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This is a revised submission. Many thanks for your consideration.

Stability and pinning synchronization analysis of fractional order delayed Cohen-Grossberg neural networks with discontinuous activations

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Abstract

This article, we explore the asymptotic stability and asymptotic synchronization analysis of fractional order delayed Cohen-Grossberg neural networks with discontinuous neuron activation functions (FCGNNDDs). First, under the framework of Filippov theory and differential inclusion theoretical analysis, the global existence of Filippov solution for FCGNNDDs is studied by means of the given growth condition. Second, by virtue of suitable Lyapunov functional, Young inequality and comparison theorem for fractional order delayed linear system, some global asymptotic stability conditions for such system is derived by limiting discontinuous neuron activations. Third, the global asymptotic synchronization condition for FCGNNDDs is obtained based on the pinning control. At last, two numerical simulations are given to verify the theoretical findings.

Keywords. *Asymptotic stability; Asymptotic Synchronization; Fractional order systems; Time-delay; Filippov's solutions; Pinning control.*

1 Introduction

Around three hundred years back, the origin of fractional order calculus was first off mentioned by Leibniz and L'Hospital and its realistic applications have been developed very slow for a long time [39]. Until recently, it has been a great research topic due to the fact many fractional order models play a crucial role in many real world objects. Comparing to an integer order dynamical system, fractional order dynamical system is more accuracy, non-local and has weakly singular kernels. As a result, fractional order calculus has been bought into various disciplines, especially modeling such as epidemic models [2], financial model [9], market dynamics [24], artificial neural networks [27], dielectric polarization [32] and so on. We realize that the next state of a system not only depends upon its current state but also upon its historical information. Since a model derived from the fractional-order equations possesses memory, it is precise to describe the states of the neurons. In the truth, the fractional calculus consolidated into the artificial neural network system can all the more likely present the dynamical attributes. In this manner, the investigation on the fractional order neural network (FNNs) dynamical behaviors in both theory and applications has turned out to be urgent and mandatory.

On the other hand, the neural networks which have promising potential for applications in pattern

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recognition, automated control, and associative memory have obtained important interest among researchers over the most recent three decades. In nature, there are numerous kinds of neural networks: recurrent networks, competitive networks, bidirectional associative memory (BAM) networks, cellular networks, Cohen-Grossberg networks and Hopfield networks which might be all computational models motivated by biological neural networks. In a digital implementation of biological neural network dynamical behaviors, the time delays unavoidably appear in the process of transmission signals and information storage for the motive that the finite switching speed of the amplifiers, see Refs [10, 34]. Consequently, the stability and synchronization of fractional-order delayed neural network dynamical behaviours have been taken into consideration by means of many research scholars and a large number of great outcomes has been gained in the existing literature [5, 21, 26, 30, 50, 51] for instance.

Cohen-Grossberg type neural network, as special case of Hopfield neural networks, was firstly coined by Cohen and Grossberg in 1983 [11]. Recently, fractional order Cohen-Grossberg neural network (FCGNNs) has attracted considerable attention owing to its widespread applications in various disciplines in pattern recognition, parallel computing, and many different areas and extensively investigated by many researchers, see [35, 36, 48, 49] for instance. Basically, the neuron activations are not continuous due to the fact signal transmission among neurons and signal outcomes are all discontinuous. In [20], it's miles discovered that neural network system with discontinuous activation functions is a more perfect model. Lamentably, the most of the previous literature focused on FNNs with continuous neuron activation functions, see [5, 31, 44, 45] for instance. However, up to now, there is little attention for synchronization analysis of FNNs with discontinuous activations have been investigated see Refs [13, 14, 15] via state feedback control, adaptive feedback control and impulsive control. Different from those control techniques, pinning control technique is more ideal because it has been applied to one neuron or the huge number of neurons instead of all neurons. To the authors knowledge, however, so far the stability and pinning synchronization problem for FCGNNs with discontinuous activations has not been tackled.

With the inspirations outlined as over, our main aim in this work is to investigate the global asymptotic stability and pinning synchronization analysis of general class of fractional order discontinuous Cohen-Grossberg neural networks with time delays via comparison theorem for the fractional order linear delayed system. The designed pinning control strategy of this paper is totally different from those in the existing synchronization results of fractional order and integer order Cohen-Grossberg neural networks. The main contributions of this paper can be highlighted in the following aspects.

1. By means of non-smooth analysis and the framework of differential inclusion theory, the global existence of a solution in the Filippov sense is established for addressed FCGNNs with time delays and discontinuous neuron activations.

2. Based on the comparison theorem for the fractional order linear delayed system, suitable fractional order Lyapunov function and 2-norm method, some sufficient condition for global asymptotic stability of FCGNNs with discontinuous neuron activations are introduced.

3. A novel pinning controller is designed to guarantee the asymptotic synchronization criteria for FCGNNs with discontinuous neuron activations.

4. In most of the FNNs in the available literature the existence of pinning control strategy and discontinuous neuron activations have not taken into consideration simultaneously. This shows the novelty of our proposed result.

Notations. In this work, $\mathbb{N}$ represents the space of natural numbers from 1 to n, $\mathbb{R}^m$ represents the space of m-D Euclidean space, respectively, and $\mathbb{R}^{m\times m}$ stands for a set of all $m \times m$ real matrices. For $p = (p_1, ..., p_m)^T \in \mathbb{R}^m$, $\|p\|_2$ is the 2-norm, which is denoted by $\|p\|_2 = \sqrt{p_1^2 + ... + p_m^2}$.

2 Preliminaries

In this section, we will recall the basic definition's and some properties concerning fractional order derivative are presented, while we introduce model formulation, assumptions and some important lemmas. Moreover, we also present the existence of solution in Filippov sense.

Definition 2.1 *[22, 29] The Riemann-Liouville fractional integral order $0 < \mu < 1$ for a function $q(t)$ is defined as*

$$I^{\mu}q(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} q(\kappa)\,\mathrm{d}\kappa,$$

where the Euler's Gamma function is denoted by $\Gamma(\mu) = \int_0^{+\infty} \exp\{-t\} t^{\mu-1} \mathrm{d}t$.

Definition 2.2 *[22, 29] The Caputo fractional-order derivative with order μ for a differential function $q(t)$ is defined as follows:*

$$D^{\mu}q(t) = \frac{1}{\Gamma(m-\mu)} \int_0^t \frac{q^{(m)}(\kappa)}{(t-\kappa)^{\mu-m+1}}\,\mathrm{d}\kappa,$$

where $t \geq 0$ and $m-1 < \mu < m \in \mathbb{Z}^+$. Especially, when $0 < \mu < 1$,

$$D^{\mu}q(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t \frac{q'(\kappa)}{(t-\kappa)^{\mu}}\,\mathrm{d}\kappa.$$

Furthermore, the following necessary properties about Caputo fractional-order derivative are provided.
Property 1 : For $m-1 < \mu < m$, we have

$$I^{\mu}\big[D^{\mu}q(t)\big] = q(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} q^{(k)}(0), \ \mu \geq 0.$$

Especially, $0 < \mu < 1$, one has

$$I^{\mu}\big[D^{\mu}q(t)\big] = q(t) - q(0).$$

Property 2 : For any arbitrary constants v_1 and v_2, then

$$D^{\mu}\Big[v_1 q_1(t) + v_2 q_2(t)\Big] = v_1 D^{\mu} q_1(t) + v_2 D^{\mu} q_2(t).$$

Definition 2.3 *[22, 29] The two parameters Mittag-Leffler function with $\mu > 0, \bar{\mu} > 0$ has expressed in the following form:*

$$\mathbb{E}_{\mu,\bar{\mu}}(\tau) = \sum_{l=0}^{+\infty} \frac{\tau^l}{\Gamma(\mu l + \bar{\mu})}.$$

For $\bar{\mu} = 1$, its Mittag-Leffler with one parameter function is shown as

$$\mathbb{E}_{\mu}(\tau) = \sum_{l=0}^{+\infty} \frac{\tau^l}{\Gamma(\mu l + 1)} = \mathbb{E}_{\mu,1}(\tau).$$

Lemma 2.4 *[1] Let $p(t) \in \mathbb{R}^m$ be a continuous and differentiable function. Then the following relationship holds*

$$D^{\mu}p^2(t) \quad \leq \quad 2p(t)D^{\mu}p(t), \ \forall\ 0 < \mu < 1, \ t \geq 0.$$

Moreover, when $p(t) \in \mathbb{R}^m$, it holds that $D^{\mu}p^T(t)p(t) \leq 2p^T(t)D^{\mu}p(t)$, $\forall\ 0 < \mu < 1$, $t \geq 0$, or the equivalent inequality $D^{\mu}\sum_{l=1}^{m} p_l^2(t) \leq 2\sum_{l=1}^{m} |p_l(t)| D^{\mu}|p_l(t)|$ holds for all $0 < \mu < 1$, $t \geq 0$.

Lemma 2.5 *[7] For any vectors* $p, q \in \mathbb{R}^m$ *and a positive definite matrix* $L \in \mathbb{R}^{m\times m}$*, then*

$$2p^T q \leq p^T Lp + q^T L^{-1} q.$$

Lemma 2.6 *[23] Let* $q_1 > 0,\ q_2 > 0,\ q_3 > 1,\ q_4 > 1$ *and* $q_3^{-1} + q_4^{-1} = 1$*, then the following relationship holds:*

$$q_1 q_2 \leq \frac{(\varepsilon q_1)^{q_3}}{q_3} + \frac{(\varepsilon^{-1} q_2)^{q_4}}{q_4}$$

where ε *is any positive real number.*

Lemma 2.7 *[37] (Fractional Halanay inequality) If the continuous function* $v(t) > 0,\ t \in \mathbb{R}$*, and*

$$\begin{cases} D^\mu v(t) \leq \delta_0 + \delta_3 v(t) + \delta_4 \sup_{t-\lambda(t)\leq\kappa\leq t} v(\kappa),\ t \geq 0, \\ v(t) = |\varrho(t)|,\ t \leq 0,\ 0 < \mu < 1, \end{cases}$$

where $\varrho(t)$ *is a bounded and continuous function, the coefficients* $\delta_0,\ \delta_3,\ \delta_4$ *satisfy that* $\delta_0,\ \delta_4 \geq 0,\ \delta_3 < 0$*, and* $-\tau \leq t - \lambda(t) \leq t$*. Let* $E_0 = \sup_{-\tau\leq\kappa\leq 0}\{|\varrho(\kappa)|\}$ *and* $v_0 = |\varrho(0)|$*. If* $\delta_3 + \delta_4 < 0$*, we have*

$$v(t) \leq E_0 + \frac{\delta_3}{\delta_3 + \delta_4} v_0 - \frac{\delta_0}{\delta_3 + \delta_4}.$$

In addition to that, $\lim_{t\to+\infty}(t - \lambda(t)) = +\infty$*, then for any given* $\theta > 0$*, there exists* $t^\star = t^\star(E_0, \theta) > 0$ *such that*

$$v(t) \leq -\frac{\delta_0}{\delta_3 + \delta_4} + \theta,\ t \geq t^\star.$$

Lemma 2.8 *[38] Consider the following delayed fractional order differential inequality*

$$\begin{cases} D^\mu p(t) \leq -\beta p(t) + \gamma p(t - \lambda),\ t > 0,\ 0 < \mu \leq 1, \\ p(\kappa) = \varrho(\kappa),\ \kappa \in [-\lambda, 0], \end{cases}$$

and delayed fractional order linear system

$$\begin{cases} D^\mu \rho(t) = -\beta \rho(t) + \gamma \rho(t - \lambda),\ t > 0,\ 0 < \mu \leq 1, \\ \rho(\kappa) = \varrho(\kappa),\ \kappa \in [-\lambda, 0], \end{cases}$$

where $p(t)$ *and* $\rho(t)$ *are continuous and non negative in* $[0, +\infty)$*, and* $\varrho(t) > 0,\ t \in [-\lambda, 0]$*. If* $\beta,\ \gamma > 0$*, then* $p(t) \leq \rho(t)$ *for all* $t \in [0, +\infty]$*.*

Lemma 2.9 *[38] If* $0 < \mu < 1$*, all the eigenvalues* λ^*s *of* $S = -\beta + \gamma$ *satisfy* $arg\big(\lambda^*\big) > \frac{\pi}{2}$ *and the characteristic equation* $det\big(\Delta(t)\big) = s^\mu + \beta - \gamma e^{-s\lambda} = 0$ *has no pure imaginary roots for any* $\lambda > 0$*, then the equilibrium point of system*

$$D^\mu p(t) = -\beta p(t) + \gamma p(t - \lambda)$$

is Lyapunov globally asymptotically stable.

