ON THE MASKED PERIODICITY OF HORADAM SEQUENCES: A GENERATOR-BASED APPROACH

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Abstract. The Horadam sequence is a general second order linear recurrence sequence, dependent on a family of four (possibly complex) parameters—two recurrence coefficients and two initial conditions. In this article we examine a phenomenon identified previously and referred to as 'masked' periodicity, which links the period of a self-repeating Horadam sequence to its initial conditions. This is presented in the context of cyclicity theory, and then extended to periodic sequences arising from recursion equations of degree three or more.

1. Introduction

A Horadam sequence $\{w_n\}_{n=0}^{\infty} = \{w_n(a, b; p, q)\}_{n=0}^{\infty}$ is defined by the recurrence

$$w_{n+2} - pw_{n+1} + qw_n = 0, \quad w_0 = a, w_1 = b, \tag{1.1}$$

where the parameters a, b, p, q are complex numbers. Horadam sequences have now been studied for over half a century, and many of their numerous properties are discussed in the survey article [5] (an update to this work is currently in press). Recently, periodicity of Horadam sequences has begun to be analyzed, and self-repeating orbits in the complex plane were first characterized in [1], arising when zeros z_1 and z_2 of the characteristic equation

$$z^2 - pz + q = 0 \tag{1.2}$$

associated with (1.1) (called generators) are linked to roots of unity. The theory developed around this idea also led to results concerning the geometric structure of self-repeating Horadam sequences whose orbits were shown to exhibit dual symmetry [4]. The theoretical framework for periodicity in complex Horadam sequences produced by higher order versions of (1.1) was also developed by Bagdasar and Larcombe [3]. Horadam sequences having a fixed period have also been enumerated [2], the result providing a first context to the OEIS Sequence No. A102309 [8].

Periodicity as described from a different mathematical viewpoint has appeared in a paper by Larcombe and Fennessey [6] who identified a certain type of phenomenon therein, termed 'masked' periodicity—in which a fully general (*arbitrary* initial values) Horadam sequence can mask, or hide, one or two special case (*specific* initial values) sequence(s) of smaller period—and the notion was explored in a separate publication [7]. In this paper we address the issue of masking purely from a generator approach, investigating self-repeating Horadam sequences that arise from distinct generators in order to confirm the presence of masking potential. Specifically, a fixed generator pair of primitive roots of unity $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$ may generate a periodic Horadam sequence of length $[k_1, k_2]$ (that is, $\text{lcm}(k_1, k_2)$), k_2, k_1, or 1, depending on the choice of the initial starting points a and b. The methodology is readily extendible to higher order Horadam type recurrence sequences, with the necessary generalized theory set down and an illustrative example given in the case of order three.

2. Theoretical Background

In this section the periodicity conditions from [1] are revisited for equal and distinct roots z_1 and z_2 of (1.2). When self-repeating, the period k of the Horadam sequence clearly depends on the generators z_1, z_2 and the initial conditions a, b. Specifically, for generators which are k_1th and k_2th roots of unity, the period of the sequence orbit may have different values, depending on the initial conditions a, b. We shall refer to this property as 'masked' periodicity.

2.1. **Degenerate case $z_1 = z_2$.** For equal roots $z_1 = z_2$ of (1.2), the general term of the Horadam sequence $\{w_n\}_{n=0}^{\infty}$ is

$$w_n = \left[a + \left(\frac{b}{z_1} - a\right)n\right] z_1^n.$$

In this case the sequence can only be periodic if $z_1 = 0$, or $b = az_1$ and z_1 is a root of unity.

If $z_1 = 0$ the sequence is constant, therefore periodic of period one.

If z_1 is a k_1th primitive root of unity, the sequence terms are $w_n = az_1^n$. The orbit is either a single point for $a = 0$, or a regular k_1-gon for $a \neq 0$. The k_1th primitive root of unity could therefore produce self-repeating sequences of periods 1 and k_1 for different values of a and b.

