On the Phenomenon of Masked Periodic Horadam Sequences

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Abstract

A recently discovered phenomenon, termed masked periodicity and observed in self-repeating Horadam sequences using matrix based methods in a study by the authors elsewhere, is considered further here. This article continues the approach, identifying both governing parameters and particular behaviour types which fall naturally into three categories convenient for explanation and illustration.

1 Introduction

A Horadam sequence is written $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a, b; p, q)\}_0^{\infty}$, being characterised by the four parameters $a, b, p, q \in \mathbf{C}$ through the defining recurrence

$$w_n = pw_{n-1} - qw_{n-2}; \qquad w_0 = a, w_1 = b, \tag{1}$$

first introduced and analysed by A.F. Horadam in the 1960s (see the two seminal papers [1,2]). The fully general nature of both the recursion itself and the initial values w_0, w_1 means that many well known, and much considered, (real) sequences can be generated by (1) as special instances, and the literature on linear recurrence equations is vast.

Whilst various mathematical properties of Horadam sequences have been studied over the years [3], it is only more recently that their potential to exhibit cyclicity has been examined in any depth. This paper gives an understanding of one interesting aspect which is merely reported and illustrated in [4] as so called 'masked' periodicity, without proper explanation. What we mean by this is that a fully general (arbitrary initial values) periodic Horadam sequence can mask, or hide, one or two special case (specific initial values) sequence(s) of smaller period. Here we detail the salient factors enabling this phenomenon to occur and determine those behaviour types possible, through which it becomes evident that the underlying causes are dictated by the defining characteristic of a primitive root of unity. We both continue, and make appeal to, the matrix based approach employed in [4], developing new results as required and providing examples (the nature of Horadam periodicity has been dealt with elsewhere [5,6] deploying alternative methods of analysis; see also [7]).

2 Theory and Results

2.1 Background Theory

Writing

$$\mathbf{A} = \mathbf{A}(p, q) = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}, \tag{2}$$

it is easily shown that, for $k \geq 0$, the recurrence (1) gives rise to the matrix power equation

$$\begin{pmatrix} w_{k+1} \\ w_k \end{pmatrix} = \mathbf{A}^k(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix}. \tag{3}$$

In terms of those distinct roots $z_{1,2} = \frac{1}{2}(p \pm \sqrt{p^2 - 4q})$ in the non-degenerate case ($p^2 \neq 4q$) of the characteristic equation $0 = \lambda^2 - p\lambda + q$ associated with (1), then $z_1 + z_2 = p$, $z_1 z_2 = q$. More importantly, we find that the kth power of the (characteristic roots) matrix

$$\mathbf{A}(z_1, z_2) = \mathbf{A}(p(z_1, z_2), q(z_1, z_2)) = \begin{pmatrix} z_1 + z_2 & -z_1 z_2 \\ 1 & 0 \end{pmatrix} \tag{4}$$

possesses eigenvectors $(z_1, 1)^T, (z_2, 1)^T$ with (resp.) eigenvalues $(z_1)^k$, $(z_2)^k$. We see this by decomposing the matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1}, \tag{5}$$

where $\mathbf{E}$ is the non-singular matrix

$$\mathbf{E}(z_1, z_2) = \begin{pmatrix} z_1 & z_2 \\ 1 & 1 \end{pmatrix} \tag{6}$$

and $\mathbf{D}$ is the diagonal matrix

$$\mathbf{D}(z_1, z_2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}. \tag{7}$$

It then follows readily that

$$\mathbf{A}^k = \mathbf{E}\mathbf{D}^k\mathbf{E}^{-1} \tag{8}$$

and, since $\mathbf{A}^k$ and $\mathbf{D}^k$ are similar matrices, the eigenvalues of $\mathbf{A}^k$ are those of $\mathbf{D}^k$ which latter are immediate as its diagonal entries $(z_1)^k$ and $(z_2)^k$, the matrix $\mathbf{E}$ containing as its columns the corresponding eigenvectors of $\mathbf{A}^k$ (we leave it as a straightforward reader exercise to check that, for $k \geq 0$, $\mathbf{A}^k \binom{z_{1,2}}{1} = \mathbf{E}\mathbf{D}^k\mathbf{E}^{-1} \binom{z_{1,2}}{1} = (z_{1,2})^k \binom{z_{1,2}}{1}$ by hand).

