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Robust generalized Mittag-Leffler synchronization of fractional order neural networks with discontinuous activation and impulses

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Abstract

Fractional order system is playing an increasingly important role in terms of both theory and applications. In this paper we investigate the global existence of Filippov solutions and the robust generalized Mittag-Leffler synchronization of fractional order neural networks with discontinuous activation and impulses. By means of growth conditions, differential inclusions and generalized Gronwall inequality, a sufficient condition for the existence of Filippov solution is obtained. Then, sufficient criteria are given for the robust generalized Mittag-Leffler synchronization between discontinuous activation function of impulsive fractional order neural network systems with (or without) parameter uncertainties, via a delayed feedback controller and M-Matrix theory. Finally, four numerical simulations demonstrated the effectiveness of our main results.

Keywords. *Generalized Mittag-Leffler synchronization; Discontinuous neural networks; Filippov solutions; Delayed feedback controller; Parameter uncertainties;.*

1 Introduction

In recent years, fractional order dynamical system has aroused interest of many researchers in the field of nonlinear science and technology. Fractional-order calculus, which generalized the classical calculus developed in the 17th century (*Podlubny*, 1999 & *Kilbas et al.*,2006). Fractional calculus investigates primarily the properties of derivatives and integrals of non-integer order. In particular, the differential equations involving fractional derivatives have important geometric interpretations. For this reason, fractional calculus is currently a rapidly growing field, in terms of both theory and applications to real world problem. More precisely, fractional calculus has been applied in various branches of science and engineering, including electromagnetic waves (*Heaviside*, 1971) and bioengineering (*Magin et al.*,2008). Compared to integer order calculus, fractional order one has infinite memory and more degrees of freedom (*Chen et al.*,2010). Moreover, fractional order is also said to be "more authentic" (*Hilfer*, 2000). Nowadays, the dynamical system of synchronization or stability of fractional order neural networks was found to play an important role in applications, such as information theory, pattern recognition, cryptography or secure communication (*Milanovic et al.*,1996; *Yang et al.*,2012; *Ren et al.*,2015).

Since the formulation of drive-response synchronization concept in 1990s by Carroll and Pecora, which means dynamical behaviors of a coupled system that realizes convergence to the matching

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spatial state, has become an important research topic in various areas. Still now, there are numerous types of synchronization concepts known in the literature, including complete synchronization (*Ding et al.*,2016; *Yang et al.*,2014; *Lu et al.*,2012), anti-synchronization (*Dessoky*, 2010), lag synchronization (*Zhan et al.*,2002), phase synchronization (*Rosenblum et al.*,1996) and others. However, few practical network systems can be synchronized directly. To address this problem, several control schemes have been introduced, such as feedback control (*Li et al.*,2017; *Cao et al.*,2017), linear feedback control (*Xiao et al.*,2016), observer based control (*Jiang et al.*,2006) and impulsive control (*Yang et al.*,2007).

Generally, time delay of the signal between the driver and response system is unavoidable because of the network traffic congestion as well as finite switching speed of signal transmission over the links, which may leads to instability, chaos, oscillation or other performance of network models (*Li et al.*,2017). Moreover, time delays are more complicated compared to other networks (*Chen et al.*, 2013; *Li et al.*,2016). Impulses, i.e., abrupt changes in state at certain times also affect the stability of the systems (*Li et al.*,2017, 2015). Generally, impulsive systems belong to two major types: first is the constant impulsive system, while the other one is time varying impulsive system. Due to measurement errors, parameter fluctuations as well as external disturbance, parameter uncertainty is unavoidable, which has important effects on the stability and synchronization capability of most real world dynamical systems. Additionally, the main application of this dynamical problem is used to secure communication, only if the drive and response systems realize synchronization can the transmitted signal be fingered out. Therefore, it is necessary to study fractional order complete synchronization of neural networks with discontinuous activations. Firstly, the global convergence of general neural networks with discontinuous activations were considered in (*Forti. M* and *Nistri.P*, 2003), while *Forti et al.*,2005 discussed the infinite gain of neural discontinuous activations. Besides, these activations are mainly applied to systems oscillating under earthquake, dry friction, power circuits and so on. Several results with respect to synchronization of discontinuous neural networks have been reported in the literature(*Lu et al.*,2005, *Xiao et al.*,2006, *Liu et al.*,2011, *Liu et al.*,2014). On the other hand, *Wang et al.*,2016, some parameter uncertain models of integer order delayed neural networks with discontinuous activations are discussed, while *Ding et al.*,2016 investigated Mittag-Leffler synchronization of neural networks with discontinuous activation functions by using M-matrix theory and non smooth analysis. However, there are few results of synchronization of fractional neural networks with discontinuous activation. To our best of knowledge, there is no results published in robust generalized Mittag-Leffler synchronization of delayed neural network systems (GMSDNNs) with (or without) parameter uncertainties. This model is more general and can be extended beyond the study of integer order discontinuous dynamical systems.

Inspired by the above analysis and discussions, our main aim in this paper, is to study the generalized Mittag-Leffler synchronization of delayed fractional order neural networks(GMSDNNs) with discontinuous activations. The crucial novelty of this paper is further summarized as follows:

- In the sense of Caputo fractional order derivative of $0 < \alpha < 1$, based on the growth condition and non smooth analysis, we have proved the global existence of Filippov solution.
- A delayed feedback controller is designed which includes the constant time delay terms and discontinuous term.
- By means of M-matrix theory, Lyapunov stability theory and proposed discontinuous control scheme, the algebraic sufficient condition for generalized Mittag-leffler synchronization is addressed, and we improved the fractional order continuous activation synchronization methods. Moreover, an important feature presents in our paper is that the improved result is still true for integer order robust exponential synchronization of delayed neural networks with discontinuous (continuous) activations with impulses.

The rest of the paper is organized as follows. In *Section* 2, some basic definitions and preliminaries are given including the problem formulation are introduced. In *Section* 3, The existence of Filippov solution is provided and derive the sufficient criteria for the robust generalized Mittag-Leffler synchronization between drive and response neural network systems. *Section* 4 consider the four numerical examples to validate the theoretical obtained results, conclusions are drawn in *Section* 5.

2 Model Description and Preliminaries

Notations: Throughout this paper, $\mathbb{R}$ is the space of real number, $\mathbb{N}_+$ is the set of positive integers and $\mathbb{C}$ is the space of complex numbers. For a vector $x \in \mathbb{R}^n$, we shall use the norm $\|x\| = \|.\|_1 = \sum_{i=1}^{n} |x_i|$. The signum function applied for a vector $\text{sgn}(x) = [\text{sgn}(x_1), \text{sgn}(x_2), ..., \text{sgn}(x_n)]^T$ is given by

$$\text{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

Also, $\mathbb{R}^{n\times n}$ denotes the set of all $n \times n$ real matrices. For a square matrix $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, we consider the absolute value given by the formula $|A| = (|a_{ij}|)_{n\times n} \in \mathbb{R}^{n\times n}$. In addition $C^n([t_0, +\infty), \mathbb{R})$ denotes the space consisting of n-order continuous differentiable functions from $[t_0, +\infty)$ into $\mathbb{R}$.
For our further presentation and convenience, we set the following notations:

$$\begin{aligned}
\underline{D} &= diag(\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_n),\ \hat{a}_{ij} = \max\{|\underline{a}_{ij}|, |\bar{a}_{ij}|\},\ \ \hat{b}_{ij} = \max\{|\underline{b}_{ij}|, |\bar{b}_{ij}|\}, \\
K_1 &= diag(\kappa_{1,1},\ \kappa_{1,2}, ..., \kappa_{1,n}),\ K_2 = diag(\kappa_{2,1}, \kappa_{2,2}, \ldots, \kappa_{2,n}),\ \ K_3 = diag(\kappa_{3,1},\ \kappa_{3,2},\ \ldots...,\kappa_{3,n}), \\
E_1 &= (\hat{a}_{ij}p_j)_{n\times n},\ \ E_2 = \Big((\hat{a}_{ij} + \hat{b}_{ij})p_j\Big)_{n\times n},\ E_3 = (|a_{ij}|p_j)_{n\times n},\ \ E_4 = \Big((|a_{ij}| + |b_{ij}|)p_j\Big)_{n\times n}, \\
F_1 &= diag\Big\{\sum_{j=1}^{n} \hat{b}_{1j}p_j, \sum_{j=1}^{n} \hat{b}_{2j}p_j, \ldots.., \sum_{j=1}^{n} \hat{b}_{nj}p_j\Big\}, \\
F_2 &= diag\Big\{\sum_{j=1}^{n}(\hat{a}_{1j} + \hat{b}_{1j})q_j, \sum_{j=1}^{n}(\hat{a}_{2j} + \hat{b}_{2j})q_j, \ldots, \sum_{j=1}^{n}(\hat{a}_{nj} + \hat{b}_{nj})q_j\Big\}, \\
F_3 &= diag\Big\{\sum_{j=1}^{n} \hat{a}_{1j}q_j, \sum_{j=1}^{n} \hat{a}_{2j}q_j, \ldots...., \sum_{j=1}^{n} \hat{a}_{nj}q_j\Big\}, \\
M_1 &= diag\Big\{\sum_{j=1}^{n} |b_{1j}|p_j, \sum_{j=1}^{n} |b_{2j}|p_j, \cdots\cdots \sum_{j=1}^{n} |b_{nj}|p_j\Big\}, \\
M_2 &= diag\Big\{\sum_{j=1}^{n}(|a_{1j}| + |b_{1j}|)q_j, \sum_{j=1}^{n}(|a_{2j}| + |b_{2j}|)q_j, \ldots, \sum_{j=1}^{n}(|a_{nj}| + |b_{nj}|)q_j\Big\}.
\end{aligned}$$

In this section we recalled some key definitions, assumption and some basic lemmas.

