LOCAL MAXIMIZERS OF GENERALIZED CONVEX VECTOR-VALUED FUNCTIONS

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ABSTRACT. Any local maximizer of an explicitly quasiconvex real-valued function is actually a global minimizer, if it belongs to the intrinsic core of the function's domain. In this paper we show that similar properties hold for componentwise explicitly quasiconvex vector-valued functions, with respect to the concepts of ideal, strong and weak optimality. We illustrate these results in the particular framework of linear fractional multicriteria optimization problems.

1. Introduction

Generalized convexity plays an important role in both scalar and vector optimization, variational inequalities, equilibrium problems or game theory (see, e.g., the books by Avriel *et al.* [6], Cambini and Martein [13], Crouzeix [16], Göpfert *et al.* [20], Jahn [21, 22], Luc [28], or Stoer and Witzgall [35]).

In particular, quasiconvex functions were studied using normal operators and generalized differentials by Aussel and Hadjisavvas [5], and by Linh and Penot [26, 27]. Optimality conditions for quasiconvex vector optimization were formulated in terms of radial epiderivatives by Flores-Bazán [19] and more recently by Ait Mansour and Riahi [3, 4]. (Semi-)strictly quasiconvex functions were studied by Daniilidis and Garcia Ramos [17] by a variational approach. Generalized convex functions were investigated by Ait Mansour and Aussel [1, 2] in the context of quasimonotone variational inequalities.

Among the various classes of generalized convex functions known in the literature, the (semistrictly/explicitly) quasiconvex functions are of special interest, as they preserve some fundamental properties of convex functions:

• every semistrictly quasiconvex real-valued function satisfies the so-called "*local min - global min*" property, i.e., its local minimizers are actually global minimizers (see, e.g., Ponstein [31, Theorem 2]);

• every explicitly quasiconvex real-valued function satisfies the so-called "*local max - global min*" property, namely any local maximizer is a global minimizer, whenever it belongs to the intrinsic core of the function's domain (see, e.g., Bagdasar and Popovici [7, Theorem 3.1]).

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The aim of this paper is to generalize these two extremal properties for vector functions with values in $\mathbb{R}^m$, with respect to the concepts of ideal, strong and weak optimality, currently used in multiobjective optimization. We show that the "*local min - global min*" property can be actually extended to different classes of semistrictly quasiconvex vector-valued functions, while the "*local max - global min*" property can be generalized for componentwise explicitly quasiconvex vector functions. As applications of these results we obtain new insights on the structure of ideal, strong and weakly optimal solution sets in multicriteria linear fractional programming.

The paper is organized as follows. In Section 2 we recall some notions of convex analysis and vector (i.e. multicriteria) optimization, and we present two preliminary results (Lemmas 2.2 and 2.4).

In Section 3 we study several notions of semistrict/explicit quasiconvexity for vector functions. We establish the relationship between these notions and characterize some of them in terms of level sets. Also, a new characterization of componentwise explicitly quasiconvex vector functions (Theorem 3.17) is given by combining the componentwise semistrict quasiconvexity with a notion of explicit quasiconvexity proposed by Luc and Schaible [29].

Section 4 contains our main results. The vectorial counterparts of the "*local min - global min*" property (Lemmas 4.4, 4.11 and 4.16) rely on the concept of C-quasiconvexity in the sense of Jahn [21, 22]. These lemmas allow us to prove the vectorial counterparts of the "*local max - global min*" property (Theorems 4.6, 4.12 and 4.17). They show that any ideal, strong or weak local maximizer, which belongs to the intrinsic core of the domain, is an ideal, strong or weak global minimizer, respectively. Among other results, we show that the vectorial "*local min - global min*" properties do not ensure the componentwise semistrict quasiconvexity of continuous componentwise quasiconvex functions, in contrast to the scalar case (see, e.g., Elkin [18]).

In Section 5 we apply the main results obtained in the previous section to vector functions whose scalar components are linear fractional, i.e., ratios of affine functions. Multicriteria linear fractional optimization problems have been intensively studied from both theoretical and practical points of view (see, e.g., Cambini and Martein [12], Göpfert *et al.* [20, Section 4.4], Stancu-Minasian [34, Chapter 6] or Yen [36]). By observing that every linear fractional function, as well as its opposite, are explicitly quasiconvex, we can revert the role of minimizers and maximizers in our vectorial counterparts of "*local min - global min*" and "*local max - global min*" properties, obtaining "*local max - global max*" and "*local min - global max*" type properties. By this approach we obtain Theorem 5.1, which gives new insights on the structure of ideal, strong and weakly optimal solution sets, in a general setting. They are illustrated in the particular framework of bicriteria linear fractional optimization problems (Examples 5.3 and 5.4).

Section 6 contains some concluding remarks concerning further possible extensions of our results to appropriate classes of generalized vector-valued or set-valued functions.