Lemma 2.10 *[43] Suppose the locally integrable and non negative function $E(\kappa)$ on $\kappa \in [0,T)$, $T \leq +\infty$ and the nondecreasing continuous function $F(\kappa) \leq \mathcal{K}$ defined on $\kappa \in [0,T)$, where $\mathcal{K} > 0$ is positive scalar. If non-negative locally integrable function, $J(\kappa)$ satisfies*

$$J(\kappa) \leq E(\kappa) + F(\kappa)\int_0^t (\kappa - t)^{\beta-1} J(t)\mathrm{d}t$$

on $\kappa \in [0,T)$, we have

$$J(\kappa) \leq E(\kappa)\mathbb{E}_\mu\Big[F(\kappa)\Gamma(\beta)\kappa^\beta\Big]$$

where $\beta > 0$ is a positive constant, $\mathbb{E}_\mu$ is a Mittag-Leffler one parameter function and $\Gamma(\cdot)$ is a Gamma function.

Lemma 2.11 *[47] Let $v(t) \in C^1\big([0,+\infty),\mathbb{R}\big)$ be a continuous and differentiable function, the following inequality satisfies almost everywhere.*

$$D^\mu|v(t)| \leq sgn(v(t))D^\mu v(t),\ 0 < \mu < 1.$$

3 System formulation and existence of solutions in Filippov sense

In this manuscript, we consider a general class of fractional order Cohen-Grossberg neural networks with delays and discontinuous neuron activation function (FCGNNDDs) described by the following expression:

$$\begin{cases} D^\mu p_l(t) = -a_l\big(p_l(t)\big)\Bigg[b_l\big(p_l(t)\big) - \sum_{j=1}^m c_{lj}h_j(p_j(t)) - \sum_{j=1}^m d_{lj}h_j(p_j(t-\lambda)) - k_l\Bigg], \\ p_l(\kappa) = \varrho_l(\kappa),\ \kappa \in [-\lambda, 0]. \end{cases} \tag{1}$$

for $l = 1, 2, .., m$, where m denotes the number of neurons in a network; $p_l(t)$ denote the state variable of l^{th} neuron at time t; $a_l(\cdot)$ is an amplification function and $b_l(\cdot)$ means well behaved function; c_{lj} and d_{lj} represents the synaptic connection strengths at time t and $t-\lambda$, respectively; λ is the constant time delay; $h_j(q_j)$ is the neuron activations; k_l is the constant external input.

Denote $p(t) = \big(p_1(t), ..., p_m(t)\big)^T \in \mathbb{R}^m$, $A(p(t)) = diag\big(a_1(p_1(t)), ..., a_m(p_m(t))\big)$, $B(p(t)) = \big(b_1(p_1(t)), ..., b_m(p_m(t))\big)^T$, $C = (c_{lj})_{m\times m}$, $D = (d_{lj})_{m\times m}$, $h(p(t)) = \big(h_1(p_1(t)), ..., h_m(p_m(t))\big)^T$, and $K = (k_1, ..., k_m)^T$, the vector form of Eq.(1) can be expressed with the following form:

$$\begin{cases} D^\mu p(t) = -A(p(t))\big[B(p(t)) - Ch(p(t)) - Dh(p(t-\lambda)) - K\big], \\ p(\kappa) = \varrho(\kappa),\ \kappa \in [-\lambda, 0]. \end{cases} \tag{2}$$

where $\varrho(\kappa) = \big(\varrho_1(\kappa), ..., \varrho_m(\kappa)\big)^T \in C\Big([-\lambda, 0],\ \mathbb{R}^m\Big)$ is the initial condition and the norm is defined by:

$$\|\varrho\| = \sup_{t\in[-\lambda,0]} \|\varrho(t)\|.$$

In this manuscript, the neuron activations is assumed to the sense of discontinuity form. As a result, the traditional solution for fractional order differential equations does not suitable to FCGNNDDs system (1). In this case, we need to study the concept of Filippov solutions [1, 2] of considering the fractional order discontinuous right-hand side system.

Now, we define the Fillipov set-valued map analysis [19] of $f(l)$ at $l \in \mathbb{R}^m$ as follows:

Definition 3.1 (**Filippov Regularization**). *We consider the fractional order differential system as follows:*

$$\begin{cases} D^{\mu}\phi(t) = f(t,\phi),\ t > 0, \\ \phi(0) = \phi_0,\ \phi \in \mathbb{R}^m, \end{cases} \tag{3}$$

where $f(t,\phi)$ is discontinuous in ϕ. The Fillipov set-valued map $\mathcal{H} : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ is defined as:

$$\mathcal{H}(t,\phi) = \bigcap_{\varepsilon>0} \bigcap_{\sigma(\mathcal{J})=0} \overline{\text{co}} \left[f(t, \mathcal{B}(\phi,\varepsilon) \backslash \mathcal{J}) \right]$$

where $\mathcal{B}(\phi,\varepsilon) = \{\breve{\phi}; \|\breve{\phi} - \phi\| \le \varepsilon\}$, $\mathcal{J} \subseteq \mathbb{R}^m$ and $\sigma(\mathcal{J})$ represents the Lebesgue measure of set $\mathcal{J}$. A vector function $\phi(t)$ defined on $I \subseteq \mathbb{R}$ is said to be a Filippov solution of system (2)*, if it is absolutely continuous on any subinterval a non degenerate interval $[t_1, t_2]$ of I, for a.a. $t \in I$, $\phi(t)$ satisfies the differential inclusion: $D^{\mu}\phi(t) \in \mathcal{H}(t,\phi)$*

Via the above differential inclusion analysis, the FCGNNDDs system (1) can be written by the following form:

$$\begin{cases} D^{\mu}p_l(t) \in -a_l\big(p_l(t)\big)\Big[b_l\big(p_l(t)\big) - \sum_{j=1}^m c_{lj}\mathcal{H}\big[h_j(p_j(t))\big] - \sum_{j=1}^m d_{lj}\mathcal{H}\big[h_j(p_j(t-\lambda))\big] - k_l\Big], \\ p(\kappa) = \varrho(\kappa),\ \kappa \in [-\lambda, 0]. \end{cases} \tag{4}$$

for a.a $t \ge 0$, $l = 1,2,...,m$, where $\mathcal{H}\big[h(q)\big] = \Big(\mathcal{H}\big[h_1(q_1)\big], ..., \mathcal{H}\big[h_m(q_m)\big]\Big)^T$, $\mathcal{H}\big[h_j(q_j)\big] = \Big[\min\{h_j(q_j^-),\ h_j(q_j^+)\},\ \max\{h_j(q_j^-),\ h_j(q_j^+)\}\Big]$.

From the aforementioned discussion, we define the solution of initial value problem of FCGNNDDs system (1) as below:

Definition 3.2 (**Initial Value Problem**) (**IVP**). *For any continuous function $\varrho_j : [-\lambda, 0] \to \mathbb{R}^m$ and measureable selection $\varpi_j : [-\lambda, 0] \to \mathbb{R}^m$, such that $\varpi_j(\kappa) \in \mathcal{H}\big[h_j(\varrho_j(\kappa))\big]$ for a.a $\kappa \in [-\lambda, 0]$ by an IVP corresponding to FCGNNDDs system* (1) *with initial values (ϱ_j, ϖ_j). Suppose, there is a functions $[p_j(t), \delta_j(t)] : [-\lambda, T) \to \mathbb{R}^m \times \mathbb{R}^m$, such that p is an output solution of FCGNNDDs system* (1)*on $[-\lambda, T)$ for some $T > 0$, and*

$$\begin{cases} D^{\mu}p_l(t) = -a_l\big(p_l(t)\big)\Big[b_l\big(p_l(t)\big) - \sum_{j=1}^m c_{lj}\delta_j(t) - \sum_{j=1}^m d_{lj}\delta_j(t-\lambda) - k_l\Big], \\ \delta_j(t) \in \mathcal{H}\big[h_j(p_j(t))\big],\ \text{for a.a}\ t \in [0,T) \\ p_l(\kappa) = \varrho_l(\kappa),\ \forall\ \kappa \in [-\lambda, 0], \\ \delta_l(\kappa) = \varpi_l(\kappa),\ \text{for a.a}\ \kappa \in [-\lambda, 0]. \end{cases} \tag{5}$$

Definition 3.3 *A constant vector $p^* = (p_1^*, ..., p_m^*)^T$ is said to be an equilibrium point FCGNNDDs* (1) *in the Filippov's sense, such that*

$$0 \in -a_l\big(p_l^*\big)\left[b_l\big(p_l^*\big) - \sum_{j=1}^m c_{lj}\mathcal{H}\big[h_j(p_j^*)\big] - \sum_{j=1}^m d_{lj}\mathcal{H}\big[h_j(p_j^*)\big] - k_l\right].$$

Or equivalently, there exist $\delta_l^* \in \mathcal{H}\big[h_j(p_j^*)\big]$, such that

$$0 = -a_l\big(p_l^*\big)\left[b_l\big(p_l^*\big) - \sum_{j=1}^m c_{lj}\delta_l^* - \sum_{j=1}^m d_{lj}\delta_l^* - k_l\right].$$

In order to establish our stability and synchronization results, we introduce the following assumptions.

Assumption 1. For every $l = 1, 2, ..., m$, there exist a non-negative constants $\underline{a}_l$, $\overline{a}_l$, $\underline{b}_l$, $\overline{b}_l$ and z_l, the amplification function $a_l(\cdot)$ and the behaviour function $b_l(\cdot)$ are continuous functions, which satisfying the following relationship:

$$\underline{a}_l \le a_l(p_l) \le \overline{a}_l,\ \frac{a_l(q_l) - a_l(p_l)}{q_l - p_l} \ge z_l,\ \underline{b}_l \le \frac{b_l(q_l) - b_l(p_l)}{q_l - p_l} \le \overline{b}_l,\ \forall\, p_l, q_l \in \mathbb{R},\ p_l \ne q_l.$$

Assumption 2. For every $l = 1, 2, ..., m$, there exist a non-negative constant ϕ_l, such that

$$\frac{a_l(q_l)b_l(q_l) - a_l(p_l)b_l(p_l)}{q_l - p_l} \ge \phi_l,\ \forall\, p_l, q_l \in \mathbb{R}.$$

Next, we provide the assumptions for the discontinuous neuron activations in system (1).