Definition 2.1 (*Kilbas*, 2006 & *Podlubny*, 1999). *The Caputo fractional-order derivative of order α for a function x(t)$\in C^n([t_0, +\infty))$ is defined as*

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{x^n(s)}{(t-s)^{\alpha-n+1}}\, \mathrm{d}s,$$

where $t \geq t_0$ and n is the positive integer such that $n - 1 < \alpha < n$.
Particularly, when $0 < \alpha < 1$,

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{x'(s)}{(t-s)^\alpha}\, \mathrm{d}s.$$

Definition 2.2 (*Podlubny*, 1999). *The Laplace transform of Mittag-Leffler function is*

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})\} = \frac{s^{\alpha-\beta}}{s^{\alpha}+\lambda}, \qquad (Re(s) > |\lambda|^{\frac{1}{\alpha}}),$$

where $E_{\alpha,\beta}(\cdot)$ *is the the two-parameter Mittag-Leffler function, which is defined by* $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$, $\alpha > 0,\ \beta > 0,\ z \in \mathbb{C}$, *while* s *is the variables in Laplace domain. Additionally,* $E_{\alpha,1}(\cdot)$ *denote the one-parameter Mittag-Leffler function and* $E_{1,1}(\cdot)$ *represent the exponential function.*

Consider an n-dimensional fractional order delayed system:

$$D^{\alpha}x(t) \;=\; f(t, x(t-\tau)), \tag{1}$$

where $\alpha \in (0,1),\ x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n,\ f : [0,+\infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is piecewise continuous on t. Its solution can be solved as

$$x(t) = \varsigma(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s-\tau))\,\mathrm{d}s,$$

where $x(s) = \varsigma(s),\ s \in [-\tau, 0]$ is the initial values of system (1). We consider a class of fractional order impulsive delayed neural networks with discontinuous activations as follows:

$$\begin{cases} D^{\alpha}x_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij}h_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}h_j(x_j(t-\tau_j)) + I_i,\ t \neq t_k,\ t \geq 0, \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k) = \Upsilon_{ik}(x_i(t_k)),\ k = 1, 2, ... \\ x_i(s) = \rho_i(s),\ s \in [-\tau, 0], \end{cases} \tag{2}$$

and the vector form is

$$\begin{cases} D^{\alpha}x(t) = -Dx(t) + Ah(x(t)) + Bh(x(t-\tau)) + I,\ t \neq t_k,\ t \geq 0, \\ \Delta x(t_k) = x(t_k^+) - x(t_k) = \Upsilon_{ik}(x(t_k)),\ k = 1, 2, \ldots \\ x(s) = \rho(s),\ s \in [-\tau, 0], \end{cases}$$

where $i = 1, \ldots, n (n \in \mathbb{N}_+)$, D^{α} is the Caputo fractional derivative of order α $(0 < \alpha < 1)$; $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the state vector $(t > 0)$; $D = diag(d_1, d_2, \ldots, d_n) > 0$, which stands for self connection weight matrix; $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, where a_{ij} represents connection weight matrix on the jth neurons to ith neurons; $B = (b_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, where b_{ij} represents delayed connection weight matrix on the jth neurons to ith neurons; $h(x(t)) = ((h_1(x_1(t)), ..., (h_n(x_n(t)))^T \in \mathbb{R}^n$ is the neuron nonlinear activation function at time $t > 0$; $\tau_j \geq 0$ is constant time delay; I_i corresponds to the constant external input; $\Upsilon_{ik} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is impulsive operator and the impulsive moment $t_k,\ k = 1, 2, ...$ satisfying $t_0 < t_1 < \ldots$ and $\lim_{k\to+\infty} t_k = +\infty$; $x_i(t_k^-) = \lim_{t\to t_k^-} x(t)$ and $x_i(t_k^+) = \lim_{t\to t_k^+} x(t)$ express the left and right limits on impulsive moments at time $t = t_k$. Without loss of generality, the solution of network system (2) is left continuous at time t_k. i.e., $x_i(t_k^-) = x_i(t_k)$; The uncertainties of network parameter is important factor that affects stability. In this paper, the parameter matrices $D = diag\{d_1, d_2, ..., d_n\},\ A = (a_{ij})_{n\times n},\ B = (b_{ij})_{n\times n}$ of system (2) are assumed to be norm-bounded within the following ranges:

$$\begin{aligned} D_I &= [\underline{D}, \bar{D}] = \{diag(d_i) : 0 \leq \underline{d}_i \leq d_i \leq \bar{d}_i, i = 1, 2, ..., n\}, \\ A_I &= [\underline{A}, \bar{A}] = \{diag(a_{ij}) : 0 \leq \underline{a}_{ij} \leq a_{ij} \leq \bar{a}_{ij}, i = 1, 2, ..., n\}, \\ B_I &= [\underline{B}, \bar{B}] = \{diag(b_{ij}) : 0 \leq \underline{b}_{ij} \leq b_{ij} \leq \bar{b}_{ij}, i = 1, 2, ..., n\}. \end{aligned} \tag{3}$$

Now the following assumptions about the discontinuous neuron activation function for system (2) are considered:

A(1): (i) $h_j,\ \ j = 1, 2, \ldots, n$ are piecewise continuous, i.e., h_j are continuous in $\mathbb{R}$ except a countable set of jump discontinuous points and in every compact set of $\mathbb{R}$ has only a finite number of discontinuous points.

(ii) h_j, $j = 1, 2, \dots, n$ are non decreasing and bounded.

Throughout this paper, we consider drive-response system of the corresponding system as follows:

$$\begin{cases} D^\alpha y_i(t) = -d_i y_i(t) + \sum_{j=1}^n a_{ij} h_j(y_j(t)) + \sum_{j=1}^n b_{ij} h_j(y_j(t-\tau_j)) + I_i + \theta_i(t),\ t \neq t_k,\ t \geq 0, \\ \Delta y_i(t_k) = y_i(t_k^+) - y_i(t_k) = \Upsilon_{ik}(y_i(t_k)),\ k = 1, 2, ... \\ y_i(s) = \varrho_i(s),\ s \in [-\tau, 0], \end{cases} \tag{4}$$

and the vector form is

$$\begin{cases} D^\alpha y(t) = -Dy(t) + Ah(y(t)) + Bh(y(t-\tau)) + I + \theta(t),\ t \neq t_k,\ t \geq 0, \\ \Delta y(t_k) = y(t_k^+) - y(t_k) = \Upsilon_{ik}(y(t_k)),\ k = 1, 2, ... \\ y(s) = \varrho(s),\ s \in [-\tau, 0], \end{cases}$$

where $i = 1, 2, ..., n$ $(n \in \mathbb{N}_+)$ and $y(t) = (y_1(t), y_2(t), ..., y_n(t))^T \in \mathbb{R}^n$ is the state vector of the system (4), the coefficients are same as ones of system (2) and $\theta_i(t)$ is a delayed feedback controller which is defined by

$$\begin{aligned} \theta_i(t) \quad = \quad & -\kappa_{1,i}(y_i(t) - x_i(t)) - \kappa_{2,i}\,\mathrm{sgn}(y_i(t) - x_i(t)) \sum_{j=1}^n |y_j(t-\tau_j) - x_j(t-\tau_j)| \\ & -\kappa_{3,i}\,\mathrm{sgn}(y_i(t) - x_i(t)), \end{aligned} \tag{5}$$

where $i = 1, 2, ..., n$ $(n \in \mathbb{N}_+)$ and constants $\kappa_{1,i}$, $\kappa_{2,i}$ and $\kappa_{3,i}$ are three positive gain coefficients and sgn(.) is a function defined as follows

$$\mathrm{sgn}(y_i(t) - x_i(t)) = \begin{cases} 1, & y_i(t) - x_i(t) > 0 \\ 0, & y_i(t) - x_i(t) = 0 \\ -1, & y_i(t) - x_i(t) < 0. \end{cases}$$

Since the activation functions h_j $(j = 1, 2, ..., n)$ are discontinuous, the definition of traditional solution for differential equations does not exists. In this case, we introduce the concept of Filippov solution for fractional order differential equation. Now, we consider the following fractional order differential system in vector form as follows:

$$\begin{cases} D^\alpha x(t) = h(t, x) \\ x(0) = x_0, \end{cases} \tag{6}$$

where $h(t, x)$ are discontinuous in x. In the following, we apply the framework of Filippov in discussing the solution of fractional order delayed neural network (2) .

Definition 2.3 *(Filippov, 1988). Suppose $E \subset \mathbb{R}^n$. Then $x \mapsto H(x)$ is called a set valued map from $E \hookrightarrow \mathbb{R}^n$, if for each point x of a set $E \subset \mathbb{R}^n$, there corresponds a nonempty set $H(x) \subset \mathbb{R}^n$. A set valued map H with nonempty values is said to be upper-semi-continuous at $x_0 \in E$ if for any open set containing $H(x_0)$, there exists a neighborhood M of x_0 such that $H(M) \subset N$. H(x) have closed (convex, compact) image if for each $x \in E$, $H(x)$ is closed (convex, compact).*

Definition 2.4 *(Filippov, 1988). A set valued map $H : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is defined as*

$$H(t, x) = \bigcap_{\delta > 0} \bigcap_{m(N)=0} K[h(t, B(x, \delta)/N)],$$

where K(E) is the closure of the convex hull of set E, $B(x, \delta) = \{y : \|y - x\| \leq \delta\}$, and m(N) is the lebesgue measure of a set N. A vector function $x(t)$ defined on a non degenerate interval $I \subset \mathbb{R}$ is called a Filippov solution of system (6), if it is absolutely continuous on any subinterval $[t_1, t_2]$ of I and for a.e $t \in I$, $x(t)$ satisfies the differential inclusion: $D^\alpha x(t) \in H(t, x)$.

Now we denote $H(x) = [H(x)] = ([h_1(x_1)], ..., [h_n(x_n)])$, where $K[h_i(x_i)] = [\min\{h_i(x_i^-), h_i(x_i^+)\}$, $\max\{h_i(x_i^-), h_i(x_i^+)\}]$, for $i = 1, 2, ..., n$. As $(y_i(t) - x_i(t))$ is discontinuous at $y_i(t) - x_i(t) = 0$, we denote the set valued map as follows

$$K[\operatorname{sgn}(y_i(t) - x_i(t))] = \begin{cases} 1, & y_i(t) - x_i(t) > 0 \\ [-1, 1], & y_i(t) - x_i(t) = 0 \\ -1, & y_i(t) - x_i(t) < 0. \end{cases}$$

