fully promoted, one can once again attain a peace of mind, and decide **not** to play that game. With today's internet, where one can have a website, it suffices to publish in your own website, and, even more importantly, in the most important venue for mathematical (and other) knowledge, `arxiv.org`. This would guarantee that your papers will be accessible "for ever" (well, at least, as long as in "real" journals), and the future would decide the "significance" (and correctness!) of your work."[17]

He has some interesting and sometimes controversial things to say about the way mathematics is done—especially concerning the role of computer-generated and computer-assisted proofs which he has promoted relentlessley and boldly for years—as befits someone who is old enough to have made a serious contribution to the mathematical community without worrying too much (if at all) how it regards him and his energetic *chutzpah* these days.

When all is said and done, one could do worse than take the following pithy reflection as an antidote to the outputs-orientated environment which can take the joy out of research all too easily whether we be young, old, or in between; it is fitting that it, too, comes from one of the major research statesmen who ever lived:

> *"The value of achievement lies in the achieving."*
> Albert Einstein (attributed, 1950[18])

Wise words for a personal mantra, as mathematics is a symphony scored for many kinds of voices. Anyhow, I'll just bat on I guess, stoic in the portentous shadow of what the Germans term *torschlusspanik* that now looms over my own mortal *sojourn*. So much to do, and so very little time ...

Appendix: Backdrop to the Essay

This appendix details, at some length, aspects of the essay which served as a backdrop to the main discourse but which are deserving of inclusion.[19] G.H. Hardy's 1928 Rouse Ball Lecture reveals that he placed himself quite deliberately on the unaccustomed enclave of mathematical logic and philosophy as the theme for his talk.[20] His apology to some mathematicians with little or no interest in the latter was made with only partial sincerity: "I feel that this distaste is usually based on no better foundation than an unreasoning shrinking from anything unfamiliar, the distaste of the pragmatist for truth, of the engineer for mathematics, of the pavilion critic at Lords for the in-swinger and the two-eyed stance." It's good to move away from our own comfort zones—even the greats recognise the value of doing so (perhaps that is one reason why they are great: "..., the exercise will be good for me, since it will force me to think seriously about

[17] He regards the competitive atmosphere of academia and the pressure—in order to survive—to apply for research grants and to submit articles to 'prestigious' journals (where rejection often comes with "nasty, patronizing, "reports", whose contents show that the referees, or reviewers, missed the whole point") as a real nuisance and distraction from things on which able mathematicians should be spending their time profitably.

[18] I have not seen the citation, referenced in Calaprice's aforementioned work, in its published form.

[19] One of the few serious projects to investigate how ageing mathematicians review and go about their work is summarised in the excellent 2001 essay by Hersh [9] who solicited emotive and telling self-appraisals on the matter from over sixty mathematicians; he presented his findings in an extremely well written and informative composition that I unreservedly endorse as a fine read.

[20] 'Mathematical Proof', the transcript of which appeared in the January 1929 issue (Issue No. 149) of Volume XXXVIII of the journal (of psychology and philosophy) *Mind* (pp. 1–25); we take his words from the opening section of the address.

questions which a professional mathematician like myself is apt to neglect."). Accordingly, I have espoused this position here.

Hardy proffered a view (given in the Introduction) that has been seized and quoted often by writers of history—let's see what he had to say in full:

> "I had better say something here about this question of age, since it is particularly important for mathematicians. No mathematician should ever allow himself to forget that mathematics, more than any other art or science, is a young man's game. To take a simple illustration at a comparatively humble level, the average age of election to the Royal Society is lowest in mathematics." [8, pp. 70–71],

and, in addition,

> "... I do not know an instance of a major mathematical advance initiated by a man past fifty. If a man of mature age loses interest in and abandons mathematics, the loss is not likely to be very serious either for mathematics or for himself.
>
> On the other hand the gain is no more likely to be substantial; ..." (p. 72),

whereupon he proceeded to cite three people (Newton, Painlevé and Laplace) who had left the profession (or effectively left, in the case of Laplace) and not fared too well afterwards. I tend to think he is being somewhat narrow, and rather too simplistic ("Young men should prove theorems, old men should write books" is a quote attributed to Hardy by the admired British-American theoretical and mathematical physicist Freeman John Dyson (1923–2020) at a conference in 1987). French mathematician André Weil, taking stock of the discipline in a pensive piece but a few short years after the end of World War 2, was in Hardy's camp: "There are examples to show that in mathematics an old person can do useful work, even inspired work; but they are rare, and each case fills us with wonder and admiration." [22, p. 304]. Really? He did at least acknowledge the opportunities mathematics offers for personal satisfaction to all with a desire to acquire some, writing

> "It is certain that few men of our times are as completely free as the mathematician in the exercise of their intellectual activity. Even if some State ideologies sometimes attack his person, they have never yet presumed to judge his theorems. Every time that so-called mathematicians, to please the powers that be, have tried to subject their colleagues to the yoke of some orthodoxy, their only reward has been contempt. ... Mathematics is taught the world over, well here, badly there; the exiled mathematician—and who among us can to-day feel free from the danger of exile—can find everywhere the modest livelihood which allows him to pursue his work to some extent. Even in gaol one can do good mathematics if one's courage fail him not." (p. 296),

a feeling behind F.F. Bonsall's comment that "It cannot be denied that there are some thin and pointless generalizations and some cottage industries. But we should not worry too much about this; it is better for a mathematician to stay alive by attempting something than to give up the attempt." (see p. 13 of his 1982 article 'A Down-To-Earth View of Mathematics', *Amer. Math. Month.* **89**, 8–15).

Age alone brings no *de facto* position of authority within the academy, and perhaps Adler was on the right lines in his thinking when he wrote

> "There is in mathematics no repressive establishment of older men, no traditional way to do mathematics and so to become an important member of the mathematical community, no shortage of room for the rapid progress of gifted men, none of the slavish behavior toward academic superiors which is common in many other disciplines. What is also required is the institutionalized conviction that accomplishment is important only if it advances the discipline in some significant way. Competition must exist for creative achievement only—with and for the discipline itself, rather than with competitors." [1, p. 39].

Indeed, we would do well to face the sense in which our field very much chooses its own developmental bearings. English mathematician and astronomer James W. L. Glaisher, in an address published in one of the weekly issues of the science journal *Nature* (dated September 11th, 1890, and delivered to its British Association under the Mathematics and Physics Section for which he

acted as President), lauded the easy flow or homogeneity of form so characteristic of a mathematical theory, holding it above the fragmentary nature of other types of mathematical advancement and evolution found especially in topics related to the physical and other less abstract sciences. He wrote

> "In such a theory ... the subject develops itself naturally as it proceeds; one group of results leads spontaneously to another; new and unexpected prospects open of themselves; ideas the most novel and striking, which penetrate the mind with a charm of their own, spring directly out of the subject itself. We are surprised by the wonderful connections with other subjects which unexpectedly start into existence, and by the widely different methods of arriving at the same truths; in fact, as our knowledge progresses, we continually find that results which seemed to lie far away in the interior of the subject—so remote and concealed that, at first sight, we might think that no other path except the one actually pursued could have reached them—are actually close to its edge when approached from another side, or viewed from another stand-point. We notice, too, that any great theory gives rise to its own special analysis or algebra, frequently connecting together into one whole what were hitherto merely isolated and apparently independent analytical results, and affording a reason for their existence, and also—what is often even more interesting—a reason for the non-occurrence of others, which analogy might have led us to expect."[21]