Assumption 3. For $j = 1, 2, .., m$, the neuron activation function h_j is bounded $\big(|h_j(\cdot)| \le \varsigma_j\big)$ and continuous function except for a finite number of jump discontinuities χ_k^j in every bounded interval. In addition, there exist right limit $h_j^+(\chi_k^j)$ and left limit $h_j^-(\chi_k^j)$, respectively.

Assumption 4. For each $j = 1, 2, .., m$, suppose $\mathcal{H}$ satisfies a growth condition, then there exist positive constant $\tilde{u}_j$ and $\tilde{w}_j$ such that

$$\big|\mathcal{H}[h_j(p_j(t))]\big| \;==\; \sup_{\vartheta \in \mathcal{H}[h_j(p_j(t))]} |\vartheta| \le \tilde{u}_j \big|p_j(t)\big| + \tilde{w}_j.$$

Assumption 5. For every $j = 1, 2, ..., m$, there exist a non-negative constants u_j and w_j such that

$$\big|\theta_j(t) - \delta_j(t)\big| \;\le\; u_j\big|q_j(t) - p_j(t)\big| + w_j,\ \forall\, p_j, q_j \in \mathbb{R},$$

where $\delta_j(t) \in \mathcal{H}\big[h_j(p_j(t))\big]$ and $\theta_j(t) \in \mathcal{H}\big[h_j(q_j(t))\big]$.

Theorem 3.4 *Under the assumptions (1), assumption (3) and assumption (4), then there exist at least one solution $p(t)$ of FCGNNDDs* (1) *on* $[0, +\infty)$ *in the sense of Eq.*(5).

Proof. If $p(t) \hookrightarrow -A(p(t))\big[B(p(t)) - C\mathcal{H}(p(t)) - D\mathcal{H}(p(t-\lambda)) - K\big]$ is upper semi-continuous with bounded nonempty closed convex value, the local existence of solution $(p(t))$ with initial values (ϱ, ϖ) of Eq.(5) can be guaranteed. From Assumption (1) and Assumption (4), one has

$$\begin{aligned}
\|p(t)\| \;&\le\; \|p(0)\| + \left\| \frac{1}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} \Big[-A(p(\kappa))\big[B(p(\kappa)) - C\mathcal{H}(p(\kappa)) - D\mathcal{H}(p(\kappa-\lambda) - K)\big]\Big] \,\mathrm{d}\,\kappa \right\| \\
&\le\; \|p(0)\| + \frac{1}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} \|\overline{A}\|\|\overline{B}\|p(\kappa)\| \,\mathrm{d}\,\kappa + \frac{1}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} \|\overline{A}\|\|C\| \Big[F\|p(\kappa)\| + R\Big] \,\mathrm{d}\,\kappa \\
&\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} \|\overline{A}\|\|D\| \Big[F\|p(\kappa-\lambda)\| + R\Big] \,\mathrm{d}\,\kappa + \frac{1}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} \|\overline{A}\|\|K\| \,\mathrm{d}\,\kappa \\
&\le\; \|p(0)\| + \frac{1}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} \|\overline{A}\|\|\overline{B}\|\|p(\kappa)\| \,\mathrm{d}\,\kappa + \frac{1}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} \|\overline{A}\|\|C\| \Big[F\|p(\kappa)\| + R\Big] \,\mathrm{d}\,\kappa \\
&\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} \|\overline{A}\|\|D\| \Big[F\Big(\|p(\kappa)\| + \|p(0)\|\Big) + R\Big] \,\mathrm{d}\,\kappa + \frac{1}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} \|\overline{A}\|\|K\| \,\mathrm{d}\,\kappa \\
&=\; \Big[1 + \frac{\|\overline{A}\|\|D\|F}{\Gamma(\mu+1)} t^{\mu}\Big] \|p(0)\| + \frac{1}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} \Big[\|\overline{A}\|\big[\|C\| + \|D\|\big]R + \|\overline{A}\|\|K\|\Big] \,\mathrm{d}\,\kappa \\
&\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} \Big[\|\overline{A}\|\|\overline{B}\| + \|\overline{A}\|\big[\|C\| + \|D\|\big]F\Big] \|p(\kappa)\| \,\mathrm{d}\,\kappa \\
&=\; \mathcal{M}(t) + \frac{\mathcal{N}(t)}{\Gamma(\mu)} \int_0^t (t-\kappa)^{\mu-1} \|p(\kappa)\| \,\mathrm{d}\,\kappa
\end{aligned} \tag{6}$$

where

$$\mathcal{M}(t) = \Big[1+\frac{t^{\mu}}{\Gamma(\mu+1)}\|\overline{A}\|\|D\|F\Big]\|p(0)\| + \frac{t^{\mu}}{\Gamma(\mu+1)}\Big[\|\overline{A}\|\big[\|C\|+\|D\|\big]R + \|\overline{A}\|\|K\|\Big]$$
$$\mathcal{N}(t) = \Big[\|\overline{A}\|\|\overline{B}\| + \|\overline{A}\|\big[\|C\|+\|D\|\big]F\Big],$$

$F=\max\{\tilde{u}_1,...,\tilde{u}_m\}$, $R=\max\{\tilde{v}_1,...,\tilde{v}_m\}$, $\overline{A}=diag\{\overline{a}_1,...,\overline{a}_m\}$ and $\overline{B}=diag\{\overline{b}_1,...,\overline{b}_m\}$. Moreover, $\mathcal{M}(t)$ is non decreasing function, then

$$\|p(t)\| \leq \mathcal{M}(t)\mathbb{E}_{\mu}\Big\{\mathcal{N}(t)\Gamma(\mu)t^{\mu}\Big\},$$

where Lemma 2.10 has been used. So $p(t)$ remains bounded for any positive time, which guarantees the existence of global solution in the Filippov sense and it is defined on $[0,+\infty)$. Hence the proof is completed.

4 Stability results

This section is devoted to the global asymptotic stability of FCGNNDDs system (1).

Theorem 4.1 *Under Assumptions (1)-(3) and Assumption (5), the equilibrium point of system* (1) *is global asymptotically stable if the following algebraic inequality are satisfied:*

$$\begin{aligned}\beta &= \min_{1\leq l\leq m}\Big\{2\underline{a}_l\underline{b}_l - \varepsilon_3 - \sum_{j=1}^{m}\Big[\overline{a}_l\Big[|c_{lj}|\varepsilon_1 + |d_{lj}|\varepsilon_2\Big]u_j + \frac{1}{\varepsilon_1}\overline{a}_j|c_{jl}|u_l\Big]\Big\}\\ &> \max_{1\leq l\leq m}\Big\{\sum_{j=1}^{m}\frac{1}{\varepsilon_2}\overline{a}_j|d_{jl}|u_l\Big\} = \gamma > 0, \qquad (7)\\ \hat{\pi} &= \sum_{l=1}^{m}\frac{1}{2\varepsilon_3}R_l^2 > 0,\ \alpha = -\beta\sin(\frac{\mu\pi}{2})+\gamma<0,\ \textit{where } R_l = \sum_{j=1}^{m}\overline{a}_l\Big[|c_{lj}|+|d_{lj}|\Big]w_j. \qquad (8)\end{aligned}$$

Proof. Let us take $v_l(t)=p_l(t)-p_l^*$, according to Definition 3.3 and system (1), we have

$$\begin{cases}D^{\mu}v_l(t) = -\tilde{a}_l(v_l(t))\Big[\tilde{b}_l(v_l(t)) - \sum_{j=1}^{m}c_{lj}\tilde{\delta}_j(t) - \sum_{j=1}^{m}d_{lj}\tilde{\delta}_j(t-\lambda)\Big],\\ v_l(\kappa) = \varrho_l(\kappa)-p_l^* = \iota_l(\kappa),\ \kappa\in[-\lambda,0].\end{cases} \qquad (9)$$

where $\tilde{a}_l(v_l(t)) = a_l\big(v_l(t)+p_l^*\big)$, $\tilde{b}_l(v_l(t)) = b_l\big(v_l(t)+p_l^*\big) - b_l\big(p_l^*\big)$ and $\tilde{\delta}_j(t)=\delta_j(t)-\delta_j^*$.

Consider the following Lyapunov function:

$$G(t) = \frac{1}{2}v^T(t)v(t) = \frac{1}{2}\sum_{l=1}^{m}v_l^2(t). \qquad (10)$$

From Lemma 2.4, Assumption (1) and Assumption (5), we gain

$$\begin{aligned}D^{\mu}G(t) &\leq \sum_{l=1}^{m}v_l(t)D^{\mu}v_l(t)\\ &= \sum_{l=1}^{m}(v_l(t))\tilde{a}_l(v_l(t))\Big[-\tilde{b}_l(v_l(t)) + \sum_{j=1}^{m}c_{lj}\tilde{\delta}_j(t) + \sum_{j=1}^{m}d_{lj}\tilde{\delta}_j(t-\lambda)\Big]\\ &\leq \sum_{l=1}^{m}\tilde{a}_l(v_l(t))\Bigg[-v_l^2(t)\frac{\tilde{b}_l(v_l(t))}{v_l(t)} + \sum_{j=1}^{m}|c_{lj}||v_l(t)|\Big[u_j|v_j(t)|+w_j\Big]\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{j=1}^{m} |v_l(t)||d_{lj}|\Big[u_j|v_j(t-\lambda)| + w_j\Big]\Big] \\
&\leq \; -\sum_{l=1}^{m} \underline{a}_l \underline{b}_l v_l^2(t) + \sum_{l=1}^{m}\sum_{j=1}^{m} \overline{a}_l |c_{lj}| u_j |v_l(t)||v_j(t)| \\
&\quad + \sum_{l=1}^{m}\sum_{j=1}^{m} \overline{a}_l |d_{lj}| u_j |v_l(t)||v_j(t-\lambda)|\Big] + \sum_{l=1}^{m}\sum_{j=1}^{m} \overline{a}_l \Big[|c_{lj}| + |d_{lj}|\Big] w_j |v_l(t)| \qquad (11)
\end{aligned}
$$

By using Lemma 2.6, one has

$$
\begin{aligned}
\sum_{l=1}^{m}\sum_{j=1}^{m} \overline{a}_l |c_{lj}| u_j |v_l(t)||v_j(t)| &\leq \sum_{l=1}^{m}\sum_{j=1}^{m} \overline{a}_l |c_{lj}| u_j \Big[\frac{\varepsilon_1}{2} v_l^2(t) + \frac{1}{2\varepsilon_1} v_j^2(t)\Big] \\
&= \sum_{l=1}^{m}\sum_{j=1}^{m} \Big[\overline{a}_l |c_{lj}| u_j \varepsilon_1 + \frac{1}{\varepsilon_1} \overline{a}_j |c_{jl}| u_l\Big] \frac{v_l^2(t)}{2} \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{l=1}^{m}\sum_{j=1}^{m} \overline{a}_l |d_{lj}| u_j |v_l(t)||v_j(t-\lambda)| &\leq \sum_{l=1}^{m}\sum_{j=1}^{m} \overline{a}_l |d_{lj}| u_j \Big[\frac{\varepsilon_2}{2} v_l^2(t) + \frac{1}{2\varepsilon_2} v_j^2(t-\lambda)\Big] \\
&= \sum_{l=1}^{m}\sum_{j=1}^{m} \varepsilon_2 \overline{a}_l |d_{lj}| u_j \frac{v_l^2(t)}{2} + \sum_{l=1}^{m}\sum_{j=1}^{m} \frac{1}{\varepsilon_2} \overline{a}_j |d_{jl}| u_l \frac{v_l^2(t-\lambda)}{2} \qquad (13)
\end{aligned}
$$

and

$$
\sum_{l=1}^{m} |v_l(t)| R_l \leq \sum_{l=1}^{m} \Big[\frac{\varepsilon_3}{2} v_l^2(t) + \frac{1}{2\varepsilon_3} R_l^2\Big] \qquad (14)
$$