By using the theories of differential inclusions and set valued maps, from (2) it follows that

$$\begin{cases} D^\alpha x_i(t) \in -d_i x_i(t) + \sum_{j=1}^n a_{ij} K[h_j(x_j(t))] + \sum_{j=1}^n b_{ij} K[h_j(x_j(t - \tau_j))] + I_i \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k) = \Upsilon_{ik}(x_i(t_k)), k = 1, 2, ... \\ x_i(s) = \rho_i(s),\ s \in [-\tau, 0] \end{cases}$$

for a.a. $t \geq 0,\ i = 1, 2, \dots, n$ (or) there exists $\lambda_j(t) \in K[h_j(x_j(t))]$ such that

$$\begin{cases} D^\alpha x_i(t) = -d_i x_i(t) + \sum_{j=1}^n a_{ij} \lambda_j(t) + \sum_{j=1}^n b_{ij} \lambda_j(t - \tau_j) + I_i \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k) = \Upsilon_{ik}(x_i(t_k)),\ k = 1, 2, ... \\ x_i(s) = \rho_i(s),\ s \in [-\tau, 0]. \end{cases} \tag{7}$$

for a.a. $t \geq 0,\ i = 1, 2, \dots, n$. From system (4), we have

$$\begin{cases} D^\alpha y_i(t) \in -d_i y_i(t) + \sum_{j=1}^n a_{ij} K[h_j(y_j(t))] + \sum_{j=1}^n b_{ij} K[h_j(y_j(t - \tau_j))] + I_i - \kappa_{1,i}(y_i(t) - x_i(t)) \\ \qquad -\kappa_{2,i} \operatorname{sgn}(y_i(t) - x_i(t)) \sum_{j=1}^n |y_j(t - \tau_j) - x_j(t - \tau_j)| - \kappa_{3,i} \operatorname{sgn}(y_i(t) - x_i(t)), \\ \Delta y_i(t_k) = y_i(t_k^+) - y_i(t_k) = \Upsilon_{ik}(y_i(t_k)),\ k = 1, 2, ... \\ y_i(s) = \varrho_i(s),\ s \in [-\tau, 0]. \end{cases}$$

for a.a. $t \geq 0,\ i = 1, 2, \dots, n$ (or) there exists $\mu_j(t) \in K[h_j(y_j(t))]$ and $\epsilon_i(t) \in K[\operatorname{sgn}(y_i(t) - x_i(t))]$ such that

$$\begin{cases} D^\alpha y_i(t) = -d_i y_i(t) + \sum_{j=1}^n a_{ij} \mu_j(t) + \sum_{j=1}^n b_{ij} \mu_j(t - \tau_j) + I_i - \kappa_{1,i}(y_i(t) - x_i(t)) \\ \qquad -\kappa_{2,i} \epsilon_i(t) \sum_{j=1}^n |y_j(t - \tau_j) - x_j(t - \tau_j)| - \kappa_{3,i} \epsilon_i(t) \\ \Delta y_i(t_k) = y_i(t_k^+) - y_i(t_k) = \Upsilon_{ik}(y_i(t_k)),\ k = 1, 2, ... \\ y_i(s) = \varrho_i(s),\ s \in [-\tau, 0] \end{cases} \tag{8}$$

for a.a. $t \geq 0, \quad i = 1, 2, \dots, n$. We now define the synchronization error $z_i(t) = y_i(t) - x_i(t)$. According to (7) and (8), the error system can be founded by

$$\begin{cases} D^\alpha z_i(t) = -(d_i + \kappa_{1,i}) z_i(t) + \sum_{j=1}^n a_{ij}(\mu_j(t) - \lambda_j(t)) + \sum_{j=1}^n b_{ij}(\mu_j(t - \tau_j) - \lambda_j(t - \tau_j)) \\ \qquad -\kappa_{2,i} \epsilon_i(t) \sum_{j=1}^n |z_j(t - \tau_j)| - \kappa_{3,i} \epsilon_i(t) \\ \Delta z_i(t_k) = z_i(t_k^+) - z_i(t_k) = \Upsilon_{ik}(z_i(t_k)),\ k = 1, 2, ... \\ z_i(s) = \varrho_i(s) - \rho_i(s) = \Theta_i(s),\ s \in [-\tau, 0]. \end{cases} \tag{9}$$

for a.a. $t \geq 0,\ i = 1, 2, \dots, n$.
Let us give the definition of generalized Mittag-Leffler synchronization for system (2) and (4).

Definition 2.5 *For any solutions $x(t)$ and $y(t)$ of system (2) and (4) is said to be generalized Mittag-Leffler synchronization with differential initial values denoted by $x(0)$ and $y(0)$ if there exists a two constants $\Lambda > 0$ and $\Omega > 0$ such that*

$$\|y(t) - x(t)\| \leq \Lambda \|y(0) - x(0)\| t^{-\zeta} E_{\alpha, 1-\zeta}(-\Omega t^\alpha),\ t \geq 0,\ \alpha \in (0, 1),\ -\alpha < \zeta \leq 1 - \alpha.$$

Remark 2.6 *Let $\zeta = 0$ from the above definition, it follows that $\|y(t)-x(t)\| \leq \Lambda\|y(0)-x(0)\|E_\alpha(-\Omega t^\alpha)$, $t \geq 0$. This is called the robust Mittag-Leffler synchronization.*

Remark 2.7 *Let $\Omega = 0$ from the above definition, it follows that $\|y(t) - x(t)\| \leq \Lambda\|y(0) - x(0)\|t^{-\zeta}$. This is called the power-law synchronization.*

The following lemma are useful in existence of Filippov solutions in next section.

Lemma 2.8 (Ye, et al.,2007). *For a $\beta > 0$, suppose $a(t)$ is a nonnegative, nondecreasing function locally integrable on $0 \leq t < T$ some $(T \leq +\infty)$ and $b(t) \leq M$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < T$, where M is a constant. If $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with satisfying $u(t) = a(t) + b(t)\int_0^t (t-s)^{\beta-1}u(s)\,\mathrm{d}s$ on the interval, we have $u(t) \leq a(t)E_\beta(b(t)\Gamma(\beta)t^\beta)$.*

Besides, the following Lemma plays an important role in the main results for robust generalized Mittag-Leffler synchronization of a system (2) and (4). If J is an M-matrix, then there is a positive vector $\beta \in \mathbb{R}^n$ such that $\beta^T J > 0$.

Lemma 2.9 *(Berman and Plemmons, 1979). Let $J = (t_{ij})_{n\times n}$ have non positive off-diagonal elements. Each of the following condition is equivalent to that J is an M-matrix.*
(i) All principal minors off J are positive.
(ii) All diagonal elements of J are positive and there exists a positive diagonal matrix $P = diag(p_1, p_2, \ldots, p_n)$ such that matrix JP is strictly diagonally row dominant. i.e.

$$t_{ii}p_i > \sum_{j=1, j\neq 1}^{n} |t_{ij}|p_j, \ j = 1, 2, \ldots, n.$$

(iii) All diagonal elements of T are positive and there exists a positive diagonal matrix $P = diag(p_1, p_2, \ldots, p_n)$ such that PJ is strictly diagonally column dominant. , i.e.,

$$t_{jj}p_j > \sum_{i=1, i\neq 1}^{n} p_i|t_{ij}|, \ j = 1, 2, \ldots, n.$$

3 Main results

In this section, we prove the global existence of a Filippov solution of a system (2) on $[0, +\infty)$ and to guarantee the global robust generalized Mittag Leffler synchronization criteria for such drive-response error dynamical system based on a state feedback control strategy with or without delay.

Theorem 3.1 *Suppose H satisfies a growth conditions (g.c): there exist a constants $k_i > 0$ and r_i such that*

$$|H_i(x_i)| \quad = \quad \sup_{\gamma\in H_i(x_i)} |\gamma| \leq k_i|x_i| + r_i, \quad i = 1, 2, \ldots, n \tag{10}$$

then there exists at least one solution of system (2) for any initial value $x(s) = \rho(s)$, $s \in [-\tau, 0]$ based on assumption A(1).

Proof. Because the set valued map $x(t) \hookrightarrow -Dx(t) + AH(x(t)) + BH(x(t-\tau)) + I$ is upper semi continuous with non empty compact convex values, the local existence of a solution $x(t)$ of equation (7) can be guaranteed(Filippov, 1988). For a.e $t \in [0, +\infty)$, now we have

$$\begin{aligned}\|x(t-\tau)\| &\leq \sup_{-\tau\leq s\leq t} \|x(s)\| \\ &= \sup_{-\tau\leq s\leq 0} \|x(s)\| + \sup_{0\leq s\leq t} \|x(s)\| \\ &= \|\rho(s)\| + \|x(t)\|\end{aligned}$$

According to equation (7) and (10), we obtain

$$\begin{aligned}
&\| -Dx(t)+AH(x(t))+BH(x(t-\tau))+I\| \\
&\quad\leq \|D\|\,\|x(t)\|+\|A\|\,\Big(K\|x(t)\|+R\Big)+\|B\|\,\Big(K\|(x(t-\tau)\|+R\Big)+\|I\| \\
&\quad\leq \|D\|\,\|x(t)\|+\|A\|\,\Big(K\|x(t)\|+R\Big)+\|B\|\,\Big(K(\|\rho(s)\|+\|x(t)\|)+R\Big)+\|I\| \\
&\quad\leq \|D\|\,\|x(t)\|+K\|A\|\,\|x(t)\|+\|A\|\,R+K\|B\|\,\|\rho(s)\|+K\|B\|\,\|x(t)\|+\|B\|\,R+\|I\| \\
&\quad\leq \Big[\|D\|+K(\|A\|+\|B\|)\Big]\,\|x(t)\|+\Big[(\|A\|+\|B\|)R+K\|B\|\,\|\rho(s)\|+\|I\|\Big] \\
&\quad\leq \Phi\|x(t)\|+\Psi,
\end{aligned}$$

where $K=\max\{k_1,\ldots,k_n\}$, $R=\max\{r_1,\ldots,r_n\}$, $\Phi=\|D\|+K(\|A\|+\|B\|)$, $\Psi=(\|A\|+\|B\|)R+K\|B\|\|\rho(s)\|+\|I\|$. Based on the solution expression of fractional order system, one has

$$\begin{aligned}
\|x(t)\| &= \|\rho(0)\|+\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[-Dx(s)+AH(x(s))+BH(x(s-\tau))+I]\,\mathrm{d}s\| \\
&\leq \|\rho(0)\|+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|[-Dx(s)+AH(x(s))+BH(x(s-\tau))+I]\,\mathrm{d}s\| \\
&\leq \|\rho(0)\|+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\Phi\|x(s)\|+\Psi)\,\mathrm{d}s \\
&= \|\rho(0)\|+\frac{\Psi}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s+\frac{\Phi}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x(s)\|\,\mathrm{d}s \\
&= \|\rho(0)\|+\frac{\Psi}{\Gamma(\alpha)}\Big[\frac{-(t-s)^{\alpha}}{\alpha}\Big]_0^t+\frac{\Phi}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x(s)\|\,\mathrm{d}s \\
&= \|\rho(0)\|+\frac{\Psi}{\alpha\Gamma(\alpha)}t^{\alpha}+\frac{\Phi}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x(s)\|\,\mathrm{d}s \qquad (11)
\end{aligned}$$

According to Lemma 2.8 and the inequality (11), we obtain

$$\begin{aligned}
\|x(t)\| &\leq \Big(\|\rho(0)\|+\frac{\Psi}{\alpha\Gamma(\alpha)}t^{\alpha}\Big)E_{\alpha}\Big(\frac{\Phi}{\Gamma(\alpha)}\Gamma(\alpha)t^{\alpha}\Big) \\
\|x(t)\| &\leq \Big(\|\rho(0)\|+\frac{\Psi}{\alpha\Gamma(\alpha)}t^{\alpha}\Big)E_{\alpha}(\Phi t^{\alpha}).
\end{aligned}$$

Hence, since $x(t)$ remains bounded on $[0,+\infty)$. Hence there exists at least one solution of system (2) on $[0,+\infty)$. This completes the proof of theorem.

