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Local maximum points of explicitly quasiconvex functions

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Abstract This work concerns (generalized) convex real-valued functions defined on a nonempty convex subset of a real topological linear space. Its aim is twofold. The first concerns explicitly quasiconvex functions. As a counterpart of some known results, it is shown that any local maximum point of such a function is actually a global minimum point whenever it belongs to the intrinsic core of the function's domain. Secondly, we establish a new characterization of strictly convex normed spaces by applying this property for a particular class of convex functions.

Keywords Local maximum point · Relative algebraic interior · Convex function · Explicitly quasiconvex function · Strictly convex space · Least squares problem

1 Introduction

Optimization problems involving explicitly quasiconvex objective functions, i.e., real-valued functions which are both quasiconvex and semistrictly quasiconvex, have been intensively studied in the literature, beginning with the pioneering works by Martos [6] and Karamardian [5]. These functions are of special interest since they preserve several fundamental properties of convex functions.

An interesting result concerning extended real-valued convex functions defined on a locally convex space has been established by Zălinescu [10, Proposition 2.5.8]. It states that any local maximum point located in the function's effective domain (indeed in its interior) is actually a global minimum point. Cambini and Martein [3, Exercise 2.22] pointed out that this property holds for any lower semicontinuous explicitly quasiconvex real-valued function defined on a nonempty convex subset of a finite-dimensional Euclidean space.

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After establishing a preliminary result on linear operators in Section 2, we present a counterpart of the above mentioned results in the more general context of explicitly quasiconvex real-valued functions defined on a nonempty convex subset of a real topological linear space in Section 3. More precisely, we will show that every local maximum point belonging to the intrinsic core of the function's domain is in fact a global minimum point. By applying this property for a particular class of convex functions we establish a new characterization of strictly convex normed spaces.

The paper is structured as follows. In Section 2 we present some definitions and notations used in the sequel and we establish a preliminary result on linear operators. Section 3 contains results concerning the extremal properties of generalized convex functions. In Section 4 we present two new characterizations of strictly convex normed spaces, while a result related to the least squares problem is elegantly proved and refined as a direct application.

2 Preliminaries

Throughout this paper X will be a real topological linear space. We denote by 0_X the origin of X and by $\mathcal{V}(x)$ the family of all neighborhoods of $x \in X$. Recall that (see, e.g., Boţ and Csetnek [2]) the *core* (*algebraic interior*) and the *intrinsic core* (*relative algebraic interior*) of a set $S \subseteq X$ are defined as

$$\operatorname{cor} S = \{x \in S \mid \forall\, y \in X,\ \exists\, \delta > 0 \text{ s.t. } x + [0,\delta] \cdot y \subseteq S\};$$
$$\operatorname{icr} S = \{x \in S \mid \forall\, y \in \operatorname{span}(S - S),\ \exists\, \delta > 0 \text{ s.t. } x + [0,\delta] \cdot y \subseteq S\}.$$

Notice that $\operatorname{int} S \subseteq \operatorname{cor} S \subseteq \operatorname{icr} S$ for any $S \subseteq X$. Thus every $S \in \mathcal{V}(0_X)$ is absorbing, i.e., $0_X \in \operatorname{cor} S$. When X is a locally convex space and $S \subseteq X$ is convex with $\operatorname{int} S \neq \emptyset$, then $\operatorname{int} S = \operatorname{cor} S = \operatorname{icr} S$ (see, e.g., Borwein and Lewis [1]).

Given a real topological linear space Y we denote by $L(X,Y)$ the space of all linear operators acting between X and Y. The origin of Y is denoted by 0_Y, while the origin of $L(X,Y)$ is the null operator $0_{L(X,Y)}$, i.e., $0_{L(X,Y)}(x) = 0_Y$ for all $x \in X$. The kernel and the image of any linear operator $A \in L(X,Y)$ will be denoted by $\operatorname{Ker}(A) = A^{-1}(\{0_Y\})$ and $\operatorname{Im}(A) = A(X)$, respectively.