More generally, by considering the Horadam parameters p, q to be actually *defined* in terms of two pre-chosen values ζ_1, ζ_2, say, according to

$p(\zeta_1,\zeta_2) = \zeta_1 + \zeta_2$, $q(\zeta_1,\zeta_2) = \zeta_1\zeta_2$, then *these* ζ_1, ζ_2 *values are precisely those characteristic roots of a particular Horadam sequence which they fundamentally characterise* (through, of course, the generating recurrence (1)), a fact that now opens up the problem of masked periodicity to scrutiny. It was established in [4] that, based on (3), a (necessary) condition for a Horadam sequence to have (minimum) period $\delta \geq 1$, say, is, for fixed p, q, the existence of an initial values vector $\mathbf{w} = \mathbf{w}(w_0, w_1) = (w_1, w_0)^T$ such that

$$\mathbf{A}^{\delta}\mathbf{w} \;=\; \mathbf{w}, \tag{9}$$

which is to say that $\mathbf{w}$ is an eigenvector of $\mathbf{A}^{\delta}(p,q)$ with unit eigenvalue. In the case when $\mathbf{A}^{\delta}(p,q) = \mathbf{I}_2$ (the 2×2 identity matrix) then (9) is automatically satisfied and the initial values w_0, w_1 can remain arbitrary (*i.e.*, $\mathbf{w} = (b,a)^T$), whereupon the triplet $[p,q,\delta]$ is called an identity triplet. These observations—coupled with a rudimentary knowledge of a primitive root of unity—allow us to deduce which types of masked periodicity may occur in Horadam sequences.

2.2 Behaviour Classification

We recall that an (ordinary) nth root of unity r, say, is a so called *primitive* nth root if, for $k = 1, \ldots, n$, $k = n$ is the smallest value of k for which $r^k = 1$. Thus, we see trivially that 1 is the primitive first root of unity, whilst -1 is the primitive square root of unity. In addition, $\omega_{a,b} = \frac{1}{2}(-1 \pm \sqrt{3}\,i)$ are each primitive cube roots of unity. The primitive fourth roots of unity are simply $\pm i$, whilst there are 4 primitive fifth roots of unity, being those complex ordinary fifth roots. The ordinary sixth roots of unity are seen to be $\pm 1, \pm\omega_{a,b}$, delivering just $-\omega_{a,b}$ as the two primitive sixth roots; the process continues.

We suppose that ζ_1 and ζ_2 are respective primitive nth and mth roots of unity, with $m > n \geq 2$ and ζ_1, ζ_2 distinct; *these are key assumptions which drive the analysis.* Further, let $p(\zeta_1,\zeta_2) = \zeta_1 + \zeta_2$, $q(\zeta_1,\zeta_2) = \zeta_1\zeta_2$, so that a Horadam sequence is characterised duly by these primitive roots ζ_1, ζ_2 as explained above. We proceed, therefore, on the basis that, given w_0, w_1, the Horadam recurrence (1) is in effect executed for $n \geq 2$ as $w_n = (\zeta_1 + \zeta_2)w_{n-1} - (\zeta_1\zeta_2)w_{n-2}$ and, for $\mathbf{A} = \mathbf{A}(\zeta_1,\zeta_2) = \left(\begin{smallmatrix} \zeta_1+\zeta_2 & -\zeta_1\zeta_2 \\ 1 & 0 \end{smallmatrix}\right)$, that equations (5),(8) hold in terms of the other matrices $\mathbf{D}(\zeta_1,\zeta_2)$, $\mathbf{E}(\zeta_1,\zeta_2)$ as defined by (6),(7), with $\mathbf{A}^k(\zeta_1,\zeta_2)$ having eigenvalues $(\zeta_{1,2})^k$ in relation to its eigenvectors $(\zeta_{1,2},1)^T$.