Substituting (12)-(14) into (11), one has

$$
\begin{aligned}
D^{\mu} G(t) &\leq -\sum_{l=1}^{m} \left[2\underline{a}_l \underline{b}_l + \varepsilon_3 - \sum_{j=1}^{m} \Big[\overline{a}_l |c_{lj}| u_j \varepsilon_1 + \frac{1}{\varepsilon_1} \overline{a}_j |c_{jl}| u_l + \varepsilon_2 \overline{a}_l |d_{lj}| u_j\Big]\right] \frac{v_l^2(t)}{2} \\
&\quad + \sum_{l=1}^{m} \Big[\sum_{j=1}^{m} \frac{1}{\varepsilon_2} \overline{a}_j |d_{jl}| u_l\Big] \frac{v_l^2(t-\lambda)}{2} + \sum_{l=1}^{m} \frac{1}{2\varepsilon_3} R_l^2 \\
&\leq -\beta G(t) + \gamma G(t-\lambda) + \hat{\pi}. \qquad (15)
\end{aligned}
$$

Consider the following linear system

$$
D^{\mu} H(t) = -\beta H(t) + \gamma H(t-\lambda) + \hat{\pi} \qquad (16)
$$

where $H(t) \geq 0$, $(H(t) \in \mathbb{R})$, and have the similar initial values with $G(t)$. Taking Laplace transform of (16), we have

$$
\begin{aligned}
s^{\mu} H(s) - s^{\mu-1} H(0) &= -\beta H(s) + \gamma \int_0^{+\infty} \exp\{-st\} H(t-\lambda)\, \mathrm{d}\, t + \hat{\pi} \int_0^{\infty} \exp\{-st\}\, \mathrm{d}\, t \\
&= -\beta H(s) + \gamma \int_{-\lambda}^{+\infty} \exp\{-s(\psi+\lambda)\} H(\psi)\, \mathrm{d}\, \psi + \frac{\hat{\pi}}{s} \\
&= -\beta H(s) + \gamma \exp\{-s\lambda\} \int_{-\lambda}^{+\infty} \exp\{-s\psi\} H(\psi)\, \mathrm{d}\, \psi + \frac{\hat{\pi}}{s}
\end{aligned}
$$

$$
\begin{aligned}
&= -\beta H(s) + \gamma \exp\{-s\lambda\}\Big[\int_{-\lambda}^{0} + \int_{0}^{+\infty}\Big] \exp\{-s\psi\} H(\psi)\,\mathrm{d}\,\psi + \frac{\hat{\pi}}{s} \\
&= -\beta H(s) + \gamma \exp\{-s\lambda\} \int_{-\lambda}^{0} \exp\{-s\psi\} H(\psi)\,\mathrm{d}\,\psi \\
&\quad + \gamma \exp\{-s\lambda\} H(s) + \frac{\hat{\pi}}{s}
\end{aligned}
$$

$$
\Big[s^{\mu} + \beta - \gamma \exp\{-s\lambda\}\Big] H(s) = s^{\mu-1} H(0) + \gamma \exp\{-s\lambda\} \int_{-\lambda}^{0} \exp\{-s\psi\} H(\psi)\,\mathrm{d}\,\psi + \frac{\hat{\pi}}{s} \quad (17)
$$

By virtue of Lemma 2.9 and in Eq.(17), we obtain $\Delta(s) = s^{\mu} + \beta - \gamma \exp\{-s\lambda\}$ and $\Delta(s) = det(\Delta(s)) = 0$. Now we have to prove $det(\Delta(s))$ has pure imaginary roots for any $\lambda > 0$.

Suppose $\Delta(s) = s^{\mu} + \beta - \gamma \exp\{-s\lambda\}$ has pure imaginary roots for any $\lambda > 0$. If $\sigma < 0$, $s = \sigma i = |\sigma|\big[\cos(\frac{\pi}{2}) - \sin(\frac{\pi}{2})\big]$, σ is a real constant. That is, $s = \sigma i = |\sigma|\big[\cos(\frac{\pi}{2}) + \sin(\frac{\pm\pi}{2})\big]$ into characteristic equation $s^{\mu} + \beta - \gamma \exp\{-s\lambda\} = 0$, we have

$$
|\sigma|^{\mu}\big[\cos(\frac{\mu\pi}{2}) + i\sin(\frac{\pm\mu\pi}{2})\big] + \beta - \gamma\big[\cos(\sigma\lambda) - i\sin(\sigma\lambda)\big] = 0
$$

which imply that

$$
|\sigma|^{\mu}\big[\cos(\frac{\mu\pi}{2}) + i\sin(\frac{\pm\mu\pi}{2})\big] + \beta = \gamma\big[\cos(\sigma\lambda) - i\sin(\sigma\lambda)\big] \quad (18)
$$

Splitting real and imaginary parts of Eq.(18), we gain

$$
|\sigma|^{\mu}\cos(\frac{\mu\pi}{2}) + \beta = \gamma\cos(\sigma\lambda) \quad (19)
$$

$$
|\sigma|^{\mu}\sin(\frac{\pm\mu\pi}{2}) = -\gamma\sin(\sigma\lambda) \quad (20)
$$

From (19) and (20), we obtain

$$
\Big[|\sigma|^{\mu}\cos(\frac{\mu\pi}{2}) + \beta\Big]^2 + \Big[|\sigma|^{\mu}\sin(\frac{\pm\mu\pi}{2})\Big] - \gamma^2 = 0
$$

It follows that

$$
|\sigma|^{2\mu} + \beta^2 + 2\beta|\sigma|^{\mu}\cos(\frac{\mu\pi}{2}) - \gamma^2 = 0. \quad (21)
$$

Now, we estimate the discriminant of Eq.(21), one has

$$
\begin{aligned}
\Delta &= \Big[2\beta\cos(\frac{\mu\pi}{2})\Big]^2 - 4(1)(\beta^2 - \gamma^2) \\
&= \Big[\gamma^2 - \beta^2\sin^2(\frac{\pm\mu\pi}{2})\Big].
\end{aligned}
$$

From hypothesis of our theorem, we have $\gamma < \beta\sin(\frac{\pm\mu\pi}{2})$, which follows that $\Delta < 0$. That is $det(\Delta(s))$ has no pure imaginary roots for any $\lambda > 0$. Further, we need to prove eigenvalues of matrix $J = \gamma - \beta$ satisfy $|arg(\lambda^*(J)| > \frac{\pi}{2}$. As $\gamma < \beta\sin(\frac{\pm\mu\pi}{2}) < \beta$, $0 < \mu < 1$. Therefore $\lambda^*(J)$ are negative. That is $|arg(\lambda^*(J)| > \frac{\pi}{2}$. Again by using Lemma 2.9, the equilibrium point of (16) is globally asymptotic stable. Hence $H(t) \to 0$ as $t \to +\infty$. Based on Lemma 2.8, we have $0 \leq G(t) \leq H(t) \to 0$ as $t \to +\infty$. Therefore the equilibrium point of system (1) is global asymptotically stable. This proof is ended.

Remark 4.2 *Only a few works focused on the global stability of FCGNNs. Different from the stability results in [35, 49], our standards given sufficient conditions for global asymptotic stability of FCGNNs with discontinuous neuron activations through comparison theorem for linear fractional order delayed system, at the same time as the preceding literature either concerned in continuous neuron activations or without using Lyapunov principle for the addressed network model. Furthermore, the obtained stability criteria, in terms of algebraic inequalities, is quite simple to check in practice.*

5 Pinning synchronization results

This segment, we design a class of novel pinning controller to ensure global asymptotical synchronization criteria for the FCGNNDDs.

Model (1) is consider as master system, the corresponding controlled slave system is as follows

$$\begin{cases} D^{\mu}q_l(t) = -a_l\big(q_l(t)\big)\Big[b_l\big(q_l(t)\big) - \sum_{j=1}^{m} c_{lj}h_j(q_j(t)) - \sum_{j=1}^{m} d_{lj}h_j(q_j(t-\lambda)) - k_l - E_l(t)\Big], \\ q_l(\kappa) = \breve{\varrho}_l(\kappa),\ \kappa \in [-\lambda, 0], \end{cases} \tag{22}$$

where $E_l(t)$ is pinning control inputs, $q_l(t)$ is the state variable and other parameters are similar as those in master system (1). The initial values $q_l(\kappa) \in C\big([-\lambda, 0], \mathbb{R}^n\big)$ is associated with slave system (22) and the norm is defined by

$$\|\breve{\varrho}\| = \sup_{t\in[-\lambda,0]} \|\breve{\varrho}(t)\|.$$

Based on differential inclusion analysis and from (22), we have

$$\begin{cases} D^{\mu}q_l(t) \in -a_l\big(q_l(t)\big)\Big[b_l\big(q_l(t)\big) - \sum_{j=1}^{m} c_{lj}\mathcal{H}\big[h_j(q_j(t))\big] - \sum_{j=1}^{m} d_{lj}\mathcal{H}\big[h_j(q_j(t-\lambda))\big] - k_l - E_l(t)\Big], \\ q_l(\kappa) = \breve{\varrho}_l(\kappa),\ \kappa \in [-\lambda, 0]. \end{cases}$$

for a.e. $t \geq 0$. Or equivalently there exist $\theta_j(t) \in \mathcal{H}\big[h_j(q_j(t))\big]$, the initial value problem of slave system in the following expression:

$$\begin{cases} D^{\mu}q_l(t) = -a_l\big(q_l(t)\big)\Big[b_l\big(q_l(t)\big) - \sum_{j=1}^{m} c_{lj}\theta_j(t) - \sum_{j=1}^{m} d_{lj}\theta_j(t-\lambda) - k_l - E_l(t)\Big], \\ q_l(\kappa) = \breve{\varrho}_l(\kappa),\ \kappa \in [-\lambda, 0]. \end{cases} \tag{23}$$

Define synchronization error $v_l(t) = q_l(t) - p_l(t)$. From master system (1) (*or* (5)) and slave system (22) (*or* (23)), the error system can be obtained as

$$\begin{cases} D^{\mu}v_l(t) = -a_l\big(q_l(t)\big)\Big[b_l\big(q_l(t)\big) - \sum_{j=1}^{m} c_{lj}\theta_j(t) - \sum_{j=1}^{m} d_{lj}\theta_j(t-\lambda) - E_l(t)\Big] \\ \qquad\qquad +a_l\big(p_l(t)\big)\Big[b_l\big(p_l(t)\big) - \sum_{j=1}^{m} c_{lj}\delta_j(t) - \sum_{j=1}^{m} d_{lj}\delta_j(t-\lambda)\Big] \\ v_l(\kappa) = \breve{\varrho}_l(\kappa) - \varrho_l(\kappa) = \varpi_l(\kappa),\ \kappa \in [-\lambda, 0]. \end{cases} \tag{24}$$

for $l = 1, 2, .., m$, a.e.$t \geq 0$, $v_l(\kappa) = \breve{\varrho}_l(\kappa) - \varrho_l(\kappa)$ is the initial values associated with error system (24).