Lemma 2.1 *For any $A \in L(X,Y)$ and $U \subseteq X$ the following assertions hold true*:

1° $A(\operatorname{icr} U) \subseteq \operatorname{icr} A(U)$.
2° *If* $0_X \in \operatorname{icr} U$, *then* $0_Y \in \operatorname{icr} A(U)$.
3° *If* $0_X \in \operatorname{cor} U$ (*in particular, if* $U \in \mathcal{V}(0_X)$), *then* $\mathbb{R}_+ \cdot A(U) = \operatorname{Im}(A)$.

Proof For proving 1°, consider a point $y^0 \in A(\operatorname{icr} U)$ and choose $x^0 \in \operatorname{icr} U$ such that $y^0 = A(x^0)$. Let $y \in \operatorname{span}(A(U) - A(U))$. Observing that $\operatorname{span}(A(U)-A(U)) = A(\operatorname{span}(U-U))$ due to the linearity of A, we deduce the existence of a point $x \in \operatorname{span}(U-U)$ such that $y = A(x)$. Since $x^0 \in \operatorname{icr} U$, there is a $\delta > 0$ such that $x^0 + [0,\delta] \cdot x \subseteq U$. Thus $y^0 + [0,\delta] \cdot y = A(x^0) + [0,\delta] \cdot A(x) = A(x^0 + [0,\delta] \cdot x) \subseteq A(U)$, which proves that $y^0 \in \operatorname{icr} A(U)$.

Assertion 2° is a direct consequence of 1°, since $0_Y = A(0_X)$ as $A \in L(X,Y)$. In order to prove 3°, assume that $0_X \in \operatorname{cor} U$, which implies $X = \mathbb{R}_+^* \cdot U$. By the linearity of A one obtains the relation $\mathbb{R}_+^* \cdot A(U) = A(\mathbb{R}_+^* \cdot U) = A(X) = \operatorname{Im}(A)$. $\square$

Remark 2.1 The intrinsic core cannot be replaced by the core in assertions 1° and 2°, since in general $0_X \in \operatorname{cor} U$ does not imply $0_Y \in \operatorname{cor} A(U)$. Also, if $0_X \in \operatorname{icr} U$, then the conclusion of 3° may be false, as the following example shows.

Example 2.1 Let $X = Y = \mathbb{R}^2$ with $0_X = 0_Y = (0,0)$. Consider the linear operator $A \in L(X,Y)$ defined by $A(x) = (x_1, 0)$ for every $x = (x_1, x_2) \in X$. Obviously $A(X) = \operatorname{Im}(A) = \mathbb{R} \times \{0\}$, hence $0_Y \notin \operatorname{cor} A(X) = \emptyset$, while $0_X \in \operatorname{cor} X$. Also, for $U = \operatorname{Ker}(A) = \{0\} \times \mathbb{R}$ we have $0_X \in \operatorname{icr} U$, but $\mathbb{R}_+ \cdot A(U) = \{0_Y\} \neq \operatorname{Im}(A)$.

Various notions of generalized convexity are currently used in optimization theory. Since some of them appear in the literature under different names, we recall here their definition in order to avoid any confusion. A real-valued function, $f : D \to \mathbb{R}$, defined on a nonempty convex set $D \subseteq X$, is called:

– *convex*, if for any points $x', x'' \in D$ and every number $t \in [0,1]$ we have

$$f((1-t)x' + tx'') \le (1-t)f(x') + tf(x''); \tag{2.1}$$

– *strictly convex*, if f satisfies (2.1) with strict inequality for all distinct points $x', x'' \in D$ and every number $t \in]0,1[$;
– *quasiconvex*, if for any points $x', x'' \in D$ and every number $t \in [0,1]$ we have

$$f((1-t)x' + tx'') \le \max\{f(x'), f(x'')\}; \tag{2.2}$$

– *strictly quasiconvex*, if f satisfies (2.2) with strict inequality for all distinct points $x', x'' \in D$ and every number $t \in]0,1[$;
– *semistrictly quasiconvex*, if f satisfies the strict inequality in (2.2) for any points $x', x'' \in D$ such that $f(x') \neq f(x'')$ and every number $t \in]0,1[$;
– *explicitly quasiconvex*, if it is both quasiconvex and semistrictly quasiconvex.

Notice that strictly convex functions are both convex and strictly quasiconvex; convex functions and strictly quasiconvex functions are explicitly quasiconvex; any lower semicontinuous semistrictly quasiconvex function is explicitly quasiconvex.