Instances of masking fall into three differing categories.

Case I: m and n are Coprime

The Horadam sequence $\{w_n(a,b;\zeta_1+\zeta_2,\zeta_1\zeta_2)\}_0^\infty$, with arbitrary initial values, will have period $\delta = mn$. This is by virtue of the fact that, since

$$\begin{aligned}\mathbf{D}^{mn}(\zeta_1,\zeta_2) &= \begin{pmatrix} (\zeta_1)^{mn} & 0 \\ 0 & (\zeta_2)^{mn} \end{pmatrix} = \begin{pmatrix} [(\zeta_1)^n]^m & 0 \\ 0 & [(\zeta_2)^m]^n \end{pmatrix} \\ &= \begin{pmatrix} [1]^m & 0 \\ 0 & [1]^n \end{pmatrix} = \mathbf{I}_2, \end{aligned} \tag{10}$$

we observe $\mathbf{A}^\delta = \mathbf{A}^{mn} = \mathbf{E}\mathbf{D}^{mn}\mathbf{E}^{-1} = \mathbf{E}\mathbf{I}_2\mathbf{E}^{-1} = \mathbf{E}\mathbf{E}^{-1} = \mathbf{I}_2$, consequently generating an identity triplet $[\zeta_1+\zeta_2,\zeta_1\zeta_2,mn]$. This general period mn sequence will mask two initial values specific sequences, of periods n and m, whose starting values are—up to a multiplicative constant—merely eigenvector co-ordinates. Given $m > n$, the period n sequence masked is $\{w_n(1,\zeta_1;\zeta_1+\zeta_2,\zeta_1\zeta_2)\}_0^\infty$ (for we see that whilst $\mathbf{A}^n\binom{\zeta_1}{1} = (\zeta_1)^n\binom{\zeta_1}{1} = 1\cdot\binom{\zeta_1}{1} = \binom{\zeta_1}{1}$, on the other hand $\mathbf{A}^n\binom{\zeta_2}{1} = (\zeta_2)^n\binom{\zeta_2}{1} \neq \binom{\zeta_2}{1}$ so that $\mathbf{A}^n \neq \mathbf{I}_2$). The masked period m sequence is $\{w_n(1,\zeta_2;\zeta_1+\zeta_2,\zeta_1\zeta_2)\}_0^\infty$ (in a similar manner, $\mathbf{A}^m\binom{\zeta_2}{1} = (\zeta_2)^m\binom{\zeta_2}{1} = 1\cdot\binom{\zeta_2}{1} = \binom{\zeta_2}{1}$, however $\mathbf{A}^m \neq \mathbf{I}_2$ because[1] $\mathbf{A}^m\binom{\zeta_1}{1} = (\zeta_1)^m\binom{\zeta_1}{1} \neq \binom{\zeta_1}{1}$).