2.2. **Non-degenerate case $z_1 \neq z_2$.** For distinct roots $z_1 \neq z_2$ of (1.2), the general term of Horadam's sequence $\{w_n\}_{n=0}^{\infty}$ is

$$w_n = Az_1^n + Bz_2^n, \tag{2.1}$$

where the constants A and B can be obtained from the initial condition, as

$$A = \frac{az_2 - b}{z_2 - z_1}, \quad B = \frac{b - az_1}{z_2 - z_1}. \tag{2.2}$$

The necessary periodicity condition for Horadam sequences [1, Theorem 3.2], implies the existence of a positive integer k such that

$$A(z_1^k - 1)z_1 = 0,$$
$$B(z_2^k - 1)z_2 = 0,$$

where the coefficients A, B are defined in (2.2). When $z_2 = 0$ (or similarly, $z_1 = 0$) the formula (2.1) gives $w_n = Az_1^n$, which is only periodic for $A = 0$ or when z_1 is a primitive root of unity. It will be assumed from now on that $z_1 z_2 \neq 0$.

When $AB \neq 0$, periodicity implies that the distinct generators z_1 and z_2 are roots of unity. If $A = 0$, $B \neq 0$ the sequence is given by $w_n = Bz_2^n$, which is periodic if z_2 is a root of unity (for $z_1 \in \mathbb{C}$). Similarly, for $A \neq 0$, $B = 0$ the sequence is only periodic when z_1 is a root of unity (for $z_2 \in \mathbb{C}$). Finally, when $A = B = 0$ all the sequence terms vanish. One should note that the conditions $A = 0$ and $B = 0$ are equivalent to $b = az_2$ and $b = az_1$, respectively.

2.3. **The structure of periodic Horadam orbits.** The periodic patterns recovered from Horadam sequences include all regular star polygons in the complex plane, as well as bipartite and multipartite digraphs, whose stable sets are regular polygons. For $AB \neq 0$, the geometric structure of periodic Horadam orbits is given by the following theorem.

Theorem 2.1. [4, Theorem 3.4] *Let $k_1, k_2, d \geq 2$ be natural numbers s.t. $gcd(k_1, k_2) = d$ and z_1, z_2 be k_1th and k_2th primitive roots, respectively. The orbit of the sequence $\{w_n\}_{n=0}^{\infty}$ is then a k_1k_2/d-gon, whose nodes can be divided into k_1 regular k_2/d-gons representing a multipartite graph. By duality, the nodes of the orbit can also be divided into k_2 regular k_1/d-gons.*

3. Results

As shown in [1, Theorem 3.1], a sufficient condition for periodicity is that generators are distinct roots of unity. Here we discuss the possible periods yielded by self-repeating Horadam sequences obtained from a fixed generator pair, and for different initial conditions a, b.

In what follows the notation $[k_1, k_2]$ is used for the least common multiple of k_1 and k_2. In general, two primitive roots of orders k_1, k_2 are expected to produce an orbit of length $[k_1, k_2] = k$. However, this is not always true, as the following theorem shows.

Theorem 3.1. *Consider the distinct primitive roots of unity* $z_1 = e^{2\pi i p_1/k_1}$ *and* $z_2 = e^{2\pi i p_2/k_2}$, *where* p_1, p_2, k_1, k_2 *are positive integers and let the polynomial* $P(x)$ *be defined by*

$$P(x) = (x - z_1)(x - z_2), \quad x \in \mathbb{C}. \tag{3.1}$$

The recurrence sequence $\{w_n\}_{n=0}^{\infty}$ *generated by the characteristic polynomial (3.1) and the arbitrary initial values* $w_0 = a$, $w_1 = b$ *is periodic. The following periods are possible:*

(a) $AB \neq 0$*: the period is* $[k_1, k_2]$*;*
(b) $A = 0$, $B \neq 0$*: the period is* k_2*;*
(c) $A \neq 0$, $B = 0$*: the period is* k_1*;*
(d) $A = B = 0$*: the period is* 1 *(constant sequence).*

Proof. (a) For $AB \neq 0$, the general term of the sequence given by formula (2.1) is a non-degenerated linear combination of z_1^n and z_2^n, hence the period is $[k_1, k_2]$.
(b) Here Az_1^n does not feature in (2.1), hence the orbit is the regular star polygon $\{k_2/p_2\}$.
(c) Similarly to point (b), here the orbit is the regular star polygon $\{k_1/p_1\}$.
(d) When $A = B = 0$, one has $b = az_1 = az_2$, hence, $a(z_2 - z_1) = 0$. Since z_1 and z_2 are distinct, this implies $a = b = 0$ and the sequence is constant. □

Note that the only possible masked periods for $k_1 = k_2 = k$ may have lengths k and 1.