Remark 3.2 *Existence of Filippov solution with fractional order discontinuous activation system but without delays has been also proved in* (*Zhang* et al.,2016).

The following two assumption are given to obtain robust generalized Mittag-Leffler synchronization of an error system (9).
A(2). For all $i=1,2,\ldots,n$, suppose there exist nonnegative constants p_i and $q_i>0$ such that $\lambda_i(t)\in K[h_i(x_i(t))]$, $\mu_i(t)\in K[h_i(y_i(t))]$, the following inequality holds:

$$|\lambda_i(t)-\mu_i(t)|\leq p_i|y_i(t)-x_i(t)|+q_i.$$

A(3). The functions Υ_{ik} are such that

$$\Upsilon_{ik}(z_i(t_k))=-\sigma_{ik}z_i(t_k),\qquad 0<\sigma_{ik}<2,\ \ i=1,2,\ldots,n,\ \ k=1,2,\ldots.$$

Remark 3.3 *Suppose there exist one discontinuous point in (A2), the constant* $p_i > 0$ *for* $i = 1, 2, \ldots, n$ *when the activation functions are Lipschitz-continuous on* $\mathbb{R}$ *then* $q_i = 0$*. In this case we can choose the controller*

$$\theta_i(t) = -\kappa_{1,i}(y_i(t) - x_i(t)). \tag{12}$$

This implies that the results of the paper are also applicable to corresponding models with continuous activation functions.

Theorem 3.4 *Suppose* $A(1)$*,* $A(2)$ *and* $A(3)$ *holds if* $P_1 = \underline{D} + K_1 - E_1$ *is an M-matrix,* $P_2 = nK_2 - F_1$ *and* $P_3 = K_3 - F_2$ *are two positive diagonal matrices, then the error system* (9) *is robust generalized Mittag-Leffler synchronization based on the controller* (5)*.*

Proof. Assume that $x(t) = (x_1(t), \ldots, x_n(t))^T$ and $y(t) = (y_1(t), \ldots, y_n(t))^T$ are any solution of a system (2) and (4) with initial values $x(0) = (x_1(0), \ldots, x_n(0))^T$ and $y(0) = (y_1(0), \ldots, y_n(0))^T$, respectively. Since $\underline{D} + K_1 - E_1$ is an M-matrix and $nK_2 - F_1$ and $K_3 - F_2$ are positive diagonal matrices, there exist positive constants $\varphi_i (i = 1, 2, \ldots, n)$ by Lemma 2.9 such that

$$\varphi_i\Big((\underline{d}_i + \kappa_{1,i}) - \sum_{j=1}^{n} \hat{a}_{ji} p_i\Big) > 0, \tag{13}$$

$$\varphi_i\Big(n\kappa_{2,i} - \sum_{j=1}^{n} \hat{b}_{ij} p_j\Big) > 0, \tag{14}$$

$$\varphi_i\Big(\kappa_{3,i} - \sum_{j=1}^{n} (\hat{a}_{ij} + \hat{b}_{ij}) q_j\Big) > 0, \tag{15}$$

Consider the following Lyapunov function

$$V(t) = \sum_{i=1}^{n} \varphi_i |z_i(t)|, \tag{16}$$

where $z_i(t)$ are complete synchronization of errors dynamical system (9).
Suppose $z_i(0) = 0,\ i = 1 \ldots n$, then $D^\alpha |z_i(t)| = 0$.
If $z_i(0) < 0,\ i = 1 \ldots n$, then

$$D^\alpha |z_i(t)| = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{|z_i(t)|'}{(t-s)^\alpha}\,\mathrm{d}s = -\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z_i'(t)}{(t-s)^\alpha}\,\mathrm{d}s = -D^\alpha |z_i(t)|.$$

If $z_i(0) > 0,\ i = 1 \ldots n$, then

$$D^\alpha |z_i(t)| = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{|z_i(t)|'}{(t-s)^\alpha}\,\mathrm{d}s = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z_i'(t)}{(t-s)^\alpha}\,\mathrm{d}s = D^\alpha |z_i(t)|.$$

Hence, we know $D^\alpha |z_i(t)| = \mathrm{sgn}(z_i(t)) D^\alpha z_i(t)$.

When $t = t_k,\ t \geq 0$ and assumption $A(3)$, we obtain

$$\begin{aligned} V(t_k^+) &= \sum_{i=1}^{n} \varphi_i |z_i(t_k^+)| \\ &= \sum_{i=1}^{n} \varphi_i |z_i(t_k) + \Upsilon_{ik}(z_i(t_k))| \\ &= \sum_{i=1}^{n} \varphi_i |z_i(t_k) - \sigma_{ik}(z_i(t_k))| \\ &= \sum_{i=1}^{n} \varphi_i |1 - \sigma_{ik}||z_i(t_k)| \\ &< \sum_{i=1}^{n} \varphi_i |z_i(t_k)| = V(t_k), \quad k = 1, 2, \ldots\ldots \end{aligned}$$

When $t \neq t_k,\ t \geq 0$ for the time derivative of V(t) along the solution of error system (9), we obtain

$$\begin{aligned} D^\alpha V(t) &= \sum_{i=1}^{n} \varphi_i D^\alpha |z_i(t)| \\ &\leq \sum_{i=1}^{n} \varphi_i \operatorname{sgn}(z_i(t)) D^\alpha z_i(t) \\ &= \sum_{i=1}^{n} \varphi_i \operatorname{sgn}(z_i(t)) \Big\{ -(d_i + \kappa_{1,i}) z_i(t) + \sum_{j=1}^{n} a_{ij}(\mu_j(t) - \lambda_j(t)) + \sum_{j=1}^{n} b_{ij}(\mu_j(t - \tau_j) \\ &\quad -\lambda_j(t - \tau_j)) - \kappa_{2,i}\epsilon_i(t) \sum_{j=1}^{n} |z_j(t - \tau_j)| - \kappa_{3,i}\epsilon_i(t) \Big\}. \end{aligned} \tag{17}$$

According to assumption $(A2)$ and the above inequality (17) can be converted to

$$\begin{aligned} D^\alpha V(t) \leq & \sum_{i=1}^{n} \varphi_i \Big[-(d_i + \kappa_{1,i})|z_i(t)| + \sum_{j=1}^{n} |a_{ij}|(p_j|z_j(t)| + q_j) \\ & + \sum_{j=1}^{n} |b_{ij}|(p_j|z_j(t - \tau_j)| + q_j) - \kappa_{2,i} \sum_{j=1}^{n} |z_j(t - \tau_j)| - \kappa_{3,i} \Big] \end{aligned}$$

where $\epsilon_i(t) = \operatorname{sgn}(z_i(t))$ if $z_i(t) \neq 0$, which $\epsilon_i(t)$ can be arbitrary chosen in $[-1, 1]$, if $z_i(t) = 0$.

$$
\begin{aligned}
D^{\alpha}V(t) &= \sum_{i=1}^{n}\varphi_i\Big[-(\underline{d}_i+\kappa_{1,i})|z_i(t)|+\sum_{j=1}^{n}\hat{a}_{ij}(p_j|z_j(t)|+q_j) \\
&\quad +\sum_{j=1}^{n}\hat{b}_{ij}(p_j|z_j(t-\tau_j)|+q_j)-\kappa_{2,i}\sum_{j=1}^{n}|z_j(t-\tau_j)|-\kappa_{3,i}\Big] \\
&= \sum_{i=1}^{n}\varphi_i\Big[-(\underline{d}_i+\kappa_{1,i})|z_i(t)|+\sum_{j=1}^{n}\hat{a}_{ij}(p_j|z_j(t)|)+\sum_{j=1}^{n}\hat{a}_{ij}q_j \\
&\quad +\sum_{j=1}^{n}\hat{b}_{ij}p_j|z_j(t-\tau_j)|+\sum_{j=1}^{n}\hat{b}_{ij}q_j-\kappa_{2,i}\sum_{j=1}^{n}|z_j(t-\tau_j)|-\kappa_{3,i}\Big] \\
&= -\sum_{i=1}^{n}\Big[\varphi_i(\underline{d}_i+\kappa_{1,i})-\sum_{j=1}^{n}\varphi_j\hat{a}_{ji}p_i\Big]|z_i(t)|-\sum_{i=1}^{n}\Big[\varphi_i(\kappa_{2,i}-\sum_{j=1}^{n}\hat{b}_{ij}p_j)\Big]|z_j(t-\tau_j)| \\
&\quad -\sum_{j=1}^{n}\varphi_i\Big[\kappa_{3,i}-\sum_{j=1}^{n}(\hat{a}_{ij}+\hat{b}_{ij})q_j\Big]
\end{aligned}
$$

Combining the Eq.(13) − (15), we obtain

$$
\begin{aligned}
D^{\alpha}V(t) &\le -\sum_{i=1}^{n}\Big[\varphi_i(\underline{d}_i+\kappa_{1,i})-\sum_{j=1}^{n}\varphi_j\hat{a}_{ji}p_i\Big]|z_i(t)| \\
&\le -\Omega\sum_{i=1}^{n}\varphi_i|z_i(t)|,
\end{aligned}
$$

where $\Omega=\min_{1\le i\le n}\Omega_i,\ \Omega_i=(\underline{d}_i+\kappa_{1,i})-\sum_{j=1}^{n}\hat{a}_{ji}p_i$. Therefore, the above inequality can be rewritten as $D^{\alpha}V(t)\le-\Omega V(t)$. There exists a positive $\xi(t)$ satisfying $D^{\alpha}V(t)+\xi(t)=-\Omega V(t)$. Taking Laplace transform on both sides, we get

$$
\begin{aligned}
\mathcal{L}(D^{\alpha}V(t))+L(\xi(t)) &= \mathcal{L}(-\Omega V(t)) \\
s^{\alpha}V(s)-s^{\alpha-1}V(0)+\int_0^{+\infty}e^{-st}\xi(t)dt &= -\Omega\int_0^{+\infty}e^{-st}V(t)dt \\
s^{\alpha}V(s)-s^{\alpha-1}V(0)+\xi(s) &= -\Omega V(s) \\
(s^{\alpha}+\Omega)V(s) &= V(0)s^{\alpha-1}-\xi(s) \\
V(s) &= \frac{V(0)s^{\alpha-1}-\xi(s)}{s^{\alpha}+\Omega}. \qquad (18)
\end{aligned}
$$

Let $\alpha<\tilde{\alpha}<\alpha+1$ and

$$
\tilde{\xi}(s)=L[\xi(s)] = \int_0^{+\infty}e^{-st}\xi(t)dt=\xi(s)+V(0)[s^{\alpha-\tilde{\alpha}}-s^{\alpha-1}]. \qquad (19)
$$