Example I: $m=3$, $n=2$

We choose $\zeta_1 = -1$ (the primitive square root of unity) and $\zeta_2 = \frac{1}{2}(-1+\sqrt{3}i)$ (a primitive cube root of unity). Then $p(\zeta_1,\zeta_2) = \zeta_1+\zeta_2 = \frac{1}{2}(-3+\sqrt{3}i)$, $q(\zeta_1,\zeta_2) = \zeta_1\zeta_2 = \frac{1}{2}(1-\sqrt{3}i)$, with $\{w_n(a,b;\frac{1}{2}(-3+\sqrt{3}i),\frac{1}{2}(1-\sqrt{3}i))\}_0^\infty = \{a, b, -\frac{1}{2}(a+3b)+\frac{\sqrt{3}}{2}(a+b)i, b-\sqrt{3}(a+b)i, a+\sqrt{3}(a+b)i, -\frac{1}{2}(3a+b)-\frac{\sqrt{3}}{2}(a+b)i,\ldots\}$ a general sequence of predicted period $mn = 6$. Setting $b = \zeta_1 a = -a$ recovers an expected masked period $n=2$ sequence $\{w_n(a,-a;\frac{1}{2}(-3+\sqrt{3}i),\frac{1}{2}(1-\sqrt{3}i))\}_0^\infty = \{a,-a,\ldots\}$, whilst

[1] Consider $(\zeta_1)^m$. For m,n as given there exists an integer $\alpha = \alpha(m,n) = \lfloor m/n \rfloor \geq 1$ such that $0 < m - \alpha n < n$. It then follows that $(\zeta_1)^m = (\zeta_1)^{\alpha n}(\zeta_1)^{m-\alpha n} = [(\zeta_1)^n]^\alpha(\zeta_1)^{m-\alpha n} = [1]^\alpha(\zeta_1)^{m-\alpha n} = (\zeta_1)^{m-\alpha n} \neq 1$.

if instead $b = \zeta_2 a = \frac{1}{2}(-1+\sqrt{3}i)a$, a masked sequence $\{w_n(a, \frac{1}{2}(-1+\sqrt{3}i)a; \frac{1}{2}(-3+\sqrt{3}i), \frac{1}{2}(1-\sqrt{3}i))\}_0^\infty = \{a, \frac{1}{2}(-1+\sqrt{3}i)a, -\frac{1}{2}(1+\sqrt{3}i)a, \ldots\}$ of period $m = 3$ is revealed, both masked sequences having the appropriate starting values according to the preceding theory.

Case II: n is a Divisor of m

Suppose, for some integer $\beta > 1$, that $m = \beta n > n$. Noting that

$$\begin{aligned}\mathbf{D}^m(\zeta_1, \zeta_2) &= \begin{pmatrix} (\zeta_1)^m & 0 \\ 0 & (\zeta_2)^m \end{pmatrix} = \begin{pmatrix} [(\zeta_1)^n]^\beta & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} [1]^\beta & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}_2, \end{aligned} \tag{11}$$

then this time $\mathbf{A}^m = \mathbf{E}\mathbf{D}^m\mathbf{E}^{-1} = \mathbf{E}\mathbf{I}_2\mathbf{E}^{-1} = \mathbf{I}_2$, and $\{w_n(a, b; \zeta_1+\zeta_2, \zeta_1\zeta_2)\}_0^\infty$ will be a general period m Horadam sequence which masks a single initial values specific sequence $\{w_n(1, \zeta_1; \zeta_1+\zeta_2, \zeta_1\zeta_2)\}_0^\infty$ of period n ($\mathbf{A}^n \neq \mathbf{I}_2$ in this case, since $\mathbf{A}^n\binom{\zeta_1}{1} = (\zeta_1)^n\binom{\zeta_1}{1} = \binom{\zeta_1}{1}$ but $\mathbf{A}^n\binom{\zeta_2}{1} = (\zeta_2)^n\binom{\zeta_2}{1} \neq \binom{\zeta_2}{1}$); again, we have a natural identity triplet appearing, which is $[\zeta_1+\zeta_2, \zeta_1\zeta_2, m]$.