On the same theme, Raymond G. Ayoub, editor of a 2004 anthology of essays from renowned mathematicians, wrote in his introductory remarks of the way mathematics itself decides which pieces it retains and which are discarded (as if *une volonté générale*, if you will, has a secondary function); this makes a bit of a mockery of our striving to leave something of ourselves behind if that is where the priority lies, and on a bad day prompts us to know our place, as they say:[22]

> "Although there is a certain randomness in the choice of structure or property which is chosen for study, the history of mathematics shows that there is a ruthless selectivity that takes place, and in a manner that is far from being understood. The discipline casts off those parts that do not give it its proper nourishment. This selectivity is made, not by the practitioners but in a curious way, by the subject itself—the whole mathematical organism deprives the mathematician of his or her autonomous role, retaining only those parts that it deems essential. A hazardous activity then, is to try to predict the fate of a mathematical structure or object and one of the nightmares of a mathematician is to contemplate the future fate of his or her work!"

It has been conventional (though misguided) wisdom that mathematical breakthroughs are most often made in a moment of brilliance by a born genius at a young age, rather than by a seasoned practitioner after decades of work. The media reinforces the stereotype of youthful mathematical creativity and precociousness, cementing Hardy's comments into our minds and perpetuating an illusion that is decorated by stories of those mathematicians robbed of the chance to build a durable and profitable career far too early in life.[23] Andrew Wiles was 42

[21] See p. 464 of his Opening Address (Section A—Mathematics and Physics), *Nature* **XLII** (Issue No. 1089), 464–468 (1890).

[22] This quotation is taken from p. ix of *Musings of the Masters: An Anthology of Mathematical Reflections*, Mathematical Association of America, Washington (2004), where he continues thus: "There is the story, possibly apocryphal, of Bertrand Russell's dream in which he saw a phantom walking through the stacks of a library with a roaring fire below. The phantom examines a book taken from the shelf and either returns it to the shelf or casts it into the flames. Coming upon the *Principia*, a monumental work of Russell and Whitehead, the phantom pauses, [and] hesitates for a moment—at which time Russell awoke from his dream!"; G.H. Hardy himself recounts the same story [8, p. 83].

[23] As a teenager Évariste Galois solved a longstanding problem involving polynomial roots—and in doing so provided the foundations for Galois theory and group theory—only to die from wounds suffered in a duel aged 20. Niels Abel made his numerous discoveries (also looking at polynomial roots) while living in poverty, and passed away at 26 from tuberculosis. Perhaps the best known is Srinivasa Ramanujan, who discovered a vast array of theorems and identities but sadly died aged 32 after moving to England to work with Hardy and subsequently developing health problems; the list goes on (Englishman Frank Ramsey (26; developed jaundice after an operation), Frenchman René Gâteaux (26; killed in active combat during World War 1), Russian Pavel Urysohn (26; drowned while swimming), German Gotthold Eisenstein (29; died after being weakened by imprisonment), Englishman William Clifford (33; succumbed to tuberculosis)). Although she was a little older, one of the most tragic cases is that of Maryam Mirzakhani—the only woman, and the first Iranian, to be honoured with the Fields Medal (cited for "her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces")—who in her prime was defeated by breast cancer at just 40. There are plenty of others who belong to this unfortunate group, robbed of their chance to live and work as a mathematician.