Taking inverse Laplace transform on both sides, we get

$$
\begin{aligned}
\mathcal{L}^{-1}\{\tilde{\xi}(s)\} &= \mathcal{L}^{-1}\Big\{\xi(s)+V(0)[s^{\alpha-\tilde{\alpha}}-s^{\alpha-1}]\Big\} \\
\tilde{\xi}(t) &= \xi(t)+V(0)\Big[\frac{t^{\tilde{\alpha}-\alpha-1}}{\Gamma(\tilde{\alpha}-\alpha)}-\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\Big],
\end{aligned}
$$

By virtue of Eq.(18), it follows that

$$
V(s) = \frac{V(0)s^{\alpha-\tilde{\alpha}}-\tilde{\xi}(s)}{s^{\alpha}+\Omega}.
$$

Taking inverse Laplace transform on both sides, we get

$$\begin{aligned}\mathcal{L}^{-1}\{V(s)\} &= \mathcal{L}^{-1}\Big\{\frac{V(0)s^{\alpha-\tilde{\alpha}} - \tilde{\xi}(s)}{s^{\alpha}+\Omega}\Big\} \\ &= V(0)t^{\tilde{\alpha}-1}E_{\alpha,\tilde{\alpha}}(-\Omega t^{\alpha}) - \tilde{\xi}(t) * [t^{\alpha-1}E_{\alpha,\alpha}(-\Omega t^{\alpha})].\end{aligned}$$

Note that both function $E_{\alpha,\alpha}(-\Omega t^{\alpha})$ and $t^{\alpha-1}$ are non negative.
It follows that $\tilde{\xi}(t) * [t^{\alpha-1}E_{\alpha,\alpha}(-\Omega t^{\alpha})] > 0$. Then there exists a constant $t_1 > 0$ and $\delta > 0$ such that

$$V(t) = V(0)t^{\tilde{\alpha}-1}E_{\alpha,\tilde{\alpha}}(-\Omega t^{\alpha}) \; \forall\, t \geq t_1 \geq 0,$$

and $1 + \alpha - \delta < \tilde{\alpha} < 1 + \alpha$. Hence

$$V(t) \leq V(0)t^{-\zeta}E_{\alpha,1-\zeta}(-\Omega t^{\alpha}) \; \forall\, t \geq t_1 > 0, \tag{20}$$

where $-\zeta = \tilde{\alpha} - 1$. On the other hand, we see that

$$\begin{aligned}V(0) &= \sum_{i=1}^{n} \varphi_i |z_i(0)| \\ &\leq \max_{1\leq i\leq n} \varphi_i \sum_{i=1}^{n} |z_i(0)| = \varphi_{\max} \sum_{i=1}^{n} |z_i(0)| \\ V(t) &= \sum_{i=1}^{n} \varphi_i |z_i(t)| \\ &\geq \min_{1\leq i\leq n} \varphi_i \sum_{i=1}^{n} |z_i(t)| = \varphi_{\min} \sum_{i=1}^{n} |z_i(t)|\end{aligned}$$

The inequality (20) can be rewritten as

$$\begin{aligned}\varphi_{\min} \sum_{i=1}^{n} |z_i(t)| &\leq \varphi_{\max} \sum_{i=1}^{n} |z_i(0)| \Big[t^{-\zeta}E_{\alpha,1-\zeta}(-\Omega t^{\alpha})\Big] \\ \|z(t)\| &\leq \Lambda \|z(0)\| \Big[t^{-\zeta}E_{\alpha,1-\zeta}(-\Omega t^{\alpha})\Big],\end{aligned}$$

where $\Lambda = \frac{\varphi_{\max}}{\varphi_{\min}}$. According to (16), then we obtain

$$\|y(t) - x(t)\| \leq \Lambda \|y(0) - x(0)\| \Big[t^{-\zeta}E_{\alpha,1-\zeta}(-\Omega t^{\alpha})\Big], \; \forall\, t \geq t_1 > 0.$$

Then by using definition 2.5, drive system (2) and response (4) are robust generalized Mittag-Leffler synchronized. This completes the proof of the theorem.

Corollary 3.5 *Under $A(1)$, $A(2)$ and $A(3)$ holds, if there exists a positive scalar $\kappa_{1,i}$, $\kappa_{2,i}$ and $\kappa_{3,i}$ such that*

$$\Big((\underline{d}_i + \kappa_{1,i}) - \sum_{j=1}^{n} \hat{a}_{ji}p_i\Big) > 0, \; \Big(n\kappa_{2,i} - \sum_{j=1}^{n} \hat{b}_{ij}p_j\Big) > 0, \; \Big(\kappa_{3,i} - \sum_{j=1}^{n} (\hat{a}_{ij} + \hat{b}_{ij})q_j\Big) > 0, \; i, j = 1 \ldots n,$$

then the error system (9) is robust generalized Mittag-Leffler synchronization via state delayed feedback controller (5) .

Similarly, we can choose the following feedback discontinuous controller:

$$\theta_i(t) = -\kappa_{1,i} z_i(t) - \kappa_{3,i}\,\mathrm{sgn}(z_i(t)) \tag{21}$$

where $\kappa_{1,i}$ and $\kappa_{3,i}$ are constant gain coefficients. By using the above feedback controller, we have the following result.

Theorem 3.6 *Under assumptions* $A(1)$, $A(2)$ *and* $A(3)$ *if* $Q_1 = \underline{D} + K_1 - E_2$ *is an M-matrix,* $Q_2 = K_3 - F_2$ *is a positive diagonal matrix, then the error system* (9) *is robust generalized Mittag-Leffler synchronization via state feedback controller* (21).

Proof. Assume that the solution of the system (2) and the response system (4) are $x(t) = (x_1(t), \dots, x_n(t))^T$ and $y(t) = (y_1(t), \dots, y_n(t))^T$ with initial values $x(0) = (x_1(0), \dots, x_n(0))^T$ and $y(0) = (y_1(0), \dots, y_n(0))^T$ respectively.
Since $\underline{D} + K_1 - E_2$ is an M-matrix and $K_3 - F_2$ is positive diagonal matrices, there exist positive constants $\varphi_i (i = 1, 2, ..., n)$ by Lemma 2.9 such that

$$\Big(\varphi_i(\underline{d}_i + \kappa_{1,i}) - \sum_{j=1}^{n} \varphi_j(\hat{a}_{ji} + \hat{b}_{ji})p_i\Big) > 0,\ \varphi_i\Big(\kappa_{3,i} - \sum_{j=1}^{n}(\hat{a}_{ij} + \hat{b}_{ij})q_j\Big) > 0,\ i = 1 \dots n,$$

According to the rest of the proof is similar to Theorem 3.4. Hence omitted the theorem.
The following corollary are directly obtained from Theorem 3.6 are still correct.

Corollary 3.7 *Under assumptions* $A(1)$, $A(2)$ *and* $A(3)$, *if there exists a positive scalar* $\kappa_{1,i}$ *and* $\kappa_{3,i}$ *such that*

$$(\underline{d}_i + \kappa_{1,i}) - \sum_{j=1}^{n}(\hat{a}_{ji} + \hat{b}_{ji})p_i > 0,\ \kappa_{3,i} - \sum_{j=1}^{n}(\hat{a}_{ij} + \hat{b}_{ij})q_j > 0,\ i = 1 \dots n,$$

then the error system (9) *is robust generalized Mittag-Leffler synchronization based on the controller* (21).

When the system (2) and (4) has without delays based on the controller (21), we have the following result which can be obtained from Theorem 3.4.

Corollary 3.8 *Under assumptions* $A(1)$, $A(2)$ *and* $A(3)$ *holds if* $R_1 = \underline{D} + K_1 - E_1$ *is an M-matrix,* $R_2 = K_3 - F_3$ *is a positive diagonal matrix, then the error system* (9) *is robust generalized Mittag-Leffler synchronization based on the controller* (21).

Proof. Assume that the solution of drive system (2) and response system (4) are $x(t) = (x_1(t), \dots, x_n(t))^T$ and $y(t) = (y_1(t), \dots, y_n(t))^T$ with initial values $x(0) = (x_1(0), \dots, x_n(0))^T$ and $y(0) = (y_1(0), \dots, y_n(0))^T$, respectively.
Since $\underline{D} + K_1 - E_1$ is an M-matrix and $K_3 - F_3$ is positive diagonal matrices, then there exist positive constants $\varphi_i (i = 1, 2, \dots, n)$ by lemma 2.9 such that

$$\varphi_i\Big((\underline{d}_i + \kappa_{1,i}) - \sum_{j=1}^{n} \hat{a}_{ji}p_i\Big) > 0,\ \varphi_i\Big(\kappa_{3,i} - \sum_{j=1}^{n} \hat{a}_{ij}q_j\Big) > 0.$$

According to the rest of the proof is similar to Theorem 3.4. Hence omitted the corollary.

When a neuron activation are satisfies common Lipschitz continuous function, the assumption $A(2)$ is replaced by the following conditions,
(**J**) For all $x, y \in \mathbb{R}$, suppose there exists a positive scalar $p_i > 0$ such that the following conditions are established:

$$|h_i(x) - h_i(y)| \le p_i|x - y|,\ i = 1, \dots, n.$$

As a special case of theorem 3.4, we have the following synchronization result.

Corollary 3.9 *Suppose* $A(1)$, $A(3)$ *and* (J) *holds, the drive system* (2) *and response system* (4) *is robust generalized Mittag-Leffler synchronization based on the controller* (12) *if the following conditions are satisfied:*

$$\pi_1 = \min\Big[(d_i + \kappa_{1,i}) - \sum_{j=1}^{n} \hat{a}_{ji} p_i\Big] > \max\Big[\sum_{j=1}^{n} \hat{b}_{ji} p_i\Big] = \pi_2 > 0.$$

Proof. The synchronization error system between (2) and (4), we obtain

$$\begin{cases} D^\alpha z_i(t) = -d_i z_i(t) + \sum_{j=1}^{n} a_{ij}[h_j(y_j(t)) - h_j(x_j(t))] \\ \qquad\qquad + \sum_{j=1}^{n} b_{ij}[h_j(y_j(t-\tau_j)) - h_j(x_j(t-\tau_j))] + I_i + \kappa_{1,i}) z_i(t),\ t \neq t_k,\ t \geq 0, \\ \Delta z_i(t_k) = z_i(t_k^+) - z_i(t_k) = \Upsilon_{ik}(z_i(t_k)),\ k = 1, 2, ... \\ z_i(s) = \Theta_i(s),\ s \in [-\tau, 0]. \end{cases}$$

Consider a non-negative function: $V(t) = \sum_{j=1}^{n} \varphi_i |z_i(t)|$

According to proof Theorem 3.4, we can easily obtain $V(t_k^+) \leq V(t_k)$. Based on Assumption J, we get

$$\begin{aligned} D^\alpha V(t) &= \sum_{i=1}^{n} \varphi_i D^\alpha |z_i(t)| \\ &\leq \sum_{i=1}^{n} \varphi_i \Big[-(d_i + \kappa_{1,i})|z_i(t)| + \sum_{j=1}^{n} |a_{ij}| p_j |z_j(t)| + \sum_{j=1}^{n} |b_{ij}| p_j |z_j(t-\tau_j)|\Big] \\ &= -\sum_{i=1}^{n} \varphi_i \Big[(d_i + \kappa_{1,i}) - \sum_{j=1}^{n} \hat{a}_{ji} p_i\Big] |z_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n} \varphi_i \hat{b}_{ji} p_i |z_i(t-\tau_i)| \\ &\leq -\pi_1 V(t) + \pi_2 \sup_{t-\tau \leq m \leq t} V(m) \end{aligned}$$

For any solution of the synchronization error system $z_i(t)$ satisfies the Razumikhin condition, for more details[Ivanka, 2014]. Therefore $V(m) \leq V(t),\ t - \tau \leq m \leq t$. It can immediately follows that $D^\alpha V(t) \leq -[\pi_1 - \pi_2]V(t)$, we can select $\Omega = \min_{1 \leq i \leq n}[\pi_1 - \pi_2] > 0$ such that $D^\alpha V(t) \leq -\Omega V(t)$. The rest of the proof are similar in Theorem 3.4. Hence the corollary is completed.