3 Extremal properties of generalized convex functions

Given a function $f : D \to \mathbb{R}$, defined on a nonempty subset D of X, recall that $x^0 \in D$ is a local minimum point of f if there exists $V \in \mathcal{V}(x^0)$ such that

$$f(x^0) \le f(x),\ \forall\, x \in V \cap D. \tag{3.1}$$

If (3.1) holds for $V = X$, then x^0 becomes a global minimum point of f. Similarly, x^0 is a local maximum point of f if there exists $V \in \mathcal{V}(x^0)$ such that

$$f(x^0) \ge f(x),\ \forall x\ \in V \cap D \tag{3.2}$$

and x^0 is a global maximum point of f if (3.2) holds for $V = X$. As usual, we denote by $\operatorname{argmin}_{x \in D} f(x)$ the set of all global minimum points of f.

Lemma 3.1 *Let $f : D \to \mathbb{R}$ be a semistrictly quasiconvex function. If $x^0 \in D$ is a local minimum point of f, then it actually is a global minimum point.*

Proof Assume that x^0 is a local minimum point of f and choose a neighborhood $V \in \mathcal{V}(x^0)$ for which (3.1) holds true. Suppose by the contrary that x^0 is not a global minimum point of f. Then there exists $x' \in D$ such that $f(x') < f(x^0)$. Since $V - x^0 \in \mathcal{V}(0_X)$ is absorbing we have $0_X \in \operatorname{cor}(V - x^0)$. Thus we can find $\delta > 0$ such that $[0, \delta] \cdot (x' - x^0) \subseteq V - x^0$. Choosing any number $t \in]0, \min\{1, \delta\}[$ we get $(1-t)x^0 + tx' = x^0 + t(x' - x^0) \in D \cap V$. The semistrict quasiconvexity of f implies $f((1-t)x^0 + tx') < \max\{f(x^0), f(x')\} = f(x^0)$, contradicting (3.1). □

Remark 3.1 In the particular case when X is a finite-dimensional Euclidean space, we recover from Lemma 3.1 a classical result by Ponstein [7, Theorem 2]. In fact, several results of this pioneering paper can be extended to our general framework. We just mention here that, according to [7, Theorem 3], if a local minimum point x^1 of a quasiconvex function $f : D \subseteq X = \mathbb{R}^n \to \mathbb{R}$ is not a global minimum, then f is constant in the intersection of some neighborhood of x^1 and the line segment between x^1 and any global minimum point x^2. Following Ponstein's work, Greenberg and Pierskalla [4, Table II, 11.b] claimed that "every local minimum of a quasiconvex function f is a global minimum or f is constant in a neighborhood of the local minimum." However, the reader should notice that the latter statement is wrong, as the example below shows.

Example 3.1 Let $X = \mathbb{R}$ and consider the function $f : D = \mathbb{R}_+ \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1[\\ 1 & \text{if } x \in [1, 2[\\ x - 1 & \text{if } x \in [2, \infty[. \end{cases}$$

Obviously f is quasiconvex and $x^1 = 2$ is a local minimum point of f which is not a global minimum. The function f is constant on $[1, 2]$, i.e., the intersecton of $V = [1, 3] \in \mathcal{V}(x^1)$ and the line segment $[0, 2]$ joining x^1 and the global minimum point $x^2 = 0$. However, f is not constant in any neighborhood of x^1.

Theorem 3.1 *Let $f : D \to \mathbb{R}$ be an explicitly quasiconvex function. If $x^0 \in \operatorname{icr} D$ is a local maximum point of f, then x^0 is a global minimum point of f.*

Proof Assume that $x^0 \in \operatorname{icr} D$. Since x^0 is a local maximum point of f, we can choose a neighborhood V of x^0 satisfying (3.2). We will first prove that f is constant on $V \cap D$, i.e., $f(x) = f(x^0)$ for all $x \in V \cap D$.