Novel pinning control is a strategy which simply requires a small fraction of neurons with small pinning control strength to achieve asymptotical synchronization for the entire system. Without loss of generality, we will choose ζ neurons from all neurons are controlled directly and the model of pinning controller $E_l(t),\ l = 1, 2, ..., m$ in slave system is designed as

$$E_l(t) = \begin{cases} \breve{E}_l(t) = -\eta\,\mathrm{sgn}\{v_l(t)\} \times \left[\frac{\sum_{l=1}^{m}|v_l(t)|}{\sum_{l=1}^{\zeta}|v_l(t)|}\right]\left(\sum_{j=1}^{m}|v_j(t)|\right), & \text{if } l = 1, 2, ..., \zeta \\ 0, & \text{if } l = \zeta+1, \zeta+2, ..., m. \end{cases} \tag{25}$$

where $\eta > 0$ is an adjustable constant, $\breve{E}_1(t)$ denote general control input which can be applied in each node, and $E_l(t)$ is corresponding pinning control strategy to realize synchronization. It implies that, there are $m - \zeta$ neurons are pinning controlled indirectly.

In some situation, the master system (1) cannot be completely synchronized to the slave system (22) with designed pinning control law (25). In this case, quasi-synchronization will be considered in this paper. The concept of quasi-synchronization is defined as follows

Definition 5.1 *Master-slave systems* (1) *and* (25) *are said to achieve quasi-synchronization with error bound* $\theta > 0$ *if there exists a compact set* M *such that, for any initial condition* $\varpi(\kappa) \in C\Big([\lambda, 0], \mathbb{R}^m\Big)$, *the error* $v(t) = q(t) - p(t)$ *converges to*

$$\Upsilon = \{\|v(t)\| \leq \theta\},\ t \to +\infty,$$

where $v(t) = [v_1(t), ..., v_m(t)]^T$, $q(t) = [q_1(t), ..., q_m(t)]^T$, $p(t) = [p_1(t), ..., p_m(t)]^T$ *and* $\varpi(\kappa) = [\varpi_1(\kappa), ..., \varpi_m(\kappa)(t)]^T$.

Theorem 5.2 *Suppose Assumptions (1)-(3) and (5) holds. The master* (1) *and slave system* (22) *with designed pinning control law* (25) *can achieve quasi synchronization with error bound* $\sqrt{\frac{2\hat{\pi}}{\xi - \gamma}}$ *if the following algebraic inequalities are satisfied:*

$$\begin{aligned}
\xi &= \min_{1\leq l\leq m}\Bigg\{2\eta + 2\phi_l - 2z_l k_l - \varepsilon_3 - 2\sum_{j=1}^{m}\Big[|c_{lj}| + |d_{lj}|\Big]\varsigma_j z_l - \sum_{j=1}^{m}\Big[\bar{a}_l|c_{lj}|u_j\varepsilon_1 + \frac{1}{\varepsilon_1}\bar{a}_j|c_{jl}|u_l\Big] \\
&\quad - \sum_{j=1}^{m}\varepsilon_2\bar{a}_l|d_{lj}|u_j\Bigg\} - \max_{1\leq l\leq m}\Bigg\{\sum_{j=1}^{m}\frac{1}{\varepsilon_2}\bar{a}_j|d_{jl}|u_l\Bigg\} = \gamma > 0, \\
\hat{\pi} &= \sum_{l=1}^{m}\frac{1}{2\varepsilon_3}R_l^2 > 0,\ \textit{where}\ R_l = \sum_{j=1}^{m}\bar{a}_l\Big[|c_{lj}| + |d_{lj}|\Big]w_j.
\end{aligned}$$

Proof. Consider the following Lyapunov function:

$$G(t) = \frac{1}{2}v^T(t)v(t) = \frac{1}{2}\sum_{l=1}^{m}v_l^2(t) \tag{26}$$

From Lemma 2.5, Lemma 2.11, Assumptions $(1) - (3)$ and (5), we gain

$$\begin{aligned}
D^\mu G(t) &= D^\mu \sum_{l=1}^{m}\frac{1}{2}|v_l(t)|^2 \\
&\leq \sum_{l=1}^{m}|v_l(t)|D^\mu|v_l(t)| \\
&\leq \sum_{l=1}^{m}|v_l(t)|\,\mathrm{sgn}(v_l(t))D^\mu v_l(t) \\
&= \sum_{l=1}^{m}|v_l(t)|\,\mathrm{sgn}(v_l(t))\Bigg[-\Big[a_l\big(q_l(t)\big)b_l\big(q_l(t)\big) - a_l\big(p_l(t)\big)b_l\big(p_l(t)\big)\Big] \\
&\quad + \Big[a_l\big(q_l(t)\big) - a_l\big(p_l(t)\big)\Big]k_l + \sum_{j=1}^{m}a_l\big(q_l(t)\big)c_{lj}[\theta_j(t) - \delta_j(t)] \\
&\quad + \sum_{j=1}^{m}a_l\big(q_l(t)\big)c_{lj}\delta_j(t) - \sum_{j=1}^{m}a_l\big(p_l(t)\big)c_{lj}\delta_j(t) + \sum_{j=1}^{m}a_l\big(q_l(t)\big)d_{lj}[\theta_j(t-\lambda) - \delta_j(t-\lambda)] \\
&\quad + \sum_{j=1}^{m}a_l\big(q_l(t)\big)d_{lj}\delta_j(t-\lambda) - \sum_{j=1}^{m}a_l\big(p_l(t)\big)d_{lj}\delta_j(t-\lambda) + E_l(t)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{l=1}^{m}\Big[-\phi_l+z_lk_l\Big]v_l^2(t)+\sum_{l=1}^{m}|v_l(t)|\Big\{\sum_{j=1}^{m}\bar{a}_l|c_{lj}|\big[u_j|v_j(t)|+w_j\big]\Big\} \\
&\quad +\sum_{l=1}^{m}|v_l(t)|\Big\{\sum_{j=1}^{m}|a_l\big(q_l(t)-a_l\big(p_l(t)\big)||c_{lj}||\delta_j(t)|\Big\}+\sum_{l=1}^{m}|v_l(t)|\Big\{\sum_{j=1}^{m}\bar{a}_l|d_{lj}| \\
&\quad \times\big[u_j|v_j(t-\lambda)|+w_j\big]\Big\}+\sum_{l=1}^{m}|v_l(t)|\Big\{\sum_{j=1}^{m}|a_l\big(q_l(t)-a_l\big(p_l(t)\big)| \\
&\quad \times|d_{lj}||\delta_j(t-\lambda)|\Big\}-\sum_{l=1}^{\zeta}|v_l(t)|\eta\,\mathrm{sgn}\{v_l(t)\}\times\left[\frac{\sum_{l=1}^{m}|v_l(t)|}{\sum_{l=1}^{\zeta}|v_l(t)|}\right]\left(\sum_{j=1}^{m}|v_j(t)|\right) \\
&\leq -\sum_{l=1}^{m}\big[\phi_l-z_lk_l\big]v_l^2(t)-\eta\sum_{l=1}^{m}v_l^2(t)+\sum_{l=1}^{m}\sum_{j=1}^{m}\Big[|c_{lj}|+|d_{lj}|\Big]\varsigma_jz_lv_l^2(t) \\
&\quad +\sum_{l=1}^{m}\sum_{j=1}^{m}\bar{a}_l|c_{lj}|u_j|v_l(t)||v_j(t)|+\sum_{l=1}^{m}\sum_{j=1}^{m}\bar{a}_l|d_{lj}|u_j|v_l(t)||v_j(t-\lambda)| \\
&\quad +\sum_{l=1}^{m}|v_l(t)|\left[\sum_{j=1}^{m}\big[|c_{lj}|+|d_{lj}|\big]\bar{a}_lw_j\right]
\end{aligned}
\tag{27}
$$

Then, substituting (12)-(14) into (27), we obtain

$$
\begin{aligned}
D^{\mu}G(t) &\leq -\sum_{l=1}^{m}\big[\eta+\phi_l-z_lk_l\big]v_l^2(t)+\sum_{l=1}^{m}\sum_{j=1}^{m}\Big[|c_{lj}|+|d_{lj}|\Big]\varsigma_jz_lv_l^2(t)+\sum_{l=1}^{m}\left[\frac{\varepsilon_3}{2}v_l^2(t)+\frac{1}{2\varepsilon_3}R_l^2\right] \\
&\quad +\sum_{l=1}^{m}\sum_{j=1}^{m}\bar{a}_l|c_{lj}|u_j\left[\frac{\varepsilon_1}{2}v_l^2(t)+\frac{1}{2\varepsilon_1}v_j^2(t)\right]+\sum_{l=1}^{m}\sum_{j=1}^{m}\bar{a}_l|d_{lj}|u_j\left[\frac{\varepsilon_2}{2}v_l^2(t)+\frac{1}{2\varepsilon_2}v_j^2(t-\lambda)\right] \\
&= -\frac{1}{2}\sum_{l=1}^{m}\Bigg[2\eta+2\phi_l-2z_lk_l-\varepsilon_3-2\sum_{j=1}^{m}\Big[|c_{lj}|+|d_{lj}|\Big]\varsigma_jz_l-\sum_{j=1}^{m}\Big[\bar{a}_l|c_{lj}|u_j\varepsilon_1+\frac{1}{\varepsilon_1}\bar{a}_j|c_{jl}|u_l\Big] \\
&\quad -\sum_{j=1}^{m}\varepsilon_2\bar{a}_l|d_{lj}|u_j\Bigg]v_l^2(t)+\frac{1}{2}\sum_{l=1}^{m}\sum_{j=1}^{m}\frac{1}{\varepsilon_2}\bar{a}_j|d_{jl}|u_lv_l^2(t-\lambda)+\sum_{l=1}^{m}\frac{1}{2\varepsilon_3}R_l^2 \\
&= -\xi G(t)+\gamma G(t-\lambda)+\hat{\pi} \\
&\leq -\xi G(t)+\gamma\sup_{t-\lambda\leq\kappa\leq t}G(\kappa)+\hat{\pi}
\end{aligned}
\tag{28}
$$

From the condition of Theorem 5.2, we obtain $\xi-\gamma>0$. Based on the Fractional Halanay inequality Lemma 2.7, we have

$$
\|v(t)\|_2\leq\sqrt{\frac{2\hat{\pi}}{\xi-\gamma}},\ t\to+\infty. \tag{29}
$$

Thus, we can conclude that the error system (24) converges to the region Υ containing the origin, where

$$
\Upsilon=\Big\{v(t):\|v(t)\|_2\leq\sqrt{\frac{2\hat{\pi}}{\xi-\gamma}}\Big\},\ t\to+\infty, \tag{30}
$$

which indicates that the master system (1) and slave system (22) with pinning control law (25) achieve quasi synchronization with error bound $\sqrt{\frac{2\hat{\pi}}{\xi-\gamma}}$. This proof is ended.

When the neuron activation function is taken to be a common Lipschitz-type, Assumption (5) can be replaced with the following condition:

Assumption (5A). Suppose there exist constants $u_j > 0$, the following inequalities are established

$$\begin{aligned} |h_j(p) - h_j(q)| &\leq u_j|p-q|, \ \forall, \ p, q \in \mathbb{R}, \ j = 1, 2, ..., m, \\ |h_j(p)| &\leq \varpi_j, \ j = 1, 2, ..., m. \end{aligned}$$

As a special case of Theorem 5.2, we provide the corresponding result.