Example II(a): $m = 6$, $n = 3$

We choose $\zeta_1 = -\frac{1}{2}(1+\sqrt{3}i)$ (a primitive cube root of unity) and $\zeta_2 = \frac{1}{2}(1-\sqrt{3}i)$ (a primitive sixth root of unity), giving $p(\zeta_1, \zeta_2) = -\sqrt{3}i$, $q(\zeta_1, \zeta_2) = -1$. Then the resulting general period 6 sequence $\{w_n(a, b; -\sqrt{3}i, -1)\}_0^\infty = \{a, b, a-\sqrt{3}bi, -2b-\sqrt{3}ai, -2a+\sqrt{3}bi, b+\sqrt{3}ai, \ldots\}$ is seen to mask the period 3 sequence $\{w_n(a, -\frac{1}{2}(1+\sqrt{3}i)a; -\sqrt{3}i, -1)\}_0^\infty = \{a, -\frac{1}{2}(1+\sqrt{3}i)a, -\frac{1}{2}(1-\sqrt{3}i)a, \ldots\}$ with initial values vector $\mathbf{w} = (-\frac{1}{2}(1+\sqrt{3}i)a, a)^T = (\zeta_1 a, a)^T$.

Example II(b): $m = 4$, $n = 2$

This is another Case II m, n pair for which $p = -(1+i)$, $q = i$ on choosing the primitive square root of unity -1 for ζ_1 and a primitive fourth root of unity $-i$ for ζ_2. The resulting general period 4 sequence $\{w_n(a, b; -(1+i), i)\}_0^\infty = \{a, b, -b-(a+b)i, -a+(a+b)i, \ldots\}$ masks the initial values specific sequence $\{w_n(a, -a; -(1+i), i)\}_0^\infty = \{a, -a, \ldots\}$, of period 2, for which $w_1 = \zeta_1 a = -a$.

We complete the classification with a final case requiring inclusion.

Case III: m and n Possess a Lowest Common Multiple $> m$

This case assumes, for given $m > n$, that neither $n|m$ nor that m, n are co-prime, whence they have a lowest common multiple l, say $(m, n < l < mn)$. Then, for integers γ_u, γ_l, we can write $l/n = \gamma_u$, $l/m = \gamma_l$, where $1 < \gamma_l < \gamma_u$. Noting that $(\zeta_1)^l = [(\zeta_1)^n]^{\gamma_u} = [1]^{\gamma_u} = 1 = [1]^{\gamma_l} = [(\zeta_2)^m]^{\gamma_l} = (\zeta_2)^l$ it follows trivially that $\mathbf{A}^l = \mathbf{I}_2$, confirming the existence of a general period l sequence $\{w_n(a, b; \zeta_1 + \zeta_2, \zeta_1\zeta_2)\}_0^\infty$ which creates an identity triplet $[\zeta_1 + \zeta_2, \zeta_1\zeta_2, l]$ and will mask the two initial values specific sequences $\{w_n(1, \zeta_{1,2}; \zeta_1 + \zeta_2, \zeta_1\zeta_2)\}_0^\infty$ of period n, m.

Example III: $m = 6$, $n = 4$

We choose $\zeta_1 = i$ (a primitive fourth root of unity) and $\zeta_2 = \frac{1}{2}(1 + \sqrt{3}i)$ (a primitive sixth root of unity), so that $p(\zeta_1, \zeta_2) = \frac{1}{2} + (1 + \frac{\sqrt{3}}{2})i$, $q(\zeta_1, \zeta_2) = \frac{1}{2}(-\sqrt{3} + i)$. Then the general sequence $\{w_n(a, b; \frac{1}{2} + (1 + \frac{\sqrt{3}}{2})i, \frac{1}{2}(-\sqrt{3} + i))\}_0^\infty$ (whose terms are omitted here, but are given in full in the Appendix for convenience) has period $12 = \mathrm{lcm}(6, 4)$, as expected masking the period 4 sequence $\{w_n(a, \zeta_1 a; \frac{1}{2} + (1 + \frac{\sqrt{3}}{2})i, \frac{1}{2}(-\sqrt{3} + i))\}_0^\infty = \{a, ai, -a, -ai, \ldots\}$ and also the period 6 sequence $\{w_n(a, \zeta_2 a; \frac{1}{2} + (1 + \frac{\sqrt{3}}{2})i, \frac{1}{2}(-\sqrt{3} + i))\}_0^\infty = \{a, \frac{1}{2}(1 + \sqrt{3}i)a, \frac{1}{2}(-1 + \sqrt{3}i)a, -a, -\frac{1}{2}(1 + \sqrt{3}i)a, \frac{1}{2}(1 - \sqrt{3}i)a, \ldots\}$.