When $D = \underline{D} = \bar{D}, A = \underline{A} = \bar{A}, B = \underline{B} = \bar{B}$, the system (2) and (4) is without the parameter uncertainties, then we have the following results.

Theorem 3.10 *Suppose* $A(1)$, $A(2)$ *and* $A(3)$, *holds and if* $V_1 = D + K_1 - E_3$ *is an M-matrix,* $V_2 = nK_2 - M_1$ *and* $V_3 = K_3 - M_2$ *are two positive diagonal matrices, then error system* (9) *is a generalized Mittag-Leffler synchronization based on the controller* (5).

The proof of Theorem 3.10 is similar to Theorem 3.4.

The following corollary are directly obtained from Theorem 3.10 still correct.

Corollary 3.11 *Suppose* $A(1)$, $A(2)$ *and* $A(3)$ *holds, if exists a positive scalar* $\kappa_{1,i}$, $\kappa_{2,i}$ *and* $\kappa_{3,i}$ *such that*

$$\Big((d_i + \kappa_{1,i}) - \sum_{j=1}^{n} |a_{ji}| p_i\Big) > 0,\ \Big(n\kappa_{2,i} - \sum_{j=1}^{n} |b_{ij}| p_j\Big) > 0,\ \Big(\kappa_{3,i} - \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|) q_j\Big) > 0,\ i = 1, 2, \ldots, n,$$

then the error system (9) *is generalized Mittag-Leffler synchronization based on the controller* (5).

Theorem 3.12 *Under assumptions* $A(1)$, $A(2)$ *and* $A(3)$ *holds if* $W_1 = D + K_1 - E_4$ *is an M-matrix,* $W_2 = K_3 - M_2$ *is a positive diagonal matrix, then the error system* (9) *is generalized Mittag-Leffler synchronization based on the controller* (21).

Proof. Assume that the solution of the drive system (2) and the response system (4) are $x(t) = (x_1(t), \ldots, x_n(t))^T$ and $y(t) = (y_1(t), \ldots, y_n(t))^T$ with initial values $x(0) = (x_1(0), \ldots, x_n(0))^T$ and $(0) = (y_1(0), \ldots, y_n(0))^T$, respectively. Since $D + K_1 - E_4$ is an M-matrix and $K_3 - M_2$ is positive diagonal matrices, there exist positive constants $\varphi_i (i = 1, 2, \ldots, n)$ by Lemma 2.9 such that

$$\Big(\varphi_i(d_i + \kappa_{1,i}) - \sum_{j=1}^{n} \varphi_j(|(a_{ji}| + |b_{ji}|)p_i\Big) > 0,\ \varphi_i\Big(\kappa_{3,i} - \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)q_j\Big) > 0.$$

According to the rest of the proof is similar to Theorem 3.4. Hence omitted the theorem.

Corollary 3.13 *Under $A(1)$, $A(2)$ and $A(3)$ holds, if there exists a scalar $\kappa_{1,i}$, $\kappa_{2,i}$ and $\kappa_{3,i}$ such that*

$$\Big((d_i + \kappa_{1,i}) - \sum_{j=1}^{n}(|a_{ji}| + |b_{ji}|)p_i\Big) > 0,\ \Big(\kappa_{3,i} - \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)q_j\Big) > 0,\ i, j = 1, 2, \ldots, n,$$

then the error system (9) is generalized Mittag-Leffler synchronization based on the controller (21).

Remark 3.14 *A novel discontinuous feedback controller (5) plays an essential role to realize the generalized mittag Leffler synchronization goal, which contains two different discontinuous terms such as, $\kappa_{2,i}[\mathrm{sgn}(z_i(t))]$ and $\kappa_{2,i}[\mathrm{sgn}(z_i(t))]\sum_{j=1}^{n}|z_j(t-\tau_j)|$. The role of the discontinuous term $\kappa_{2,i}[\mathrm{sgn}(z_i(t-\tau_j))]$ dealing with the uncertain states is different in between the controlled drive-response synchronization goal, while the other term $[\mathrm{sgn}\,\kappa_{2,i}(z_i(t))]\sum_{j=1}^{n}|z_i(t)|$ is reduced due to the influence of time delay considered in the network model. So our proposed controller are more effective, when compared with other continuous feedback controller $\kappa_{1,i}z_i(t)$.*

Remark 3.15 *Suppose the parametric uncertainties term in the system (2) and (4) is ignored, and the activation function are treated to be a continuous one then the obtained main results of Theorem 3.4 is a generalized Mittag-Leffler synchronization. Then by using Remark 2.6 and Lemma 4 in Ref [8], the solution of the drive-response system in (2) and (4) is Mittag-Leffler synchronization. In conclusion, these results has been discussed already by Ivanka in 2014 but by using simple state feedback control techniques. Hence our proposed method shows some novelty over the works in the Refs (Ding* et al.,2016; *Ivanka* et al.,2014; *Zhang* et al.,2016).

4 Numerical Examples

Here four examples are provided to illustrate the results obtained in the previous section.

Example 4.1 *Consider the two dimensional uncertain fractional order impulsive delayed neural network with discontinuous activation as*

$$D^{\alpha}x(t) = -Dx(t) + Ah(x(t)) + Bh(x(t-\tau)) + I,\ t \neq t_k,\ t \geq t_0.$$

where $\alpha = 0.96$ and $\tau = 3$. Let $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^2$, $D \in D_I = [\underline{D}, \bar{D}]$, $A_I = [\underline{A}, \bar{A}]$, $B \in B_I = [\underline{B}, \bar{B}]$, $I = [0 \quad 0]^T$ with

$$\underline{D} = \begin{bmatrix} 0.24 & 0 \\ 0 & 0.17 \end{bmatrix},\quad \bar{D} = \begin{bmatrix} 0.38 & 0 \\ 0 & 0.43 \end{bmatrix},\quad \underline{A} = \begin{bmatrix} 0.71 & 1.43 \\ -1.2 & 0.41 \end{bmatrix},$$
$$\bar{A} = \begin{bmatrix} 1.3 & 1.76 \\ 0.3 & 0.51, \end{bmatrix},\quad \underline{B} = \begin{bmatrix} -0.94 & -0.21 \\ -0.8 & 0.12 \end{bmatrix},\quad \bar{B} = \begin{bmatrix} 0.31 & -0.14 \\ 0.15 & 0.49 \end{bmatrix}$$

and

$$\begin{cases} x_1(t_k^+) = \frac{3x_1(t_k)}{5}, k = 1, 2, \ldots \\ x_2(t_k^+) = \frac{2x_2(t_k)}{5}, k = 1, 2, \ldots \end{cases}$$

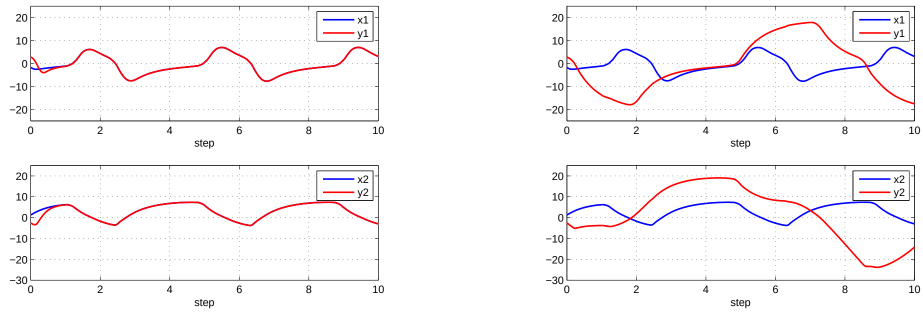

Figure 1: Controlled time evolution of state variables $x(t)$, $y(t)$ and Uncontrolled time evolution of state variables $x(t)$, $y(t)$ in Example 4.1.

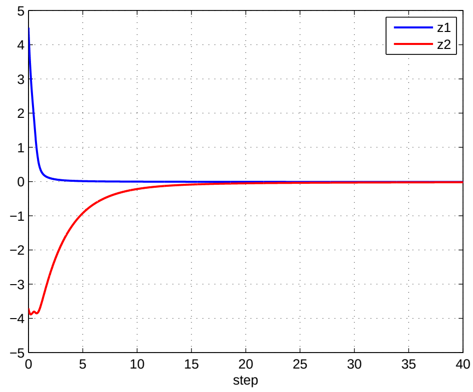

Figure 2: Synchronization error evolution of state variables $z(t)$ in Example 4.1.