The following example illustrates the result of Theorem 3.1.

Example 3.2. *Consider the primitive roots of unity* $z_1 = e^{2\pi i 2/3}$ *and* $z_2 = e^{2\pi i 1/4}$*. By Theorem 3.1, the periodic Horadam orbit produced by formula* (2.1) *may have the lengths 1, 3, 4 or 12. The distinct periodic orbits produced in this example are sketched in Figure 1 (a)-(d).*

Theorem 3.1 also has the following consequence.

Proposition 3.3. *Let* $k, k_1, k_2 \geq 2$ *be natural numbers such that* $[k_1, k_2] = k$*. Then there exists a periodic Horadam recurrent sequence* $\{w_n\}_{n=0}^{\infty}$ *of length* k*, which for different initial conditions* $w_0 = a$, $w_1 = b$ *masks orbits of period* k_1 *and* k_2*, respectively.*

Proof. One may clearly select the distinct roots $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$, where p_1, p_2, k_1, k_2 are integers. Following Theorem 3.1, the sequence generated by (2.1) will have length k for $AB \neq 0$, with A, B given by (2.2). □

The property stated in Proposition 3.3 is illustrated by the following example.

Example 3.4. *For* $k = 30$*, we can show that the primitive generator pair* $z_1 = e^{2\pi i \frac{1}{3}}$ *and* $z_2 = e^{2\pi i \frac{7}{10}}$ *can mask periods of length* 3 *and* 10*, while the pair* $z_1 = e^{2\pi i \frac{1}{2}}$ *and* $z_2 = e^{2\pi i \frac{11}{15}}$ *can mask periods of length* 2 *or* 15*, respectively. These examples are sketched in Figure 1 (e)–(h).*