Remark 1 The period of any masking sequence, as described, is a minimal one for that particular sequence, as is the case for any sequence(s) unmasked from it (since period is non-unique—any sequence of period δ also has period $2\delta, 3\delta, 4\delta, \ldots$).

Remark 2 The three cases considered can, of course, be re-stated in terms of highest common factor criteria thus: Case I corresponds to $\mathrm{hcf}(m, n) = 1$; Case II corresponds to $\mathrm{hcf}(m, n) = n \geq 2$; Case III corresponds to $2 \leq \mathrm{hcf}(m, n) < n$.

Remark 3 In this paper we have assumed $m > n$, for in the case $n = m$ no masking is evident. Here the characteristic roots ζ_1, ζ_2 are still assumed distinct, though now of the same primitive order (when, for example, $\zeta_{1,2} = \frac{1}{2}(-1 \pm \sqrt{3}i)$ (order 3) or $\zeta_{1,2} = \pm i$ (order 4)), and the sequence $\{w_n(a, b; \zeta_1 + \zeta_2, \zeta_1\zeta_2)\}_0^\infty$ has period m. Setting $b = \zeta_{1,2}a$ produces merely two period m instances of it, and no masked sequence of smaller period.

3 On the Matrix $\mathbf{A}(p,q)$

The 2015 publication [4] by the authors is, we believe, the first to appeal to the matrix $\mathbf{A}(p,q)$ (2) (characterised by the Horadam recursion parameters p,q, and for which $\mathbf{A}(1,-1)$ reduces to the so called Q-matrix studied widely in connection with the Fibonacci numbers) in examining Horadam sequence periodicity; the presentation here follows as a natural one, of course, and employs the matrix accordingly. Not surprisingly, however, it has been seen before in the literature as we now indicate for completeness—this places the paper in a slightly wider context within the arena of second order recurrence sequence theory, giving an overview of the (somewhat intermittent) usage of $\mathbf{A}(p,q)$ which itself is not without an element of interest and felt worthy of inclusion.