Suppose by contrary that $f(x') \neq f(x^0)$ for some $x' \in V \cap D$. By (3.2), we have

$$f(x') < f(x^0). \tag{3.3}$$

Since $V \in \mathcal{V}(x^0)$, we have $V - x^0 \in \mathcal{V}(0_X)$. Thus $V - x^0$ is absorbing, hence $0_X \in \operatorname{cor}(V - x^0)$. This guarantees the existence of $\delta_1 > 0$ such that

$$[0, \delta_1] \cdot (x^0 - x') \subseteq V - x^0. \tag{3.4}$$

As $x^0 \in \operatorname{icr} D$, and $x^0 - x' \in D - D \subseteq \operatorname{span}(D - D)$, there is $\delta_2 > 0$ such that

$$x^0 + [0, \delta_2] \cdot (x^0 - x') \subseteq D. \tag{3.5}$$

Consider the point $x'' = x^0 + \delta(x^0 - x')$, where $\delta = \min\{\delta_1, \delta_2\}$. From (3.4) and (3.5), it is clear that $x'' \in V \cap D$, therefore

$$f(x'') \leq f(x^0). \tag{3.6}$$

On its turn, x^0 can be written as $x^0 = (1-t)x' + tx''$, where $t = \frac{1}{\delta+1} \in \,]0,1[$. If $f(x') = f(x'')$, then $f(x^0) \leq \max\{f(x'), f(x'')\} = f(x')$ by quasiconvexity of f, which contradicts (3.3). Otherwise, if $f(x') \neq f(x'')$, then from the semistrictly quasiconvexity of f one has $f(x^0) < \max\{f(x'), f(x'')\}$. On the other hand, (3.3) and (3.6) show that $\max\{f(x'), f(x'')\} \leq f(x^0)$, which yields a contradiction.

Thus f is constant on $V \cap D$, hence x^0 is a local minimum point of f. Finally, by means of Lemma 3.1, we conclude that x^0 is a global minimum point of f. □

Remark 3.2 By considering the classical interiority concept in the particular case when $X = \mathbb{R}^n$, Cambini and Martein [3, Exercise 2.22] pointed out that a non-constant lower semicontinuous semistrictly quasiconvex function cannot have an interior local maximum point which is not a local minimum. The above theorem extends this result.

Theorem 3.1 has the following direct consequences, their proofs being omitted.

Corollary 3.1 *Let $f : D \to \mathbb{R}$ be an explicitly quasiconvex function which possesses a local maximum point $x^0 \in D$. The following assertions hold true*:

1° *If $x^0 \in \operatorname{icr} D$, then there is a neighborhood W of x^0 such that*

$$\operatorname*{argmin}_{x \in D} f(x) = W \cap D. \tag{3.7}$$

2° *If $x^0 \in \operatorname{int} D$, then $x^0 \in \operatorname{int}\left(\operatorname{argmin}_{x \in D} f(x)\right)$.*

Corollary 3.2 *Let $f : D \to \mathbb{R}$ be an explicitly quasiconvex function. If D is open, then the set of all local maximum points of f actually is* $\operatorname{int}\left(\operatorname{argmin}_{x \in D} f(x)\right)$.

Corollary 3.3 *Let $f : D \to \mathbb{R}$ be a convex function, defined on a nonempty convex subset of a real locally convex space X. If $x^0 \in \operatorname{int} D$ is a local maximum point of f, then x^0 is a global minimum point of f.*

Remark 3.3 Corollary 3.3 may be seen as a particular instance of the well-known result by Zălinescu [10, Proposition 2.5.8] already stated in the Introduction.

4 Applications

In this section $(Y, \|\cdot\|)$ will be a real normed space. Recall that $(Y, \|\cdot\|)$ is said to be *strictly convex* if for any distinct points $y', y'' \in Y$ with $\|y'\| = \|y''\| = 1$ we have $\|y' + y''\| < 2$. It is well-known that the following assertions are equivalent (see, e.g., Zălinescu [10, Th. 3.7.2]):

a) $(Y, \|\cdot\|)$ is strictly convex.
b) For any real number $p > 1$, the function $\|\cdot\|^p$ is strictly convex.
c) $\|(1-t)y' + ty''\| < 1$ for all distinct $y', y'' \in Y$ with $\|y'\| = \|y''\| = 1$ and $t \in \,]0,1[$.

Remark 4.1 Obviously, for any real normed space $(Y, \|\cdot\|)$ the norm $\|\cdot\| : Y \to \mathbb{R}$ is a convex function. However, if $Y \neq \{0_Y\}$, then the function $\|\cdot\|$ is not strictly convex, even if $(Y, \|\cdot\|)$ is strictly convex, since for $y' = 0_Y$, $y'' \in Y \setminus \{y'\}$ and $t \in]0,1[$ we have $\|(1-t)y' + ty''\| \not< (1-t)\|y'\| + t\|y''\|$. Notice also that $\|\cdot\|$ is indeed strictly convex in the trivial case when $Y = \{0_Y\}$.