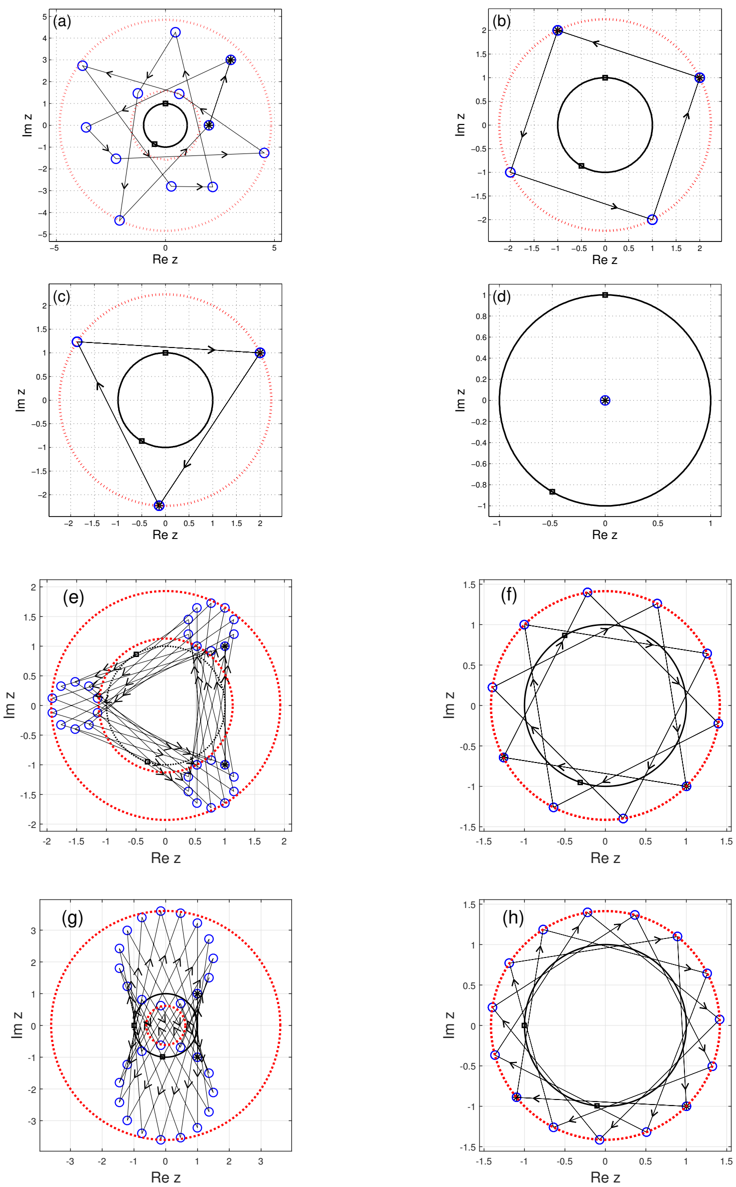

FIGURE 1. Terms of sequence $\{w_n\}_{n=0}^N$ (circles) obtained from (2.1) for the generators $z_1 = e^{2\pi i \frac{1}{3}}$ and $z_2 = e^{2\pi i \frac{1}{4}}$ (squares) and initial conditions (stars) (a) $a = 2$, $b = 3 + 3i$ $(AB \neq 0)$; (b) $a = 2$, $b = 2e^{2\pi i \frac{1}{4}}$ $(A = 0)$; (c) $a = 2$, $b = 2e^{2\pi i \frac{2}{3}}$ $(B = 0)$; (d) $a = b = 0$ $(A = B = 0)$; Then, $z_1 = e^{2\pi i \frac{1}{3}}$, $z_2 = e^{2\pi i \frac{7}{10}}$ and (e) $a = 1 - i$, $b = 1 + i$ $(AB \neq 0)$; (f) $a = 1 - i$, $b = az_2$ $(A = 0)$; Finally, $z_1 = e^{2\pi i \frac{1}{2}}$, $z_2 = e^{2\pi i \frac{11}{15}}$ and (g) $a = 1 - i$, $b = 1 + i$ $(AB \neq 0)$; (h) $a = 1 - i$, $b = az_2$ $(A = 0)$. Arrows indicate the direction of the orbit. Boundaries of annulus $U(0, ||A| - |B||, |A| + |B|)$ (solid line) with A, B from (2.2) and the unit circle (dotted line) are also plotted.

4. The Masked Periodicity of Generalized Horadam Sequences

In this section we investigate the phenomenon of masked periodicity for generalized Horadam sequences, based on the periodicity conditions formulated by Bagdasar and Larcombe in [3]. Findings are illustrated by an example involving third order recurrences.

Let $m \geq 2$ be a natural number, $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{c} = (c_1, \ldots, c_m)$ be vectors of complex numbers and let $\{w_n(\mathbf{a}; \mathbf{c})\}_{n=0}^{\infty}$ be the sequence defined by the recurrence

$$w_n = c_1 w_{n-1} + c_2 w_{n-2} + \cdots + c_m w_{n-m}, \quad m \leq n \in \mathbb{N}, \tag{4.1}$$

satisfying the initial conditions $w_{i-1} = a_i, i = 1, \ldots, m$. It may be assumed without loss of generality that the order of the recurrence cannot be reduced, therefore $c_m \neq 0$.

The characteristic equation of (4.1) is given by

$$\lambda^n = c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_{m-1} \lambda^{n-m+1} + c_m \lambda^{n-m}, \quad n \in \mathbb{N},$$

which for a non-zero value of λ, is equivalent to

$$\lambda^m = c_1 \lambda^{m-1} + c_2 \lambda^{m-2} + \cdots + c_{m-1} \lambda + c_m.$$