Corollary 5.3 *Under Assumptions (1)-(2) and (5A), master system* (1) *and slave system* (22) *with pinning control law* (25) *are globally asymptotically synchronized if the following algebraic inequalities is satisfied:*

$$\begin{aligned} \xi &= \min_{1\leq l\leq m}\Big\{2\eta + 2\phi_l - 2z_l k_l - \varepsilon_3 - 2\sum_{j=1}^{m}\Big[|c_{lj}| + |d_{lj}|\Big]\varpi_j z_l - \sum_{j=1}^{m}\Big[\overline{a}_l|c_{lj}|u_j\varepsilon_1 + \frac{1}{\varepsilon_1}\overline{a}_j|c_{jl}|u_l\Big] \\ &\quad - \sum_{j=1}^{m}\varepsilon_2\overline{a}_l|d_{lj}|u_j\Big\} > \gamma = \max_{1\leq l\leq m}\Big\{\sum_{j=1}^{m}\frac{1}{\varepsilon_2}\overline{a}_j|d_{jl}|u_l\Big\} > 0, \\ \alpha &= -\xi\sin(\frac{\mu\pi}{2}) + \gamma < 0. \end{aligned}$$

Remark 5.4 *Reviewing existing works, there are numerous outcomes on the global asymptotic synchronization analysis of integer-order discontinuous Cohen-Grossberg neural networks [16, 17] via pinning control policy and fractional order discontinuous neural networks [8, 46] have been studied extensively over the past few years. Recently, fractional order Cohen-Grossberg neural networks have been received much more interest from lot of researchers, see Ref [48, 49]. However, there are no results at present to study the pinning controller for synchronization analysis of fractional order Cohen-Grossberg neural networks with delays and discontinuous neuron activation function (FCGNNDDs) as far as we know. In view of this, we have proposed the global asymptotical pinning synchronization analysis of FCGNNDDs.*

Remark 5.5 *In order to shed light on how to design a suitable pinning controller in application perspective to obtain global asymptotically synchronization, we take an example for the application of Theorem 5.2, we're able to the layout following steps:*

Table 1: The Algorithm to design the pinning control strategy

Algorithm
step.1: Initialize the system parameters C, D.
step.2: Randomly choosing ζ of pinned neurons from all neurons.
step.3: Select the appropriate values ϕ_l, z_l, $\bar{a}_l$ u_j, w_j, ς_j to Assumptions (1)-(3) and Assumption (5).
step.4: Choose the control strengths η.
step.5: Given ε_1, ε_2, ε_3 and solve to get ξ, γ, $\hat{\pi}$.
step.6: Check whether $\xi > \gamma > 0$ and $\hat{\pi} > 0$. If success, the procedure further moves to next level. Otherwise the procedure turns back to adjust the control strengths in step 4.
step.7: Based on the proper control strengths, we design a novel pinning control.

Remark 5.6 *Yang et al. [41] dealt with the global Mittag-Leffler stability and synchronization analysis of fractional order neural networks with linear threshold neurons in quaternion field by designing simple linear feedback control. Yang et al. [42] discussed the global asymptotical synchronization analysis of fractional order neural networks with time delays in complex field by designing discontinuous feedback control. It is seen that in all the aforementioned references, the authors controllers are applied to every neuron of FNNs, which could be very high priced and impractically. However in our paper, we have used the pinning control technique which is more effective than the control techniques used by the authors in [41, 42], because it has been applied to one neuron or the huge number of neurons instead of all neurons, which greatly reduces the control costs and consumption.*

6 Numerical Examples

This segment provides two examples to indicate the advantages of the obtained stability and synchronization results in previous segments.

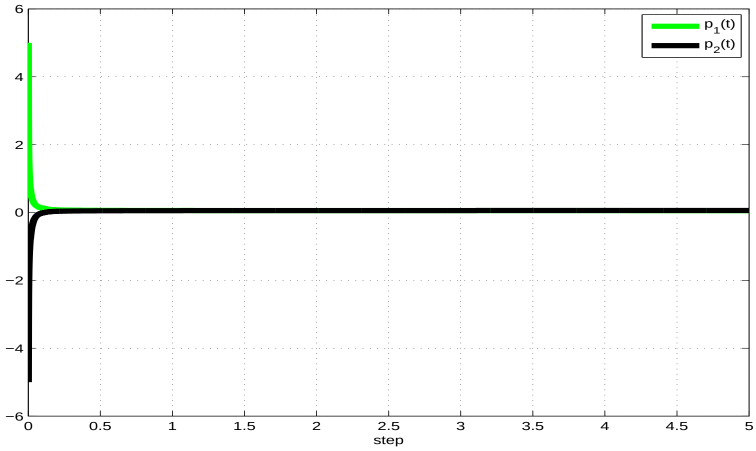

Figure 1: The equilibrium point of (31) is global asymptotically stable.

Example 6.1 *Consider the following two-state FCGNNDDs:*

$$D^{0.95}p_l(t) = -a_l\big(p_l(t)\big)\left[b_l\big(p_l(t)\big) - \sum_{j=1}^{2} c_{lj}h_j(p_j(t)) - \sum_{j=1}^{2} d_{lj}h_j(p_j(t-\lambda)) - k_l\right], \tag{31}$$

where $l = 1,2,\ \lambda = 0.5,\ k_1 = k_2 = 0.5,\ a_1(p) = a_2(p) = 1 + \frac{2}{1+p^2},\ b_1(p) = b_2(p) = 8 + \tanh(p),$ $h_j(p) = 0.005 * \sin(p) + 0.09 * \mathrm{sgn}(p),\ j = 1,2$ *and*

$$C = (c_{lj})_{2\times 2} = \begin{bmatrix} 0.3 & -1 \\ -0.7 & 0.5 \end{bmatrix}, \quad D = (d_{lj})_{2\times 2} = \begin{bmatrix} 1.5 & -0.8 \\ 0.6 & 1.2. \end{bmatrix}$$

Based on Assumptions (1) and (5), we have $\underline{a}_1 = \underline{a}_1 = 0.5,\ \overline{a}_1 = \overline{a}_2 = 2,\ u_1 = u_2 = 0.5$ *and* $w_1 = w_2 = 1$. *It is easy to estimate* $\beta = 3.3,\ \gamma = 1.05,\ \hat{\pi} = 10.7$ *with* $\varepsilon_1 = \varepsilon_2 = 2$ *and* $\varepsilon_3 = 3$, *that is the conditions presented in Theorem 4.1,* $-\beta + \gamma < 0$ *holds. Therefore, the equilibrium point of FCGNNDDs* (31) *is global asymptotically stable. In Fig.1 presents the time responses of the state variables in* (31) *with initial conditions* $p(0) = (5, -5)^T$. *So, these simulations confirm the validity of proposed Theorem 4.1.*

Example 6.2 *Consider the following FCGNNDDs with m=3:*

$$D^{0.998}p_l(t) = -a_l\big(p_l(t)\big)\Bigg[b_l\big(p_l(t)\big) - \sum_{j=1}^{3} c_{lj}h_j(p_j(t)) - \sum_{j=1}^{3} d_{lj}h_j(p_j(t-\lambda)) - k_l\Bigg], \tag{32}$$

where $l = 1, 2, 3,\ \lambda = 2,\ k_1 = k_2 = k_3 = 0.02,\ a_1(p_1) = 0.7 + \frac{0.9}{1+p_1^2},\ a_2(p_2) = 0.9 + \frac{1.3}{1+p_2^2},\ a_3(p_3) = 1 + \frac{1.5}{1+p_3^2},\ b_j(p) = 3.5 + \sin(p),\ h_j(p) = \tanh(p) + \mathrm{sgn}(p),\ j = 1, 2, 3$ *and*

$$C = \big(c_{lj}\big)_{3\times3} = \begin{bmatrix} 3 & 1 & 0.5 \\ 1 & 3.5 & 2 \\ -1.5 & -2 & 2.1 \end{bmatrix}, \quad D = \big(d_{lj}\big)_{3\times3} = \begin{bmatrix} 1.5 & 1 & -1.35 \\ -2 & 0.5 & 2.25 \\ 2.5 & -4 & -0.5 \end{bmatrix}$$

The initial conditions of system (32) *is taken as* $p(\kappa) = \big(p_1(\kappa), p_2(\kappa), p_3(\kappa)\big)^T = (-2, 3, 5)^T,\ \kappa \in [-2, 0)$. *Based on Assumptions (1)-(3) and (5), we have* $\bar{a}_1 = \bar{a}_2 = \bar{a}_3 = 1,\ \phi_1 = \phi_2 = \phi_3 = 1.5,\ z_1 = z_2 = z_3 = 0.5\ u_1 = u_2 = u_3 = 0.5,\ w_1 = w_2 = w_3 = 0.2$ *and* $\varsigma_1 = \varsigma_2 = \varsigma_3 = 1$.

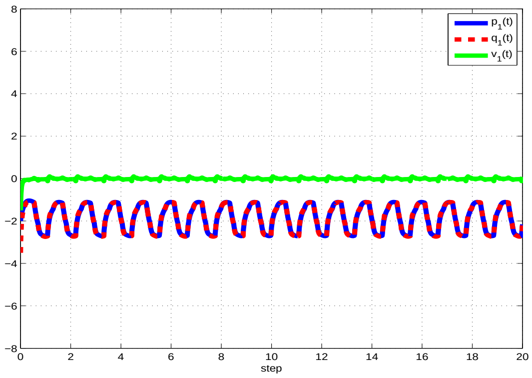

Figure 2: The trajectories of synchronization states $p_1(t)$ vs. $q_1(t)$ and their error $v_1(t)$ under the pinning control.

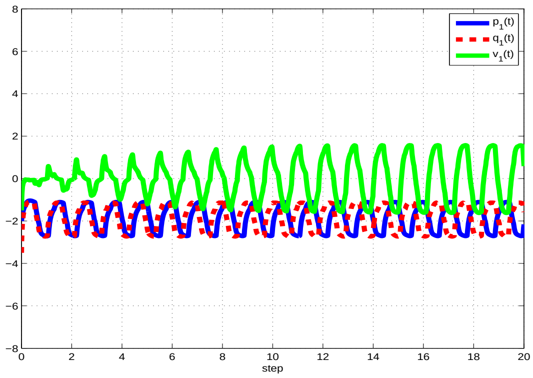

Figure 3: The trajectories of synchronization states $p_1(t)$ vs. $q_1(t)$ and their error $v_1(t)$ without control inputs.

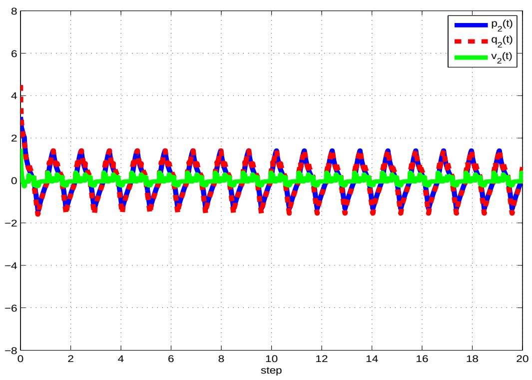

Figure 4: The trajectories of synchronization states $p_2(t)$ vs. $q_2(t)$ and their error $v_2(t)$ under the pinning control.