In 1995 Melham and Shannon used the matrix $\mathbf{A}_{k,x} = x\mathbf{A}^k$ as the basis of an extension of summation identities involving Fibonacci and Lucas numbers (found previously by Filipponi and Horadam) to accommodate the more general fundamental and primordial sequences $\{U_n\}_0^\infty = \{w_n(0,1;p,q)\}_0^\infty$ and $\{V_n\}_0^\infty = \{w_n(2,p;p,q)\}_0^\infty$ of longstanding interest [8, Theorem 1, p.16]; specialising results so as to apply to Chebyshev polynomials of first and second kind, they also produced new series summations involving the trigonometric sine and cosine functions. Other identities for terms of $\{U_n\}_0^\infty, \{V_n\}_0^\infty$, via further generalisations of $\mathbf{A}$, were formulated in [9] that same year. An important motivation for Melham and Shannon was a 1976 paper by Walton [10], some of whose results they generalised in [8]. Walton developed infinite series evaluations based on expressions for $\sin(\mathbf{X}), \cos(\mathbf{X})$, where $\mathbf{X}$ is a *general* 2×2 matrix with the properties $\mathrm{Tr}\{\mathbf{X}\} = p$, $\mathrm{Det}\{\mathbf{X}\} = q$, as possessed by $\mathbf{A}(p,q)$ (this guarantees that the eigenvalues of $\mathbf{X}$ satisfy the characteristic equation associated with the Horadam recurrence, and was significant in the analysis). His work drew on a seminal publication by Barakat [11] who, in 1964, showed that integral powers and inverse powers of the matrix $\mathbf{X}$ could be related to terms of the fundamental sequence as the dependency $\mathbf{X}^{\pm n} = \mathbf{X}^{\pm n}(U_{\pm n}, U_{\pm(n-1)}, q; \mathbf{X}, \mathbf{I}_2)$ (with the order 2 identity matrix written $\mathbf{I}_2$). Using that for $\mathbf{X}^n$, in conjunction with the exponential matrix expansion $\exp(\mathbf{X}) = \sum_{n\geq 0} \mathbf{X}^n/n!$, he was first able to construct summation formulas for terms of the fundamental sequence covering both distinct and equal characteristic root cases, and then produce one for those of the primordial sequence (and a further one for the fundamental sequence) from relations between the two sequences. Note that—as a sign of the times—these sequences were termed first and second Lucas polynomials (in p, q) by Barakat, who created an interesting line of enquiry which is perhaps little known and frames historically our use of the matrix $\mathbf{A}(p, q)$.[2] It was considered earlier still by Rosenbaum in which—motivated by the potential use of matrix methods in solving certain problems arising from linear recursion equations—he formulated in 1959 [12] degenerate and non-degenerate characteristic root instance closed forms of $w_n(a, b; p, -q)$ using the nth power of $\mathbf{A}(p, -q)$ in an elementary argument driven by matrix diagonalisation; this has been repeated (unaware of it, one would presume, and quite understandably) more than once by authors since then. Another relatively early publication to note is one from 1974 by Waddill [13]. He defined a sequence $\{U_n\}_0^\infty = \{w_n(U_0, U_1; r, -s)\}_0^\infty$ and, using the matrix $\mathbf{A}(r, -s)$, developed a range of formulas involving terms of this sequence and the initial values instance $\{K_n\}_0^\infty = \{w_n(0, 1; r, -s)\}_0^\infty$, also reproducing some known results in a new fashion (and remarking, too, that the "use of matrices adapts itself very nicely for generalizing some of the identities involving sums of Fibonacci numbers"). The introduction of $\mathbf{A}(p, -q)$ is mentioned very briefly in a more recent work by He and Shiue [14, Remark 2.4, p.4] who note the importance of its eigenvalues and eigenvectors and their relationship to the Horadam characteristic roots, giving one identity for terms of $\{a_n\}_0^\infty = \{w_n(a_0, a_1; p, -q)\}_0^\infty$ derived briefly from a general power of $\mathbf{A}$.

Not wishing to provide a full survey, as such we have nevertheless given a flavour of early interest shown in the matrix $\mathbf{A}(p, q)$, and its role in those types of works still remains pertinent to some even today. By way of example, we see that G. Cerda [15] has given results for properties of powers of $\mathbf{A}$, making connections with the Binet closed forms of the fundamental and primordial sequences. A subsequent publication [16] offers a host of

[2]Barakat—referring to a 1947 offering by André Herpin in the journal *Comptes Rendus*—describes a method of expressing an arbitrary integral power of the matrix $\mathbf{X}$ in terms of "Lucas polynomials" by writing it as a linear combination of Pauli spin matrices to which operators are applied.

further identities for the same sequences through the introduction of a so called generalised Lucas matrix $\mathbf{V}(p,q)$ for which he also formulates results and uses in combination with $\mathbf{A}(p,q)$. The matrix is a natural one to form an association with (1), and it has in fact been extended in dimension to characterise higher order linear recurrence equations (see, *e.g.*, [17]).