when he published a final proof of Fermat's Last Theorem, having spent years in single-minded solitude working on it. His peak powers sit in what is thought by many to be a golden 30–45 age window where wonderful things may be done (Grigori Perelman, for example, was 36 when he released his detailed proof sketch for a conjecture of William Thurston (who asserted that all closed 3-manifolds admit a certain kind of geometric decomposition) which eventually led to the affirmative resolution of Poincaré's Conjecture), but the flip side to all of this is someone like the Chinese-American academic Zhang Yitang who quite suddenly published an important result in number theory, in his late 50s, after years of ostensible inactivity (he established the first finite bound on the least gap between consecutive primes that is attained infinitely often, making a big step towards answering the so called Twin Prime Conjecture which at present remains an open proposition); other late(r) stage contributors are Apery, Levinson, Gelfand, Weierstrass, Rado, Dyson, de Branges, Kolmogorov, Cauchy, Euler, Ratner, Novikov, Möbius, Mihailescu, Zariski, Gauss, Furtwängler, Erdős, Weil, Wintenberger, de Moivre, Chapman, Besicovitch, Davenport, Borwein, Guy, Fraenkel, and so on. Hersh gave over part of Chapter 7 of his popular co-authored book *Loving + Hating Mathematics: Challenging the Myths of Mathematical Life* (Princeton University Press, Princeton (2011)) to the issue of age, where—as well as being offered an update on the survey reported in [9] a decade earlier—we read that Louis Joel Mordell (Hardy's successor at Cambridge) published almost half of his 270 papers after retiring at 75, and that results by Littlewood—who outlived Hardy by some thirty years—continued to appear in print until he was 87. Some areas of mathematics might even be said to militate against youth simply because there is so much material to assimilate and absorb before even tackling a problem of worth.

Adding to the frame of reference here, note that since the 1950s life span studies of achievement have concentrated for the most part on three core topics: (a) age curves that specify how creative output varies over a career, (b) connections between early impact, longevity, and output rate, and (c) relations between quantity and quality of works produced. There are many parameters in play, with cultural and discipline contrasts considered—along with critical methodology issues that confront researchers teasing out fact from artefact, myth from reality—in attempts to address speculations and make discoveries about age and accomplishment in different sectors that have piqued the interest of developmental psychologists since the 19th century (the earliest work in this regard is thought to be George M. Beard's *Legal Responsibility in Old Age, Based on Researches Into the Relation of Age to Work*, read before the Medico-Legal Society of New York at a regular meeting in March 1873). The field has expanded in its theoretical interpretations (both extrinsic and intrinsic) and its models into which longitudinal data is used (most subjects exhibit the classic and not unexpected rise in productivity to a peak with (domain-specific) age, followed by a post peak decline), within which mathematics harbours nuances arising from some of the singular working practices distinctive and occasionally peculiar to itself. Thus it is, then, that there is a small industry of formal studies into age and the effect it has on such things as inspiration, originality, ideation and fecundity, alongside which this essay—a mix of the observational, evidential and subjective—sits in relation to the habits, intellectual predispositions, sensibilities, needs and expectations of mathematicians as they move through the transitional phases of professional employment from entrance onto the academic stage until curtain close. German philosopher, writer and polymath Friedrich Wilhelm Nietzsche (1844–1900) wrote that (at least) while profundity of thought belongs to youth, clarity of thought belongs to old age.

Noting the words of optimism from American mathematician and philosopher Norbert Wiener (in an autobiographical work),

> "... mathematics is a field in which one's blunders tend to show very clearly and can be corrected or erased with a stroke of the pencil. It is a field which has often been compared with chess, but differs from the latter in that it is only one's best moments that count and not one's worst. A single inattention may lose a chess game, whereas a single successful approach to a problem, among many which have been relegated to the wastebasket, will make a mathematician's reputation." [23, p. 21],

we should strive to do our very best—guarding against complacency before mental decline rears its head or caducity raps at the door—since, in the words of Russian revolutionary, political theorist and politician Leon Trotsky (this a single May 8th entry from Notebook 2 of his published 1935 diary),

> "Old age is the most unexpected of all things that happen to a man." [21, p. 99];

it is this quote I think, over and above any other I've given, that moved me to produce the essay.