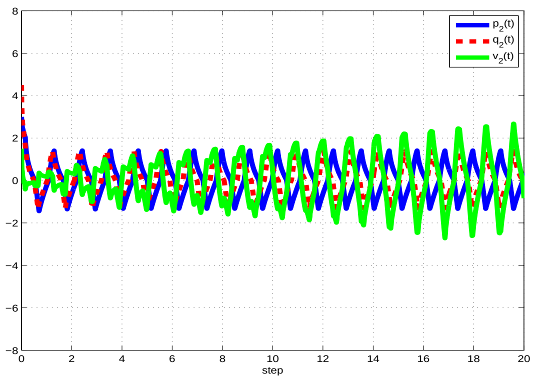

Figure 5: The trajectories of synchronization states $p_2(t)$ vs. $q_2(t)$ and their error $v_2(t)$ without control inputs.

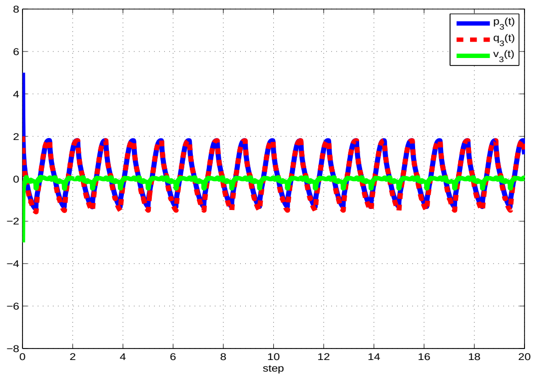

Figure 6: The trajectories of synchronization states $p_3(t)$ vs. $q_3(t)$ and their error $v_3(t)$ under the pinning control.

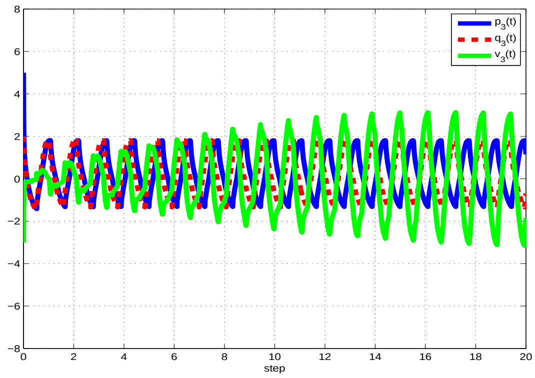

Figure 7: The trajectories of synchronization states $p_3(t)$ vs. $q_3(t)$ and their error $v_3(t)$ without control inputs.

The corresponding slave system is defined as

$$D^{0.998}q_l(t) = -a_l\big(q_l(t)\big)\Big[b_l\big(q_l(t)\big) - \sum_{j=1}^{3} c_{lj}h_j(q_j(t)) - \sum_{j=1}^{3} d_{lj}h_j(q_j(t-\lambda)) - k_l - E_l(t)\Big] \tag{33}$$

which shares the similar parameter values of the master system (32)*.* $E_l(t)$ *is pinning control inputs and two neurons are under control, i.e.,* $\zeta = 2$*. In other words, in this example, first two neurons are selected as directly control, and third neuron is pinning controlled neuron.*

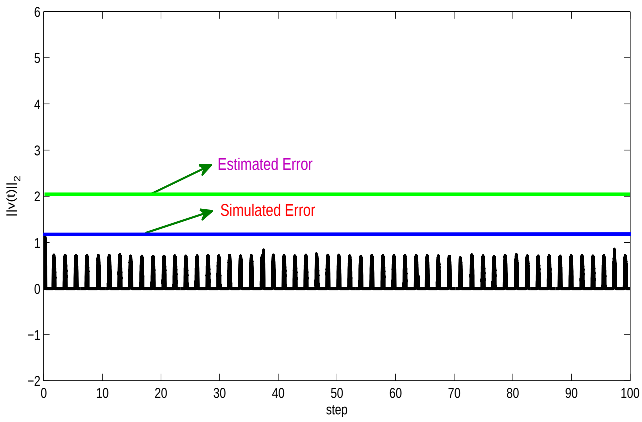

Figure 8: Synchronization error norm $\|v(t)\|$ with controller.

Next, selecting the gain of controller in (25) *with parameter* $\eta = 13$*, that satisfy the conditions presented in Theorem 5.2. The initial conditions of slave system is taken as* $q(\kappa) = \big(q_1(\kappa), q_2(\kappa), q_3(\kappa)\big)^T = (-3.5, 4.5, 2)^T,\ \kappa \in [-2, 0)$*. In simulations, Figures 2 and 4 depicts the trajectories of the directly controlled synchronization states* $p_1(t),\ p_2(t)$ *and* $q_1(t),\ q_2(t)$ *for master system and the corresponding slave system and their errors* $v_1(t),\ v_2(t)$*. The trajectories of the directly controlled synchronization states* $p_3(t),\ q_3(t)$ *and their error* $v_3(t)$ *are provided in Figure 6. The trajectories of* $p_1(t),\ p_2(t),\ p_3(t)$ *and* $q_1(t),\ q_2(t),\ q_3(t)$ *and their errors* $v_1(t),\ v_2(t),\ v_3(t)$ *without control inputs are depicted in Fig-*

ures 3, 5 and 7. It is easy to estimate $6.325 = \xi > \gamma = 3$, $\hat{\pi} = 7.095$ *with* $\varepsilon_1 = 0.6$, $\varepsilon_2 = 1$, $\varepsilon_3 = 1$, *that is the conditions presented in Theorem 5.2,* $\xi - \gamma > 0$ *holds. Thus, the synchronization between the master system and slave system with* $m = 3$ *can be achieved quasi-synchronization with estimated error* $\sqrt{\frac{2\hat{\pi}}{\xi-\gamma}} = 2.066$, *which is displayed in Fig. 8.*

Example 6.3 *Consider the following FCGNNDDs with m=4:*

$$D^{0.99}p_l(t) = -a_l\big(p_l(t)\big)\Big[b_l\big(p_l(t)\big) - \sum_{j=1}^{4} c_{lj}h_j(p_j(t)) - \sum_{j=1}^{4} d_{lj}h_j(p_j(t-\lambda)) - k_l\Big], \qquad (34)$$

where $l = 1,2,3,4$, $\lambda = 0.8$, $k_1 = k_2 = k_3 = 0.5$, $a(p) = diag\{0.8, 0.8, 0.8\}$, $b_j(p) = p$, $h_j(p) = \tanh(p)$, $j = 1,2,3,4$ *and*

$$C = (c_{lj})_{4\times4} = \begin{bmatrix} 1.4 & -2.2 & -0.2 & -0.2 \\ -0.1 & 1.5 & -1.2 & 0.8 \\ -0.5 & -1.2 & 1.3 & 1.1 \\ -1 & -0.2 & 1 & 1.1 \end{bmatrix}, \quad D = (d_{lj})_{4\times4} = \begin{bmatrix} -1 & -0.5 & 1.2 & -2 \\ 2.1 & 1.1 & -1.6 & -0.5 \\ 1.2 & -1.3 & -1 & 1.3 \\ 0.3 & 0.1 & 1.3 & -0.3 \end{bmatrix}.$$

The initial conditions of system (34) is taken as $p(\kappa) = \big(p_1(\kappa), p_2(\kappa), p_3(\kappa), p_4(\kappa)\big)^T = (1.5, 2, -2.5, -0.5)^T$, $\kappa \in [-0.8, 0)$. *Based on Assumptions (1)-(2) and (5A), we have* $\bar{a}_1 = \bar{a}_2 = \bar{a}_3 = 0.5$, $\phi_1 = \phi_2 = \phi_3 = 1$, $u_1 = u_2 = u_3 = 0.6$, $z_1 = z_2 = z_3 = 1$ *and* $\varpi_1 = \varpi_2 = \varpi_3 = 0.2$. *The corresponding slave system*

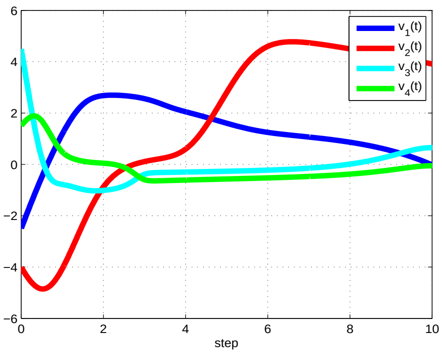

Figure 9: Synchronization error without control

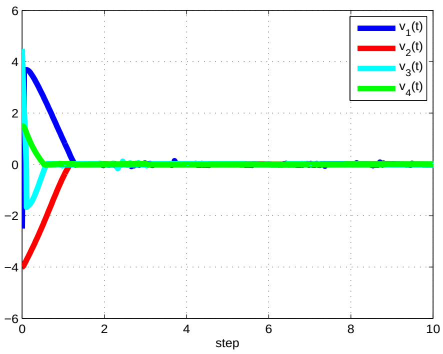

Figure 10: Synchronization error with controller (25)

is defined as

$$D^{0.99}q_l(t) = -a_l\big(q_l(t)\big)\Big[b_l\big(q_l(t)\big) - \sum_{j=1}^{4} c_{lj}h_j(q_j(t)) - \sum_{j=1}^{4} d_{lj}h_j(q_j(t-\lambda)) - k_l - E_l(t)\Big]. \qquad (35)$$

The initial conditions of slave system is taken as $q(\kappa) = \big(q_1(\kappa), q_2(\kappa), q_3(\kappa), q_4(\kappa)\big)^T = (-1, -2, 2, 1)^T$, $\kappa \in [-0.8, 0)$. *If there is no control inputs, the evolution of synchronization error* $v_1(t)$, $v_2(t)$, $v_3(t)$ *and* $v_4(t)$ *are shown in Fig. 9, which implies that system (34) and system (35) can not be global asymptotically synchronized. Under the pinning control inputs (25), three neurons are under control, i.e.,* $\zeta = 3$ *and selecting* $\eta = 5.5$. *It is easy to estimate* $1.875 = \xi > \gamma = 1.02$ *with* $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1.5$, *that is the conditions presented in Corollary 5.3,* $\xi - \gamma > 0$ *holds. Under the controller (25), we can get the state trajectories of synchronization errors* $v_1(t)$, $v_2(t)$, $v_3(t)$ *and* $v_4(t)$ *which are illustrated* *by Fig.10. The states of the error system converge to zero, which shows the validity of the condition check of Corollary 5.3. Thus, the synchronization between the master system and slave system with $m = 4$ can be achieved global asymptotically synchronized via designed pinning control law.*

7 Conclusion

This article dealt with the stability and pinning synchronization analysis of fractional order delayed Cohen-Grossberg neural networks under discontinuous activations. By means of Filippov theory, differential inclusion theoretical analysis, as well as fractional order comparison theorem, the global asymptotic stability criteria of such system was investigated, and some sufficient conditions were proposed via the concept of fractional Lyapunov-functional. At the same time, a novel pinning control strategies were designed for slave systems, and global asymptotical synchronization of FCGNNDDs was obtained in the in the Filippov sense. At last, to help readers understand this article, two numerical examples are provided to demonstrate the effectiveness and validity of the presented results. Moreover, the synchronization issue has played a vital role in engineering applications, such as information sciences [6] and secure communication [12, 18, 25]. Pinning control techniques can be suitable for various glorious dynamics such as FNNs [15, 21, 39], memristor based FNNs [48, 50], fractional order T-S fuzzy neural networks [36, 51] and fractional order complex networks [28, 35]. In the near future, we will try to work on the synchronization results of the above mentioned problems.