Then the activation function are defined as $h(x) = \tanh(x) + 0.05\,\mathrm{sgn}\,x$, *while it is easy to see that* $p_j = 1,\ q_j = 0.5,\ j = 1, 2$ *in assumption A(2). In delayed feedback controller* (5)*, we can select the control gains are* $\kappa_{1,1} = 1.5,\ \kappa_{1,2} = 0.5,\ \kappa_{2,1} = 0.875,\ \kappa_{2,2} = 0.75,\ \kappa_{3,1} = 2.75$ *and* $\kappa_{3,2} = 1.75$*. Then by using assumption A(3) and Theorem 3.4, it easily to checked that* $0 < \sigma_{1k} = \frac{2}{5} < 2,\ 0 < \sigma_{2k} = \frac{3}{5} < 2$,

$$P_1 = \underline{D} + K_1 - E_1 = \begin{bmatrix} 0.44 & -1.76 \\ -1.2 & 0.16 \end{bmatrix},\ P_2 = nK_2 - F_1 = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.21 \end{bmatrix},\ P_3 = K_3 - F_2 = \begin{bmatrix} 0.645 & 0 \\ 0 & 0.25 \end{bmatrix}.$$

Obviously P_1 *is M-matrix,* P_2 *and* P_3 *are two positive diagonal matrix. Therefore, from Theorem 3.4, the system* (2) *and* (4) *are robust generalized Mittag-Leffler synchronization. In Fig.1, it is shown that the drive-response state evolution of* $x(t)$ *and* $y(t)$ *with or without controller by using the initial conditions* $x(0) = (-1.6, 1.2)$ *and* $y(0) = (2.9, -2.5)$*. In Fig. 2, depicts the error state of drive-response synchronization* $z(t)$ *under the delayed feedback controller* (5)*, which displays that the error system converges to zero. Obviously, the synchronization results of Theorem 3.4 are confirmed by this example.*

Example 4.2 *Consider the two dimensional uncertain fractional order impulsive delayed neural net-*

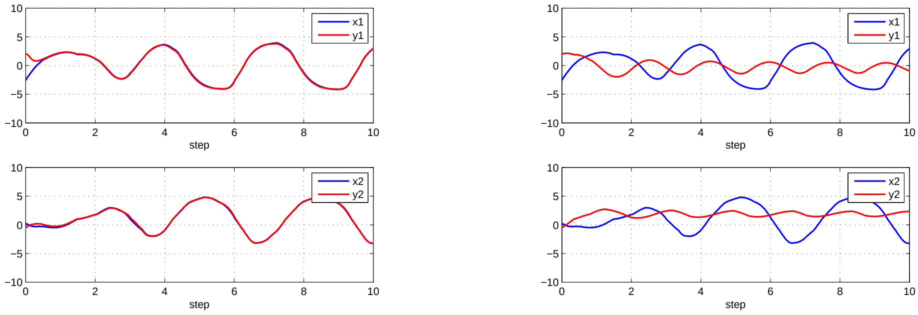

Figure 3: Controlled time evolution of state variables $x(t)$, $y(t)$ and Uncontrolled time evolution of state variables $x(t)$, $y(t)$ in Example 4.2.

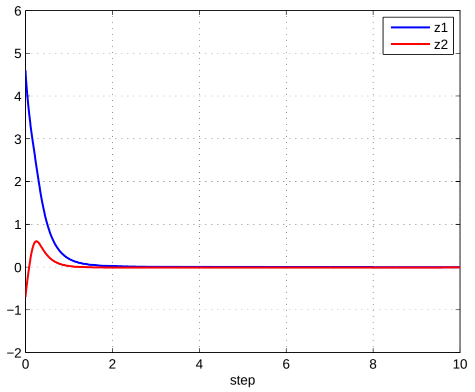

Figure 4: Synchronization error evolution of state variables $z(t)$ in Example 4.2.

work with discontinuous activation

$$D^{\alpha}x(t) = -Dx(t) + Ah(x(t)) + Bh(x(t-\tau)) + I,\ t \neq t_k,\ t \geq t_0.$$

where $\alpha = 0.97$, $\tau = 3$, $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^2$, *while the network parameters* $D \in D_I = [\underline{D}, \bar{D}]$, $A_I = [\underline{A}, \bar{A}]$, $B \in B_I = [\underline{B}, \bar{B}]$ *are given by*

$$\underline{D} = \begin{bmatrix} 0.23 & 0 \\ 0 & 1.17 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 1.28 & 0 \\ 0 & 1.61 \end{bmatrix}, \quad \underline{A} = \begin{bmatrix} 0.91 & 1.4 \\ -2.4 & -1.12 \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} 1.23 & 1.49 \\ 0.2 & 0.9, \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} -1.46 & -0.19 \\ -0.9 & 0.61 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0.9 & -1.11 \\ 1.4 & 1.7 \end{bmatrix}$$

and

$$\begin{cases} x_1(t_k^+) = \frac{x_1(t_k)}{3},\ k = 1, 2, \ldots \\ x_2(t_k^+) = \frac{x_2(t_k)}{4},\ k = 1, 2, \ldots \end{cases}$$

The activation function is chosen as $h(x) = \tanh(x) + 0.01\,\mathrm{sgn}\,x$, *while by using assumption* $A(2)$, *we have properly selected the values as* $p_j = 1$, $q_j = 0.5$, $j = 1, 2$. *In state feedback controller* (21), *we can choose the control gains as* $\kappa_{1,1} = 2.5$, $\kappa_{1,2} = 2.5$, $\kappa_{3,1} = 3.5$ *and* $\kappa_{3,2} = 3.5$. *According to assumption A(3) and Theorem 3.6, it easy to verified that* $0 < \sigma_{1k} = \frac{2}{3} < 2$, $0 < \sigma_{2k} = \frac{3}{4} < 2$,

$$Q_1 = \underline{D} + K_1 - E_2 = \begin{bmatrix} 0.04 & -2.6 \\ -3.8 & 0.85 \end{bmatrix}, \ Q_2 = K_3 - F_2 = \begin{bmatrix} 0.85 & 0 \\ 0 & 0.19 \end{bmatrix}.$$

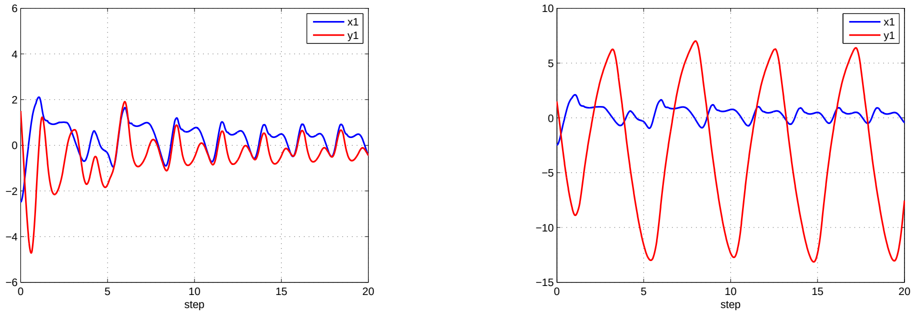

Figure 5: Controlled time evolution of state variables $x_1(t)$, $y_1(t)$ and Uncontrolled time evolution of state variables $x_1(t)$, $y_1(t)$ in Example 4.3.

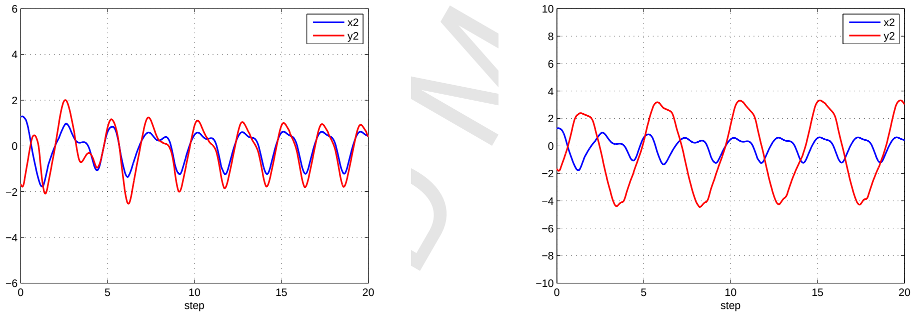

Figure 6: Controlled time evolution of state variables $x_2(t)$, $y_2(t)$ and Uncontrolled time evolution of state variables $x_2(t)$, $y_2(t)$ in Example 4.3.

Obviously, Q_1 *is M-matrix and* Q_2 *is positive diagonal matrix. Therefore, from Theorem 3.6, the system (2) and (4) are robust generalized Mittag-Leffler synchronization. In Fig.3, it is shown that the drive-response state evolution of* $x(t)$ *and* $y(t)$ *with or without controller by using the initial values* $x(0) = (-2.6, 0.2)$ *and* $y(0) = (2, -0.5)$. *In Fig.4, depicts the error state of drive-response synchronization* $z(t)$ *by means of the state feedback controller (21), which displays that the error system converges to zero. Obviously, the synchronization results of Theorem 3.6 are confirmed by simulation.*

Example 4.3 *Consider the three dimensional fractional order impulsive delayed neural network with discontinuous activation*

$$D^{\alpha}x(t) = -Dx(t) + Ah(x(t)) + Bh(x(t-\tau)) + I, \ t \neq t_k, \ t \geq t_0,$$

where $\alpha = 0.97$, $\tau = 3$. *Let* $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3$ *and the network parameters are selected*

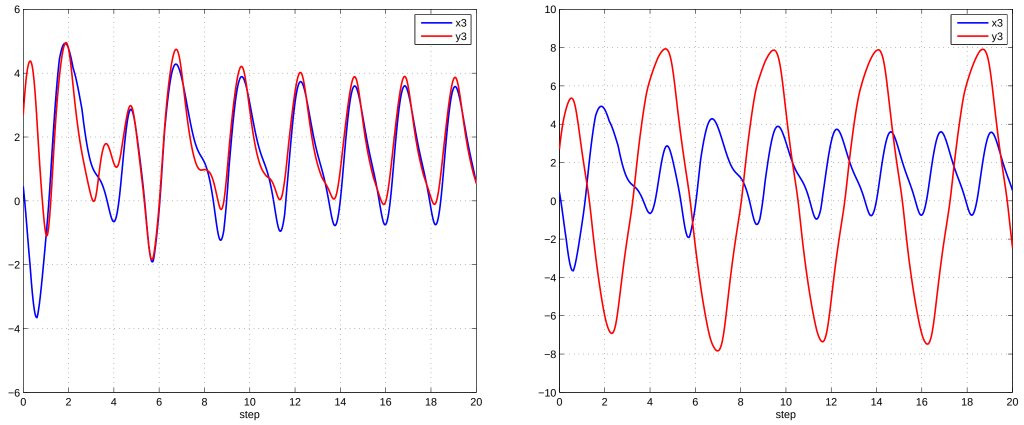

Figure 7: Controlled time evolution of state variables $x_3(t)$, $y_3(t)$ and Uncontrolled time evolution of state variables $x_3(t)$, $y_3(t)$ in Example 4.3.