For simplicity, we shall assume that the characteristic polynomial $f(x)$ defined by

$$f(x) = x^m - c_1 x^{m-1} - c_2 x^{m-2} - \cdots - c_{m-1} x - c_m, \tag{4.2}$$

has m distinct roots $z_1, \ldots, z_m$. In this case, the general term of the sequence is given by

$$w_n = A_1 z_1^n + A_2 z_2^n + \cdots + A_m z_m^n, \tag{4.3}$$

where the coefficients $A_1, \ldots, A_m$ can be obtained from the initial conditions by solving the system of linear equations

$$\begin{cases} a_1 & = A_1 + A_2 + \cdots + A_m, \\ a_2 & = A_1 z_1 + A_2 z_2 + \cdots + A z_m, \\ \cdots & \\ a_m & = A_1 z_1^{m-1} + A_2 z_2^{m-1} + \cdots + A z_m^{m-1}. \end{cases} \tag{4.4}$$

For distinct roots of the characteristic polynomial (4.2), necessary and sufficient periodicity conditions for $\{w_n\}_{n=0}^{\infty}$ have been given by Theorems 3.1 and 3.2 in [4].

Theorem 4.1. *Let $z_1, \ldots, z_m \in \mathbb{C}$ be distinct numbers and the polynomial $P(x)$ defined as*

$$P(x) = (x - z_1)(x - z_2) \cdots (x - z_m), \quad x \in \mathbb{C}. \tag{4.5}$$

The recurrent sequence $\{w_n\}_{n=0}^{\infty}$ having the characteristic polynomial (4.5) and arbitrary initial conditions $w_{i-1} = a_i \in \mathbb{C}$, $(i = 1, \ldots, m)$ is periodic if and only if there is $k \in \mathbb{N}$ positive s. t.

$$A_i(z_i^k - 1) = 0, \quad i = 1, \ldots, m,$$

where the coefficients $A_1, \ldots, A_m$ are given by the solution of (4.4).

We are now ready to formulate the masked periodicity theorem in the general case.

Theorem 4.2. *Let $m \geq 2$ and the distinct primitive roots of unity $z_j = e^{2\pi i p_j / k_j}$, $j = 1, \ldots, m$. The recurrence sequence $\{w_n\}_{n=0}^{\infty}$ generated by the characteristic polynomial (4.5), and the arbitrary initial values $w_j = a_{j+1}$, $j = 1, \ldots, m$ is periodic. If the sets I and J satisfy*

$$I \cap J = \emptyset, \quad I \cup J = \{1, \ldots, m\},$$

where the solution $A_1, A_2, \dots, A_m$ of the system (4.4) *has the property*

$$\begin{aligned} A_i &= 0, \ \textit{for} \quad i \in I, \\ A_j &\neq 0, \ \textit{for} \quad j \in J, \end{aligned}$$

then the period of the sequence is the least common multiple of the numbers k_j with $j \in J$.

Proof. For all $i \in I$, the terms corresponding to z_i^n do not feature explicitly in the general term formula (4.3). The general term is then a linear combination of non-vanishing individual terms of periods k_j, $j \in J$, hence the conclusion. □

We illustrate this result by an example involving a third order recurrence ($m = 3$).

Example 4.3. *Consider the distinct primitive roots of unity $z_1 = e^{2\pi i p_1/k_1}$, $z_2 = e^{2\pi i p_2/k_2}$, and $z_3 = e^{2\pi i p_3/k_3}$ where $p_1, p_2, p_3, k_1, k_2, k_3$ are positive integers, and the polynomial $P(x)$*

$$P(x) = (x - z_1)(x - z_2)(x - z_3), \quad x \in \mathbb{C}. \tag{4.6}$$

The recurrence sequence $\{w_n\}_{n=0}^{\infty}$ generated by the characteristic polynomial (4.6), and the arbitrary initial values $w_0 = a_1$, $w_1 = a_2$, $w_2 = a_3$ is periodic. Moreover, if A_1, A_2 and A_3 are the solutions of the system

$$\begin{cases} a_1 &= A_1 + A_2 + A_3, \\ a_2 &= A_1 z_1 + A_2 z_2 + A_3 z_3, \\ a_3 &= A_1 z_1^2 + A_2 z_2^2 + A_3 z_3^2, \end{cases} \tag{4.7}$$

the following periods are possible:

(a) $A_1 A_2 A_3 \neq 0$: the period is $[k_1, k_2, k_3]$;
(b) $A_i = 0$, $A_j A_l \neq 0$: the period is $[k_j, k_l]$ (where $i, j, l \in \{1, 2, 3\}$ are distinct);
(c) $A_i \neq 0$, $A_j = A_l = 0$: the period is k_i (where $i, j, l \in \{1, 2, 3\}$ are distinct);
(d) $A_1 = A_2 = A_3 = 0$: the period is 1 (constant sequence).