The next result represents our first characterization of strictly convex spaces.

Theorem 4.1 *For any real normed space* $(Y, \|\cdot\|)$ *the following assertions are equivalent*:

1° $(Y, \|\cdot\|)$ *is strictly convex.*
2° *There is only one subset* B *of* Y *satisfying both properties* (P1) *and* (P2) *below, namely the trivial linear subspace* $B = \{0_Y\}$.
   (P1) $\mathbb{R}_+ \cdot B$ *is a linear subspace of* Y;
   (P2) *There is* $y \in Y$ *such that* $\|y + h\| = \|y\|$ *for all* $h \in B$.

Proof For proving the implication $1° \Rightarrow 2°$, assume that $(Y, \|\cdot\|)$ is strictly convex. Obviously $B = \{0_Y\}$ satisfies both (P1) and (P2). Suppose to the contrary that there is another set, $\{0_Y\} \neq B \subseteq Y$, which also satisfies (P1) and (P2). By (P1) it follows that $B \neq \emptyset$, hence there is some $h' \in B \setminus \{0_Y\}$. The property (P1) also shows that $-h' \in \mathbb{R}_+ \cdot B$, hence $-h' = \alpha h''$ for some $\alpha \geq 0$ and $h'' \in B$. As $h' \neq 0_Y$ we have $\alpha > 0$ and $h'' \neq 0_Y$. By (P2) we can find $y \in Y$ such that $\|y + h\| = \|y\|$ for all $h \in B$. In particular, for $h \in \{h', h''\}$, we get $\|y + h'\| = \|y\|$ and $\|y + h''\| = \|y\|$. Observe that $y \neq 0_Y$ since otherwise we would have $h' = h'' = 0_Y$, a contradiction. Thus we can define $y' = \frac{1}{\|y\|}(y + h')$ and $y'' = \frac{1}{\|y\|}(y + h'')$. It is easily seen that $y' \neq y''$, $\|y'\| = \|y''\| = 1$ and $(1-t)y' + ty'' = \frac{1}{\|y\|}y$, for $t = \alpha/(1+\alpha) \in]0,1[$. By hypothesis 1° and the equivalence "$a) \Leftrightarrow c)$" mentioned at the beginning of this section, we infer that $\|(1-t)y' + ty''\| < 1$, i.e., $1 < 1$, a contradiction. Thus $B = \{0_Y\}$ is the unique subset of Y satisfying (P1) and (P2).

In order to prove the implication $2° \Rightarrow 1°$, assume that 2° holds and suppose by the contrary that 1° is not true. Then there exist distinct points $y', y'' \in Y$ with $\|y'\| = \|y''\| = 1$ such that $\|y' + y''\| \geq 2$. Since $\|y' + y''\| \leq \|y'\| + \|y''\|$ by the triangle inequality, we infer that $\|y' + y''\| = 2$. Consider the set $B = \{-h^0, 0_Y, h^0\} \subseteq Y$, where $h^0 = y' - y$. Clearly, $B \neq \{0_Y\}$, since $h^0 = \frac{1}{2}(y' - y'') \neq 0_Y$. It is easily seen that B satisfies (P1), since $\mathbb{R}_+ \cdot B = \mathbb{R} \cdot h^0$ is a (one-dimensional) linear subspace of Y. Moreover, B satisfies (P2) for the point $y = \frac{1}{2}(y' + y'') \in Y$. Indeed, since $\|y' + y''\| = 2$, we have $\|y - h^0\| = \|y\| = 1$, $\|y + 0_Y\| = \|y\| = 1$ and $\|y + h^0\| = \|y'\| = 1$, hence $\|y + h\| = \|y\|$ for every $h \in B = \{-h^0, 0_Y, h^0\}$. Since our choice of B contradicts the assumption 2°, we can conclude that 1° is true. □