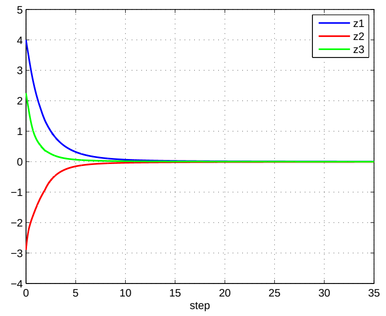

Figure 8: Synchronization error evolution of state variables $z(t)$ in Example 4.3.

as

$$D = \begin{bmatrix} 0.56 & 0 & 0 \\ 0 & 0.75 & 0 \\ 0 & 0 & 0.94 \end{bmatrix}, \quad A = \begin{bmatrix} -0.32 & 0.09 & -0.25 \\ 0.35 & -0.12 & 0.8 \\ 0.23 & -0.47 & 0.65 \end{bmatrix}, \quad B = \begin{bmatrix} 0.94 & -0.28 & -0.55 \\ 0.35 & -0.3 & 0.9 \\ 0.77 & 0.19 & -0.75 \end{bmatrix}$$

and

$$\begin{cases} x_1(t_k^+) = \frac{3x_1(t_k)}{5}, k = 1, 2, \ldots \\ x_2(t_k^+) = \frac{x_2(t_k)}{7}, k = 1, 2, \ldots \\ x_3(t_k^+) = \frac{3x_1(t_k)}{4}, k = 1, 2, \ldots \end{cases}$$

The activation function is given by $h(x) = \tanh(x) + 0.08\,\mathrm{sgn}\,x$*, while it is noted that* $p_j = 0.25,\ q_j = 0.1,\ j = 1, 2, 3$ *in A(2). In delayed feedback controller* (5)*, the control gains are designed as* $\kappa_{1,1} = 0.2,\ \kappa_{1,2} = 0.4,\ \kappa_{1,3} = 0.6,\ \kappa_{2,1} = 0.6,\ \kappa_{2,2} = 0.4,\ \kappa_{2,3} = 0.8,\ \kappa_{3,1} = 0.5,\ \kappa_{3,2} = 0.8,\ \kappa_{3,3} = 0.7$*. By means of assumption A(3) and Theorem* 3.10*, it easily to checked that* $0 < \sigma_{1k} = \frac{2}{5} < 2,\ 0 < \sigma_{2k} =$

$\frac{6}{7} < 2,\ 0 < \sigma_{3k} = \frac{1}{4} < 2,$

$$V_1 = D + K_1 - E_3 = \begin{bmatrix} 0.68 & -0.023 & -0.063 \\ -0.088 & 1.12 & -0.2 \\ -0.058 & -0.118 & 1.377 \end{bmatrix},\ V_2 = nK_2 - M_1 = \begin{bmatrix} 1.357 & 0 & 0 \\ 0 & 0.813 & 0 \\ 0 & 0 & 1.972 \end{bmatrix},$$

$$V_3 = K_3 - F_3 = \begin{bmatrix} 0.257 & 0 & 0 \\ 0 & 0.518 & 0 \\ 0 & 0 & 0.34 \end{bmatrix}$$

Obviously V_1 is M-Matrix, V_2 and V_3 are two positive diagonal matrix. Therefore, from Theorem 3.10, the system (2) and (4) are robust generalized Mittag-Leffler synchronization. In Fig.5-7, it is shown that the drive-response state evolution of $x(t)$ and $y(t)$ with or without controller by using the initial values $x(0) = (-2.5, 1.5, 0.45)$ and $y(0) = (1.5, -1.65, 2.7)$. The error state of the drive-response synchronization $z(t)$ under the delayed feedback controller (5), which displays that the error system converges to zero are presented in Fig.8. Obviously, the synchronization results of Theorem 3.10 are verified via this numerical simulation.

Example 4.4 *Consider the three dimensional fractional order impulsive delayed neural network with discontinuous activation*

$$D^{\alpha}x(t) = -Dx(t) + Ah(x(t)) + Bh(x(t-\tau)) + I, t \neq t_k, t \geq t_0,$$

where $\alpha = 0.96$ and $\tau = 3$. Let $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3$ and the network parameters are selected as

$$D = \begin{bmatrix} 0.75 & 0 & 0 \\ 0 & 0.35 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},\quad A = \begin{bmatrix} 0.3 & -0.19 & -0.28 \\ 0.74 & -1.2 & 0.47 \\ 0.65 & 0.8 & 0.35 \end{bmatrix},\quad B = \begin{bmatrix} -0.69 & -0.14 & 0.3 \\ 0.18 & -0.7 & 0.16 \\ 0.5 & 0.66 & 0.9 \end{bmatrix}$$

and

$$\begin{cases} x_1(t_k^+) = \frac{3x_1(t_k)}{4}, k = 1, 2, \dots \\ x_2(t_k^+) = \frac{3x_2(t_k)}{7}, k = 1, 2, \dots \\ x_3(t_k^+) = \frac{2x_2(t_k)}{5}, k = 1, 2, \dots \end{cases}$$

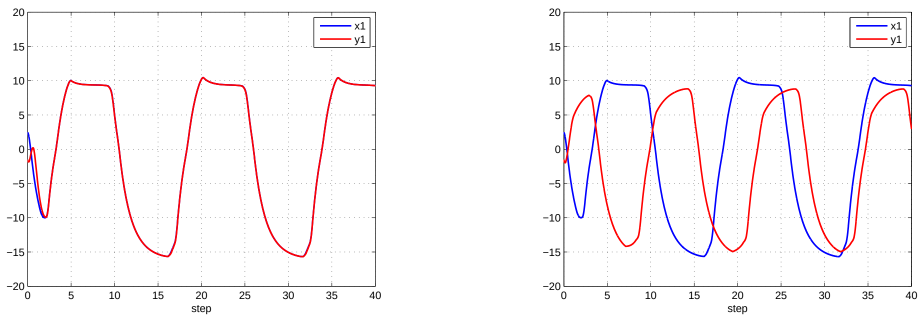

Figure 9: Controlled time evolution of state variables $x_1(t)$, $y_1(t)$ and Uncontrolled time evolution of state variables $x_1(t)$, $y_1(t)$ in Example 4.4.

The activation function is given by $h(x) = \tanh(x) + 0.08\,\mathrm{sgn}\,x$, and by using assumption $A(2)$, it is easy to see that $p_j = 0.2,\ q_j = 0.2,\ j = 1, 2, 3$. In state feedback controller (21), we can properly select

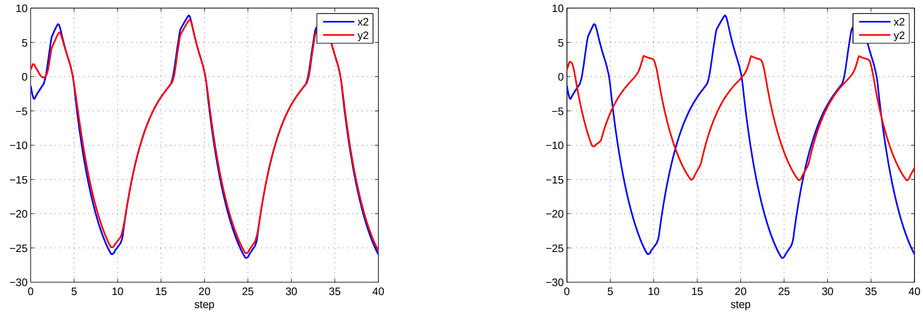

Figure 10: Controlled time evolution of state variables $x_2(t)$, $y_2(t)$ and Uncontrolled time evolution of state variables $x_2(t)$, $y_2(t)$ in Example 4.4.

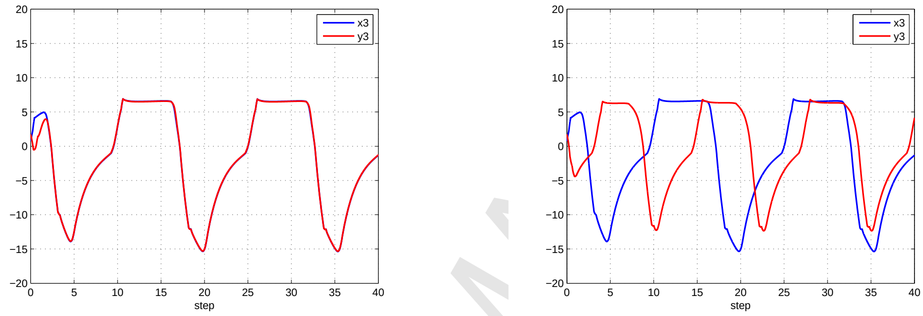

Figure 11: Controlled time evolution of state variables $x_3(t)$, $y_3(t)$ and Uncontrolled time evolution of state variables $x_3(t)$, $y_3(t)$ in Example 4.4.

the gains are $\kappa_{1,1} = 2.5$, $\kappa_{1,2} = 2.5$, $\kappa_{1,3} = 2.5$, $\kappa_{3,1} = 2.5$, $\kappa_{3,2} = 2.5$, $\kappa_{3,3} = 2.5$. *According to assumption A(3) and Theorem 3.12, it easily to checked that* $0 < \sigma_{1k} = \frac{1}{4} < 2$, $0 < \sigma_{2k} = \frac{4}{7} < 2$, $0 < \sigma_{3k} = \frac{3}{5} < 2$,

$$W_1 = D + K_1 - E_4 = \begin{bmatrix} 3.052 & -0.66 & -0.116 \\ -0.184 & 2.47 & -0.126 \\ -0.23 & -0.292 & 2.75 \end{bmatrix}, \; W_2 = K_3 - M_2 = \begin{bmatrix} 1.12 & 0 & 0 \\ 0 & 1.81 & 0 \\ 0 & 0 & 1.728 \end{bmatrix}.$$

Obviously, W_1 *is M-matrix and* W_2 *is positive diagonal matrix. Therefore, the system* (2) *and* (4) *are generalized Mittag-Leffler synchronization based on the state feedback controller* (21)*. Therefore, according to Theorem 3.12, the system* (2) *and* (4) *are robust generalized Mittag-Leffler synchronization. In Fig.9-11, it is shown that the drive-response state evolution of* $x(t)$ *and* $y(t)$ *with or without controller by using the initial values* $x(0) = (2.5, -1.3, 0.75)$ *and* $y(0) = (-1.5, 1, -1.7)$*. The error state of controlled drive-response synchronization* $z(t)$ *based on the feedback controller* (21) *are presented in Fig.12. Obviously, the synchronization results of Theorem 3.12 are verified via this numerical example.*

5 Conclusions

The concepts of Filippov solution and synchronization conditions for fractional order uncertain neural networks with discontinuous activation functions have been investigated. By means of the growth conditions, differential inclusions and the generalized Gronwall inequality, a sufficient condition for

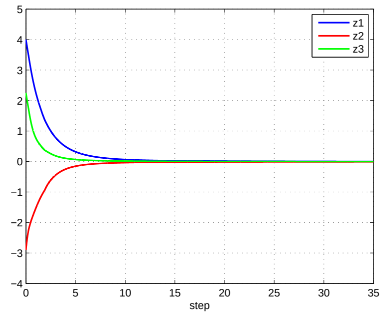

Figure 12: Synchronization error evolution of state variables $z(t)$ in Example 4.4.

the existence of Filippov solution was obtained. Moreover, sufficient criteria were given to the robust generalized Mittag-Leffler synchronization between discontinuous activation function of impulsive fractional order neural network systems with (or without) parameter uncertainties. Finally, four numerical simulations are presented to demonstrate the effectiveness of our synchronization results.

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