The progression of mathematics is neither linear nor continuous, but waxes and wanes under periods of feverish bustle and fallower times, rather like our lives. Littlewood wrote "The higher mental activities are pretty tough and resilient, but it is a devastating experience if the drive does stop, . . ." [10, p. 116], adding the sage words

> "Minor depressions will occur, and most of a mathematician's life is spent in frustration, punctuated with rare inspirations. A beginner can't expect quick results; if they are quick they are pretty sure to be poor.
> . . .
> When one has finished a substantial paper there is commonly a mood in which it seems that there is really nothing in it. Do not worry, later on you will be thinking "at least I could do something good *then*." At the end of a particularly long and exacting work there can be a strange melancholy. This, however, is romantic, and mildly pleasant, like some other melancholies."

When we are engaged in deep mathematical study we enter what is known as "lived time"—the time of our inner subjective experience, or time as felt and acted out personally. There is a degree of overlap amongst mathematicians as to how *la durée* ("duration"), as it is called, is perceived, for the discipline of mathematics requires certain states of meditation, self-situation and mental being that cut across its entirety (the term was coined by French philosopher Henri-Louis Bergson (1859–1941)). We have no say in, or control over, "objective time", however (the time of watches, calendars, and so on)—that's just an ineluctable fact, and we (as is everyone else) are presented with a Hobson's choice in dealing with it successfully or not.[24] One thing of note is that there seems to be a consensus among older mathematicians—along with a resignation—that a Shifting Baseline Syndrome has altered the environments in which we work for those younger than us, setting up a cycle of unopposed decline that is disturbing (the syndrome—which is gaining traction across different disciplines—is in this context a gradual redefinition and lowering of agreed norms for conditions associated with working educational practices due to a lack of experience, memory and/or knowledge of their past state; in other words, what is considered to be a healthy tertiary sector now would have been thought of by past generations as degraded, and what we judge to have already deteriorated will feed into what the next generation accepts without reserve).

In conclusion here, it seems to me that being a mathematician, in the (generally agreed) real sense of the word, affords entry to a fascinating and captivating world whose richness and abundance can satisfy us for a lifetime if we are able to filter out distractions and look upon it in the right way, carrying the wisdom of Halmos with us:

> "If you set out to be a mathematician, you must learn the profession, every part of it, and then work at it, profess it, live it as best you can. If you keep asking "what's there in it for me?", you're in the wrong business. If you're looking for comfort, money, fame, and glory, you probably won't get them, but if you keep trying to be a mathematician, you might." [7, p. 266].

Tennyson, in his famous poem 'Ulysses', deals with the desire to reach beyond the limits of one's field of vision and the mundane details of everyday life, the poem's hero wishing to "follow knowledge like a sinking star" and forever grow in wisdom and in learning (*"Old age hath yet his honour and his toil;/Death closes all: but something ere the end,/Some work of noble note, may yet be done,/ . . ."*)—a useful thought, and an uplifting one.[25] The refreshing waters of professional contentment, of which we may partake, are there for us to enjoy, but we should never take them for granted nor because of them overplay our position in the scheme of things for this would diminish us. Ethnographer Mark de Rond (presently Cambridge University Professor of Organisational Ethnography) spent six weeks at Camp Bastion military field hospital

[24] Interestingly, he put forward the controversial claim that only *la durée* is real, and that objective time is merely an external construction imposed upon our lives.

[25] He finishes with *"Tho' much is taken, much abides; and tho'/We are not now that strength which in old days/Moved earth and heaven, that which we are, we are;/One equal temper of heroic hearts,/Made weak by time and fate, but strong in will/To strive, to seek, to find, and not to yield."*

during the Afghanistan War, collecting material for a book on the medical teams *in situ*. What he witnessed was unimaginable suffering, loyalty and bravery, returning from his profound experiences carrying a sense of the futility—with which he still wrestles—of so much of the research that emerges from a self-absorbed and largely petty academic life. Noting the dangers of overestimating the importance of our own mathematical work, it might be beneficial to have at hand his ringing reminder that "Whatever important purpose we academics are meant to serve, what we do will make f**k-all difference to the world."[26] Overstated, yes, but the point is made.