Corollary 4.1 *Let* $A \in L(X, Y)$ *be a linear operator acting between a real topological linear space* X *and a real normed space* Y. *Assume that for some point* $b \in Y$ *the function* $f_{A,b} : X \to \mathbb{R}$, *defined as*

$$f_{A,b}(x) = \|A(x) - b\|, \ \forall\, x \in X, \tag{4.1}$$

possesses a local maximum point $x^0 \in X$. *Then the following assertions hold true*:

1° *Function* $f_{A,b}$ *attains its global minimum at* x^0*; more precisely,* $\operatorname{argmin}_{x \in X} f_{A,b}(x)$ *is a closed convex neighborhood of* x^0.
2° *If* $(Y, \|\cdot\|)$ *is strictly convex, then* $A = 0_{L(X,Y)}$.

Proof For proving 1° observe that $f_{A,b} = \|\cdot\| \circ g$ is a composition of a norm and an affine function, $g = A - b$, hence function $f_{A,b}$ is convex and continuous. By applying Corollary 3.1 for $D = X$ and $f = f_{A,b}$, we infer that $x^0 \in \operatorname{int}\left(\operatorname{argmin}_{x\in X} f_{A,b}(x)\right)$. This means that $f_{A,b}$ attains its global minimum at x^0 and $\operatorname{argmin}_{x\in X} f_{A,b}(x)$ is a neighborhood of x^0. The level set $\operatorname{argmin}_{x\in X} f_{A,b}(x) = f_{A,b}^{-1}\left(]-\infty, f(x^0)]\right)$ is closed (since $f_{A,b}$ is continuous) and convex (since $f_{A,b}$ is convex).

In order to prove 2° assume that the space $(Y, \|\cdot\|)$ is strictly convex. According to 1° we have $f_{A,b}(x) = f_{A,b}(x^0)$ for all $x \in V = \operatorname{argmin}_{x\in X} f_{A,b}(x) \in \mathcal{V}(x^0)$. By linearity of A it follows that

$$\|A(x - x^0) + A(x^0) - b\| = \|A(x^0) - b\|, \ \forall\, x \in V.$$

Denoting $U = V - x^0$, $B = A(U)$ and $y = A(x^0) - b$, the above relation becomes

$$\|h + y\| = \|y\|, \ \forall\, h \in B,$$

which shows that B satisfies the property (P2) in Theorem 4.1. On the other hand, since $V \in \mathcal{V}(x^0)$ and X is a topological linear space we have $U = V - x^0 \in \mathcal{V}(0_X)$. By Lemma 2.1 (3°) we infer that $\mathbb{R}_+ \cdot B = \operatorname{Im}(A)$, which is a linear subspace of Y. Thus the set B also satisfies the property (P1) in Theorem 4.1 and, consequently, $B = \{0_Y\}$. We conclude that $\operatorname{Im}(A) = \mathbb{R}_+ \cdot B = \{0_Y\}$, hence $A = 0_{L(X,Y)}$. □

Remark 4.2 The strict convexity of $(Y, \|\cdot\|)$ in assertion 2° of Corollary 4.1 is essential, as shown by the following example.

Example 4.1 Let $X = Y = \mathbb{R}^2$ be endowed with the uniform norm $\|\cdot\|_\infty$ and let $A \in L(X,Y)$ be the linear operator defined in Example 2.1. It is easily seen that for $b = (0,1)$ the function $f_{A,b} : \mathbb{R}^2 \to \mathbb{R}$, defined by (4.1), i.e.,

$$f_{A,b}(x) = \|A(x) - b\|_\infty = \max\{|x_1|, 1\}, \ \forall\, x = (x_1, x_2) \in \mathbb{R}^2,$$

possesses local maximum points, as for instance $x^0 = (0,0)$. However, $A \neq 0_{L(X,Y)}$. Thus, from Corollary 4.1 we recover the classical property of $(\mathbb{R}^2, \|\cdot\|_\infty)$ to be not strictly convex. Notice that, in this example, the set of all local maximum points of $f_{A,b}$ is $\operatorname{int}\left(\operatorname{argmin}_{x\in X} f_{A,b}(x)\right) =]-1,1[\times\mathbb{R}$, in view of Corollary 3.2. Observe also that, by choosing another point, $b' = (0,0)$, the function $f_{A,b'}$, given by

$$f_{A,b'}(x) = \|A(x) - b'\|_\infty = \max\{|x_1|, 0\} = |x_1|, \ \forall\, x = (x_1, x_2) \in \mathbb{R}^2,$$

doest not possess local maximum points. Indeed, the $\operatorname{argmin}_{x\in X} f_{A,b'}(x) = \{0\} \times \mathbb{R}$ has an empty interior.