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Summaries of Changes for Manuscript: Stability and pinning synchronization analysis of fractional order delayed Cohen-Grossberg neural networks with discontinuous activations

The authors would like to thank the editor and the reviewers for their careful reading and constructive comments. We greatly appreciate the anonymous referees comments and suggestions, which have helped us to improve the manuscript. In light of the comments and suggestions, we have revised the paper very carefully, and addressed all the points raised by the reviewers. We hope this version can be accepted to be published in Applied Mathematics and Computation.

Responses to Editor in Chief

Comment We have received the reports from our advisors on your manuscript, “Stability and pinning synchronization analysis of fractional order delayed Cohen-Grossberg neural networks with discontinuous activations”, submitted to Applied Mathematics and Computation.

However, I would very much like to invite you to revise your paper, seriously taking into account the comments of the reviewers, and to resubmit your revised version.

Response Thank you very much for arranging the reviewers and your kindly suggestions! According to the reviewers’ suggestions, we have checked the paper very carefully and addressed all the questions made by the reviewers. We hope this version can be accepted to be published in Applied Mathematics and Computation. Thanks again!

The specific response to the reviewers are given below.

♢ Response to Reviewer 1:

The comments are as follows:

1. The English should be polished, since some grammatical mistakes have been found. Such as: page 8 in Theorem 4.1, ”inequality” should be changed as ”inequalities”.

Ans: Thanks for your comment. We have corrected all the typo errors and grammatical errors in the revised version.

2. The reviewer thinks that the Lemma 2.4 on page 3 comes from the following paper: Duarte-Mermoud M A, Aguila-Camacho N, Gallegos J A, et al. Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems[J]. Communications in Nonlinear Science and Numerical Simulation, 2015, 22(1-3): 650-659

Ans: Thanks for your careful reading. As per your suggestion, we have changed the Lemma 2.4 in the revised version.

3. Page 6, the first line of Assumption 1: what does the "b_l" stand for?.

Ans: Thanks for your keen observation. In the revised manuscript, we have removed the "b_l".

4. The norms of vector and matrix are not explained clearly. For example: page 7, in inequality (6), the reviewer can't understand what norms of "$\underline{A}$" and "$\underline{B}$" are used.

Ans: Thanks for the productive comment. In the revised manuscript, we have added the clear explanation of the vector and matrix norms.

5. On page 7 in inequality (6), the formula is not compatible since "F" and "R" are all diagonal matrix.

Ans: Thanks a lot for your comment. We have corrected the above mentioned mistakes in the revised version.

6. On page 7 in inequality (7), how can you obtain that $\int_0^t (t-k)^{\mu-1}\|p(k)\|dk \leq \int_0^t (t-k-\lambda)^{\mu-1}\|p(k)\|dk$. The reviewer does not think so!

Ans: Thanks for your comment. We have corrected the above mentioned mistakes in the revised version.

7. On page 8, how can you say that $M(t)$ is non decreasing function when $t \geq 0$? The reviewer does not think so!

Ans: Thanks for your comment. Obviously, the function $M(t)$ is an increasing function.

8. On page 11, what does the idea of controller (25) comes from? Can you make some comparisons with results in [Neural Networks, 2018, 105: 88-103] and [Chaos, Solitons & Fractals, 2018, 110: 105-123]?.

Ans: Thanks for your comments. In the revised manuscript, we have included some comparison results about idea of controller (25), kindly see Remark 5.6.

10. According to figure 4 and Figure 6, the error is obvious, then, can you estimate the error bound for your concerned models?

Ans: Thank you very much for your comment. In the revised manuscript, we have estimated the error bounds for our concerned models.

♢ Response to Reviewer 2:

n the manuscript, the asymptotic stability and asymptotic synchronization analysis of fractional order delayed Cohen-Grossberg neural networks with discontinuous neuron activation functions (FCGNNDDs) is investigated. A novel discontinuous pinning control strategies were designed for slave systems, and global asymptotical synchronization of FCGNNDDs was obtained in the in the Filippov sense. At last, to help readers understand this article, two numerical examples are provided to demonstrate the effectiveness and validity of the presented results. The results given in the paper seem to be correct and have some contribution on related topics. However, the following comments would be helpful to improve the quality of the manuscript.

1. In section 5, the authors should give the conclusion about the minimum number of control nodes for the synchronization of the proposed model.

Ans: Thanks for your motivational comment. Our proposed results will not support by finding the minimum number of control nodes. In the near future, by using higher-degree pinning scheme, we will try to work and how to select the control nodes and also what is the minimum number of control nodes to realize the synchronization of fractional order delayed nonlinear coupled Cohen-Grossberg neural networks with discontinuous neuron activation functions.

2. In the examples, more nodes should be considered for the experiments, which can show the advantages of the proposed pinning controller.

Ans: Thanks for the innovative ideas. In the numerical examples, the increase in the number of nodes will increase the computational complexity. In order to avoid this computational complexity, we have not increased the nodes in the submitted version. Now, due to your suggestion, we have increased the number of nodes in the revised version of our manuscript, kindly see Example 6.3.

3. In the conclusion, it is necessary to explain the application or potential application value of the relevant results.

Ans: Thanks for your motivational comment. In the revised manuscript, we have added the application value of the relevant results in the conclusion section.

4. In simulation part, examples with more detailed discussion will be helpful to illustrate the advantage of the developed results.

Ans: Thanks for your valuable comment. In the revised manuscript, we have included a detailed discussion about the estimation of error bounds for the simulation part.

5. The author should carefully check the expression of the whole paper. Such as "7 conclusion" should be "7 Conclusion".

Ans: Thanks for your comment. In the revised manuscript, we have carefully corrected all those typos.

6. There are some basic references on "fractional-order neural networks" or recent advances on this topic (memrisitor) which should be considered: Finite-time projective synchronization of memristor-based delay fractional-order neural networks. Nonlinear Dynamics, 2017; Adaptive synchronization of memristor-based BAM neural networks with mixed delays. Applied Mathematics and Computation, 2018; Finite-time modified projective synchronization of memristor-based neural network with multi-links and leakage delay, Chaos, Solitons & Fractals, 2018; Finite-time stability and synchronization of memristor-based fractional-order fuzzy cellular neural networks, Communications in Nonlinear Science and Numerical Simulation, 2018.

Ans: Thanks for suggesting such innovative and fruitful publications. In the revised manuscript, we have included the above said papers in the reference section.

♢ Response to Reviewer 3:

This paper deals with global asymptotic stability and asymptotic synchronization problem of fractional order competitive neutral networks with time delays. Finally numerical examples are presented to illustrate the effectiveness of the theoretical analysis. The comments are:

1. There are lots of wrong formula. For example,

(1) Page 3, '$d\mu'$' in Definition 2.2 should be '$d\kappa'$',

(2) Page 3, $q^{(\kappa)}(t_0)$ in Property 1 should be $q^{(\kappa)}(0)$,

(3) Page 3, $E_{\beta,1}(\tau)$ in Definition 2.3 should be $E_{\mu,1}(\tau)$

Ans: Thanks for your careful reading and keen observation. In the revised manuscript, we have corrected all the above mentioned mistakes.

2. The inequality of lemma 2.6 in this manuscript is not found in Ref. [20], I think it should be in the following form.

$$q_1 q_2 \leq \frac{(\varepsilon_1 q_1)^{q_3}}{q_3} + \frac{(\varepsilon_1^{-1} q_2)^{q_3}}{q_3}$$

Ans: Thanks for the valuable comment. In the revised manuscript, we have corrected the inequality in Lemma 2.6 and also included the original reference.

3. In Eq. (6), the amplification result for $\| - A(p(\kappa)B(p(\kappa))\|$ should be $\|\bar{A}\|\|\bar{B}\|\|p(\kappa)\|$ instead of $\|\underline{A}\|\|\underline{B}\|\|p(\kappa)\|$.

Ans: Thanks for your careful reading. As per your suggestion, we have corrected the above mentioned mistakes in the revised version.

4. In Eq. (7), we can obtain that $(t - i - \lambda) < 0$ for all $i \in (t - \lambda, t)$. Thus, the following inequality does not hold.

$$\int_{-\lambda}^{t-\lambda} (t - i - \lambda)^{\mu-1} \|p(i)\| di \leq \int_{-\lambda}^{t} (t - i - \lambda)^{\mu-1} \|p(i)\| di$$

Ans: Thanks a lot for your valuable comment. In the revised manuscript, we have corrected the above mentioned mistakes and also rearranged the proof of Theorem 3.4.

5. In Eq. (27), why the following inequality is true, please give detailed proof.

$$D^\mu G(t) \leq \sum_{l=1}^{m} v_1(t) D^\mu v_1(t) \leq 2 \sum_{l=1}^{m} v_1(t)\, \mathrm{sgn}(v_1(t)) D^\mu v_1(t)$$

Ans: Thanks for the motivational comment. In the revised manuscript, we have given detailed proof of inequality (27).

6. In this paper, the authors talk about the asymptotically synchronization of the addressed models; however, in simulation example 6.2, after considering the Fig.6. we find that the drive-response systems are not synchronized well. Maybe some errors exist in this paper, please check carefully.

Ans: Thanks for your careful reading. The existence of errors is obvious and unavoidable. When compared with the submitted version, we have reduced the errors and shown the synchronization as well and also we have estimated the error bounds for our considered model in the revised manuscript.

7. In the reference section, I found that some of the references have a non-standard format, for example, [8], [15-18], [20].

Ans: Thanks for your suggestion. In the revised manuscript, we have unified all the references in the standard format.

♢ Response to Reviewer 4:

This paper studies the problem of asymptotic stability and asymptotic synchronization analysis of fractional order delayed Cohen-Grossberg neural networks with discontinuous neuron activation functions (FCGNNDDs). The following comments should be considered:

1. In Introduction part, I think the contributions of this paper should be drawn out explicitly.

Ans: Thank you very much for your valuable suggestion. As per your direction, we have shown our contribution as well in the introduction part of our revised mansucript.

2. What is the difference between 'fractional order delayed Cohen-Grossberg neural networks (FDCGNNs) and 'general delayed cohen-Grossberg neural networks (GDCGNNs)?

Ans: Thank a lot for your sensible question. The circuit configuration of GDCGNNs first-order neuron is based on a first-order integral circuit and it consists of one operational amplifier and its related common capacitor and resistors. But FDCGNNs contain fractional-order capacitors(or generalized capacitor). Otherwise, in general, GDCGNNs does not carry any information about the memory and learning mechanisms. But FDCGNNs has nonlocal property, which means the future state of a system depends not only upon its current state but also upon all its previous states. Therefore, FDCGNNs being characterized by infinite memory. The memory effect and differential operators are the main difference between FDCGNN and

GDCGNNs.

3. In Conclusion part, based on the work of this paper, the authors may propose some interesting problems as future work.

Ans: Thanks for your careful reading. As per your direction, we have included the future research topics in the conclusion section.

4. In order to improve the quality of this paper, not only the presentation of this paper should be improved as much as possible, but also the English expression should be checked carefully and improved.

Ans: Thanks for your comment. We have corrected all the typo errors, grammatical error and once again verified in the revised version.

Finally, we would like to take this opportunity to thank the editor and the reviewers again for their constructive comments and useful suggestions, and the time and efforts that they have spent in the review process. Without the expert comments made by the editor and the reviewers, the paper would not be of this quality.

Sincerely yours,

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