In addition to the theoretical result, here we also formulate the specific relations between generators and initial conditions, required for the various outcomes. The formula for the general term of sequence $\{w_n\}_{n=0}^{\infty}$ in this case can be written

$$w_n = A_1 z_1^n + A_2 z_2^n + A_3 z_3^n, \tag{4.8}$$

where constants A_1, A_2 and A_3 satisfy the system (4.7). In matrix form this becomes

$$\begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{pmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}. \tag{4.9}$$

The determinant of the 3×3 Vandermonde matrix is

$$\Delta = \det\Big(V_{3,3}(z_1, z_2, z_3)\Big) = \prod_{1 \le i < j \le 3} (z_j - z_i) = (z_2 - z_1)(z_3 - z_2)(z_3 - z_1) \neq 0,$$

hence the solutions of system (4.9) can be written as

$$\begin{aligned} A_1 &= \frac{(z_3 - z_2)\left[a_1\, z_2 z_3 - a_2(z_2 + z_3) + a_3\right]}{\Delta}, \\ -A_2 &= \frac{(z_3 - z_1)\left[a_1\, z_1 z_3 - a_2(z_1 + z_3) + a_3\right]}{\Delta}, \\ A_3 &= \frac{(z_2 - z_1)\left[a_1\, z_1 z_2 - a_2(z_1 + z_2) + a_3\right]}{\Delta}. \end{aligned} \tag{4.10}$$

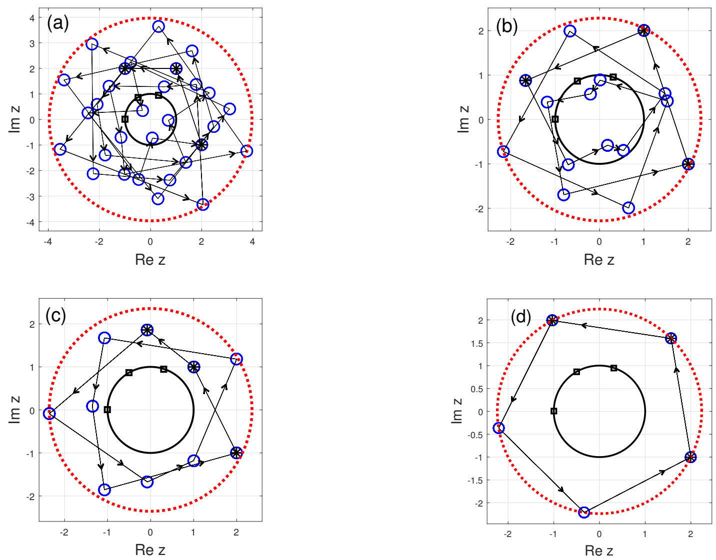

Figure 2. Terms of sequence $\{w_n\}_{n=0}^{30}$ (circles) obtained from (4.8), for the generators $z_1 = e^{2\pi i \frac{1}{2}}$, $z_2 = e^{2\pi i \frac{1}{3}}$ and $z_3 = e^{2\pi i \frac{1}{5}}$ (squares) and initial conditions (stars) for (a) $a_1 = 2-i$, $a_2 = 1+2i$, $a_3 = -1+2i$ $(A_1A_2A_3 \neq 0)$; (b) $a_1 = 2-i$, $a_2 = 1+2i$, $a_3 = a_2(z_2+z_3) - a_2z_2z_3$ $(A_1 = 0,\ A_2A_3 \neq 0)$; (c) $a_1 = 2-i$, $a_2 = 1+2i$, $a_3 = a_2(z_1+z_3) - a_2z_1z_3$ $(A_2 = 0,\ A_1A_3 \neq 0)$; (d) $a_1 = 2-i$, $a_2 = az_3$, $a_3 = a_2(z_1+z_3) - a_2z_1z_3$ $(A_1 = A_2 = 0,\ A_3 \neq 0)$. Arrows indicate the direction of the orbit. Boundaries of circle $U(0, |A_1| + |A_2| + |A_3|)$ (dotted line) with A_1, A_2, A_3 from (4.10) and the unit circle (solid line) are also plotted.