We now present our second characterization of strictly convex normed spaces.

Corollary 4.2 *For any normed space $(Y, \|\cdot\|)$ the following assertions are equivalent*:

1° $(Y, \|\cdot\|)$ *is strictly convex.*
2° *For every real topological linear space X and any $A \in L(X,Y)$ for which there is $b \in Y$ such that $f_{A,b}$ possesses a local maximum point, we have $A = 0_{L(X,Y)}$.*
3° *If $A \in L(\mathbb{R}, Y)$ and there is $b \in Y$ such that $f_{A,b}$ possesses a local maximum point, then $A = 0_{L(\mathbb{R},Y)}$.*

Proof The implication $1^\circ \Rightarrow 2^\circ$ is a straightforward consequence of Corollary 4.1, while the implication $2^\circ \Rightarrow 3^\circ$ is obvious.

For proving $3^\circ \Rightarrow 1^\circ$, assume that 3° holds true and suppose by the contrary that 1° is false. Then, there exist distinct points $y', y'' \in Y$ with $\|y'\| = \|y''\| = 1$ such that $\|y'+y''\| \geq 2$, which actually means that $\|y'+y''\| = 2$ due to the triangle inequality. Consider the function $A \in L(\mathbb{R}, Y)$, defined as $A(x) = x(y'' - y')$ for all $x \in \mathbb{R}$, let $b = -y'$ and define $f_{A,b} : X = \mathbb{R} \to \mathbb{R}$ according to (4.1), i.e.,

$$f_{A,b}(x) = \|A(x) - b\| = \|x(y'' - y') + y'\|, \ \forall x \in \mathbb{R}.$$

By the choice of y' and y'', we have $f_{A,b}(0) = f_{A,b}(1/2) = f_{A,b}(1) = 1$. On the other hand, since $f_{A,b}$ is (quasi)convex, we also have $f_{A,b}(x) \leq \max\{f_{A,b}(0), f_{A,b}(1)\}$ for all $x \in [0,1]$. It follows that $f_{A,b}(x) \leq f_{A,b}(1/2)$ for all $x \in [0,1] \in \mathcal{V}(1/2)$, hence $x^0 = 1/2$ is a local maximum point of $f_{A,b}$. As the linear function A is not the null one, this contradicts the initial assumption 3°. Thus 1° holds true. □

We conclude with an application concerning the least squares problem. The following result appeared in a technical note by Phohomsiri [8, Lemma 2.1], being proved by means of the Moore-Penrose generalized inverse of a matrix. As shown below, it can be recovered from Corollary 3.3 and refined through Corollary 4.1.

Corollary 4.3 *Given $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $b \in \mathbb{R}^m$, consider the residual function $G : \mathbb{R}^n \to \mathbb{R}$, defined for all $x \in \mathbb{R}^n$ by*

$$G(x) = \|A(x) - b\|^2,$$

where $\|\cdot\|$ represents the Euclidean norm. The extrema of G are all minima.

Proof It is easily seen that G is a convex function, since the function $f_{A,b}$ given by (4.1) is convex, $G(x) = \left[f_{A,b}(x)\right]^2$ and $f_{A,b}(x) \geq 0$ for all $x \in \mathbb{R}^n$. By applying Corollary 3.3 for $D = \mathbb{R}^n$ and $f = G$, we deduce that all local extremum points of G are global minimum points of G. □

Remark 4.3 The question on whether G possesses a local maximum point or not is not addressed explicitly by Phohomsiri [8]. Corollary 4.1 gives an answer to this question. Indeed, since $G(x) = \left[f_{A,b}(x)\right]^2$ and $f_{A,b}(x) \geq 0$ for all $x \in \mathbb{R}^n$, the functions G and $f_{A,b}$ have the same extremum points. The Euclidean normed space $(\mathbb{R}^m, \|\cdot\|)$ being strictly convex, we can deduce by Corollary 4.1 that the following assertions are equivalent:

a) G possesses a local maximum point;
b) A is the null operator.

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