References

[1] A. Adler, Reflections: mathematics and creativity, *The New Yorker* (Feb. 19th Issue), 39–45 (1972).

[2] M. Atiyah, Advice to a young mathematician (No. I), *in* T. Gowers, J. Barrow-Green and I. Leader (Eds.), *The Princeton Companion to Mathematics*, Princeton University Press, Princeton, 1000–1004 (2008).

[3] E. T. Bell, *The Development of Mathematics*, McGraw-Hill, New York (2nd edition) (1945).

[4] E. T. Bell, *Men of Mathematics*, Simon and Schuster, Inc., New York (1986).

[5] P. J. Davis and R. Hersh, *The Mathematical Experience*, Penguin, Harmondsworth (1984).

[6] P. R. Halmos, Mathematics as a creative art, *Amer. Sci.* **56**, 375–389 (1968).

[7] P. R. Halmos, *I Want To Be a Mathematician: An Automathography*, Springer Verlag, New York (1985).

[8] G. H. Hardy, *A Mathematician's Apology*, Cambridge University Press, Cambridge (2011 reprinted edition) (1940).

[9] R. Hersh, Mathematical menopause, or, a young man's game?, *The Math. Intell.* **23(3)**, 52–60 (2001).

[10] J. E. Littlewood, The mathematician's art of work, *The Math. Intell.* **1**, 112–118 (1978).

[11] W. Page, Herbert Robbins, *in* D. J. Albers and G. L. Alexanderson (Eds.), *Mathematical People: Profiles and Interviews*, Birkhäuser, Boston, 281–297 (1985).

[12] T. Paine, The Age of Reason (Part 2), *in* M. D. Conway (Ed.), *The Writings of Thomas Paine* (Vol. IV), G.P. Putnam's Sons, London, 85–195 (1908).

[13] G.-C. Rota, Ten lessons I wish I had been taught, *Not. Amer. Math. Soc.* **44**, 22–25 (1997).

[14] G.-C. Rota, *Indiscrete Thoughts*, Birkhäuser, Boston (2008 reprinted edition) (1997).

[15] G.-C. Rota, Foreword to F. Bergeron, G. Labelle and P. Leroux, *Combinatorial Species and Tree-Like Structures*, Cambridge University Press, Cambridge (1998).

[16] T. Stanway, From G.H.H. and Littlewood to XML and Maple: changing needs and expectations in mathematical knowledge management, *in* R. Hersh (Ed.), *18 Unconventional Essays on the Nature of Mathematics*, Springer, New York, 147–159 (2006).

[17] J. J. Sylvester, Note on a proposed addition to the vocabulary of ordinary arithmetic, *Nature* **37**, 152–153 (1887).

[18] J. J. Sylvester, An inquiry into Newton's rule for the discovery of imaginary roots, *in* H. F. Baker (Ed.), *The Collected Mathematical Papers of James Joseph Sylvester*, Volume II (1854–1873), Cambridge University Press, Cambridge, 376–479 (1908).

[19] J. J. Sylvester, Address to the mathematical and physical section of the British Association (Exeter British Association Report 1869), *in* H. F. Baker (Ed.), *The Collected Mathematical Papers of James Joseph Sylvester*, Volume II (1854–1873), Cambridge University Press, Cambridge, 650–661 (1908).

[20] W. P. Thurston, On proof and progress in mathematics, *Bull. Amer. Math. Soc.* **30**, 161–177 (1994).

[21] L. Trotsky, *Trotsky's Diary in Exile 1935* (translated from Russian by E. Zarudnaya), Faber and Faber, London (1958).

[22] A. Weil, The future of mathematics, *Amer. Math. Month.* **57**, 295–306 (1950).

[23] N. Wiener, *Ex-Prodigy: My Childhood and Youth*, Simon and Schuster, Inc., New York (1953).

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[26] See the *Times Higher Education* piece 'Battle Scarred: An Ethnographer at Camp Bastion' of April 2017.