(a) If A_1, A_2, A_3 are non-zero, they all appear in formula (4.8), hence the period of the sequence is given by $[k_1, k_2, k_3]$. For $k_1 = 2$, $k_2 = 3$, $k_3 = 5$ and $A_1A_2A_3 \neq 0$ the period is 30, as depicted in Figure 2 (a). For distinct z_1, z_2, z_3 the masked periodicity conditions reduce to

$$\begin{aligned} A_1 = 0: &\quad a_1\, z_2z_3 - a_2(z_2+z_3) + a_3 = 0,\\ A_2 = 0: &\quad a_1\, z_1z_3 - a_2(z_1+z_3) + a_3 = 0,\\ A_3 = 0: &\quad a_1\, z_1z_2 - a_2(z_1+z_2) + a_3 = 0. \end{aligned} \tag{4.11}$$

(b) If $a_3 = a_2(z_2+z_3) - a_1z_2z_3$, then $A_1 = 0$ and the term involving z_1^n does not feature explicitly in formula (4.8), hence the orbit period is $[k_2, k_3] = 15$ as illustrated in Figure 2 (b). A similar example is shown in Figure 2 (c), where $A_2 = 0$ and the period is 10.

(c) When two of the terms are zero (say for example $A_1 = A_2 = 0$), then the period of the sequence is given by the term z_3^n, hence it is equal to k_3. This case is illustrated in Figure 2 (d). Subtracting the first equation from the second in (4.11), one obtains

$$a_1z_3(z_2 - z_1) - a_2(z_2 - z_1) = 0,$$

which is equivalent to $a_2 = a_1z_3$. By substitution in the first line of (4.11) we obtain $a_3 = a_1z_3^2$.

(d) One may also notice that the only possibility for $A_1 = A_2 = A_3 = 0$ is $a_1 = a_2 = a_3 = 0$.

5. Summary

In this paper the role played by the choice of initial conditions in the period of self-repeating Horadam sequences has been discussed in the context of a phenomenon known as masked periodicity; this validates some earlier work in which the latter was identified and explained, and complements it through an approach based on fixed generator pairs of primitive roots of unity $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$. The period of the Horadam sequence was confirmed to be equal to $[k_1, k_2]$, k_2, k_1, or 1, depending on the choice of initial starting points a and b.

The underpinning methodology of this paper, unlike that seen previously in [6] and [7], accommodates the study of masking for *generalized* Horadam sequences in the manner described. This is readily demonstrated by the example offered which involves a Horadam type sequence of third order.

References

[1] O. D. Bagdasar and P. J. Larcombe, *On the characterization of periodic complex Horadam sequences*, The Fibonacci Quarterly, **51.1** (2013), 28–37.
[2] O. D. Bagdasar and P. J. Larcombe, *On the number of complex Horadam sequences with a fixed period*, The Fibonacci Quarterly, **51.4** (2013), 339–347.
[3] O. D. Bagdasar and P. J. Larcombe, *On the characterization of periodic generalized Horadam sequences*, J. Difference Equations and Applications, **20** (2014), 1069–1090.
[4] O. D. Bagdasar, P. J. Larcombe, and A. Anjum, *On the structure of periodic complex Horadam orbits*, Carpathian J. Mathematics, **32** (2016), 29–36.
[5] P. J. Larcombe, O. D. Bagdasar, and E. J. Fennessey, *Horadam sequences: a survey*, Bulletin of the I.C.A., **67** (2013), 49–72.
[6] P. J. Larcombe and E. J. Fennessey, *On Horadam sequence periodicity: a new approach*, Bulletin of the I.C.A., **73** (2015), 98–120.
[7] P. J. Larcombe and E. J. Fennessey, *On the phenomenon of masked periodic Horadam sequences*, Utilitas Mathematica, **96** (2015), 111–123.
[8] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

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