4 Summary

As stated in the Introduction, the quantity of published material on linear recurrences is huge. Within this, order two recursions feature heavily and the Horadam sequence has played its part, aspects of which having quite clearly engendered certain predilections amongst researchers. One rather striking characteristic of studies over the years, however, has been, for whatever reason(s), a lack of consideration given to the possible self-repeating nature of Horadam sequences (other than some work on modulo periodicity, which is a somewhat different behaviour trait) [3]. This has recently been addressed elsewhere [4-7] and—based on a particular observation in [4]—the authors are able here to develop the theme of cyclicity with reference to an interesting phenomenon termed masked periodicity in the case of non-degenerate characteristic roots, the like of which we believe to be new within the literature on second order recurrence sequences; it remains to be seen whether or not it can be described mathematically from an alternative viewpoint, or if there is any significance for the development of further Horadam sequence theory. A novel algorithmic method to generate periodic Horadam sequences, mentioned briefly also in [4], itself demands further examination and verification.

We end with one final remark worth making. Note that whilst the effect of masked periodicity has been classified and illustrated by applying the simple defining property of a primitive root of unity, its treatment here suffers no loss of generality since ordinary roots of unity can equally be used to show masking through the simple observation that any ordinary rth root of unity z_r, say, will be a primitive sth root of unity ζ_s, say, for some divisor s ($\leq r$) of r. Thus, by this natural association any pair of distinct ordinary roots fixes m, n accordingly, with masked periodicity then evident in one of the Cases I-III discussed.

Appendix

The repeated terms of the general period 12 sequence $\{w_n(a, b; \frac{1}{2} + (1 + \frac{\sqrt{3}}{2})i, \frac{1}{2}(-\sqrt{3} + i))\}_0^\infty$ of the Case III example are

$$
\begin{aligned}
w_0 &= a,\\
w_1 &= b,\\
w_2 &= \frac{1}{2}(\sqrt{3}a + b) + \left[\left(1 + \frac{\sqrt{3}}{2}\right)b - \frac{1}{2}a\right]i,\\
w_3 &= \frac{1}{2}(1 + \sqrt{3})a - \frac{1}{2}(3 + \sqrt{3})b + \left[\frac{1}{2}(1 + \sqrt{3})a + \frac{1}{2}(1 + \sqrt{3})b\right]i,\\
w_4 &= -\frac{1}{2}(1 + \sqrt{3})a - \frac{1}{2}(3 + \sqrt{3})b + \left[\frac{1}{2}(3 + \sqrt{3})a - \frac{1}{2}(3 + \sqrt{3})b\right]i,\\
w_5 &= -\left(\frac{3}{2} + \sqrt{3}\right)a + \left(1 + \frac{\sqrt{3}}{2}\right)b - \left[\frac{\sqrt{3}}{2}a + \left(\frac{3}{2} + \sqrt{3}\right)b\right]i,\\
w_6 &= (2 + \sqrt{3})b + [b - (2 + \sqrt{3})a]i,\\
w_7 &= (2 + \sqrt{3})a + [(2 + \sqrt{3})b - a]i,\\
w_8 &= \left(1 + \frac{\sqrt{3}}{2}\right)a - \left(\frac{3}{2} + \sqrt{3}\right)b + \left[\left(\frac{3}{2} + \sqrt{3}\right)a + \frac{\sqrt{3}}{2}b\right]i,\\
w_9 &= -\frac{1}{2}(3 + \sqrt{3})a - \frac{1}{2}(1 + \sqrt{3})b + \left[\frac{1}{2}(3 + \sqrt{3})a - \frac{1}{2}(3 + \sqrt{3})b\right]i,\\
w_{10} &= -\frac{1}{2}(3 + \sqrt{3})a + \frac{1}{2}(1 + \sqrt{3})b - \left[\frac{1}{2}(1 + \sqrt{3})a + \frac{1}{2}(1 + \sqrt{3})b\right]i,\\
w_{11} &= \frac{1}{2}(a + \sqrt{3}b) + \left[\frac{1}{2}b - \left(1 + \frac{\sqrt{3}}{2}\right)a\right]i. \qquad \text{(A1)}
\end{aligned}
$$

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