2. Preliminaries

2.1. **Algebraic interiors and convexity.** Throughout this paper X is a topological linear space over the field $\mathbb{R}$ of reals. The family of neighborhoods of any point $x \in X$ is denoted by $\mathcal{V}(x)$. For any points $x, y \in X$ we denote $[x, y] = \{(1-t)x + ty \mid t \in [0,1]\}$, $]x, y] = [x, y] \setminus \{x\}$, $[x, y[= [x, y] \setminus \{y\}$ and $]x, y[= [x, y] \setminus \{x, y\}$, the last three sets being nonempty whenever $x \neq y$.

Recall that the core (algebraic interior) and the intrinsic core (relative algebraic interior) of any set $D \subseteq X$ are given by (see, e.g., Holmes [23]):

$$\operatorname{cor} D = \{x \in D \mid \forall\, y \in X,\ \exists\, \delta > 0\ :\ [x,\, x + \delta y] \subseteq D\};$$
$$\operatorname{icr} D = \{x \in D \mid \forall\, y \in \operatorname{span}(D - D),\ \exists\, \delta > 0\ :\ [x,\, x + \delta y] \subseteq D\}.$$

Clearly, $\operatorname{cor} D \subseteq \operatorname{icr} D$. Moreover $\operatorname{int} D \subseteq \operatorname{cor} D$, where $\operatorname{int} D$ denotes the topological interior of a set $D \subseteq X$, since for any $x \in X$ and $V \in \mathcal{V}(x)$ we have $x \in \operatorname{cor} V$. When X is a locally convex space and $D \subseteq X$ is a convex set with nonempty interior, then $\operatorname{int} D = \operatorname{cor} D = \operatorname{icr} D$ (see, e.g., Borwein and Lewis [11]).

Remark 2.1. In the finite dimensional Euclidean space $\mathbb{R}^n$, any nonempty convex set has nonempty intrinsic core. However, this assertion is not true in general infinite dimensional spaces (see, e.g., Holmes [23]).

Lemma 2.2. *Let $D \subseteq X$ be a nonempty set, $x^0, x' \in D$ distinct points and $V \in \mathcal{V}(x^0)$. The following assertions hold:*
1° *If $x^0 \in \operatorname{icr} D$, then there is $x'' \in D \cap V$ such that $x^0 \in]x', x''[$.*
2° *If D is convex, then there is $x'' \in D \cap V$ such that $x'' \in]x^0, x'[$.*

Proof. 1° Since $V \in \mathcal{V}(x^0)$ we have $x^0 \in \operatorname{cor} V$ and there is $\delta_1 > 0$ such that

$$[x^0,\, x^0 + \delta_1(x^0 - x')] \subseteq V. \tag{2.1}$$

As $x^0 \in \operatorname{icr} D$ and $x^0 - x' \in D - D \subseteq \operatorname{span}(D - D)$, there is $\delta_2 > 0$ such that

$$[x^0,\, x^0 + \delta_2(x^0 - x')] \subseteq D. \tag{2.2}$$

Define $x'' = x^0 + \delta(x^0 - x')$ with $\delta = \min\{\delta_1, \delta_2\}$. From (2.1) and (2.2) we have $x'' \in D \cap V$. Also, $x^0 = (1-t)x' + tx'' \in]x', x''[$ for $t = 1/(\delta+1) \in]0, 1[$.

2° Since $V \in \mathcal{V}(x^0)$ we have $x^0 \in \operatorname{cor} V$. Hence $[x^0,\, x^0 + \tau(x' - x^0)] \subseteq V$ for some $\tau > 0$. Choosing $t = \min\{\tau, 1/2\}$, the point $x'' = x^0 + t(x' - x^0)$ belongs to V. The convexity of D also ensures that $x'' \in D$. □

2.2. **Multiobjective optimization.** We endow the real Euclidean space $\mathbb{R}^m$ ($m \geq 1$) with the standard (componentwise) order relation and two strict order relations, defined for any $u = (u_1, \ldots, u_m), v = (v_1, \ldots, v_m) \in \mathbb{R}^m$ by

$$u \leq v \Leftrightarrow u_i \leq v_i,\ \forall\, i \in \{1, \ldots, m\};$$
$$u \lneq v \Leftrightarrow u \leq v \ \text{ and }\ u \neq v;$$
$$u < v \Leftrightarrow u_i < v_i,\ \forall\, i \in \{1, \ldots, m\}.$$

The inverse relations are denoted by $\geq$, $\gneq$ and $>$, respectively.

Let $f = (f_1, \dots, f_m) : D \to \mathbb{R}^m$ be a function, defined on a nonempty set $D \subseteq X$. As usual in vector optimization, we define the sets of *ideal global minimizers*, *strong global minimizers*, and *weak global minimizers* of f, by

$$\begin{aligned} \mathrm{IMin}(D|f) &= \{x^0 \in D \mid \forall\, x \in D : f(x^0) \leq f(x)\}; \\ \mathrm{Min}(D|f) &= \{x^0 \in D \mid \nexists\, x \in D : f(x) \lneq f(x^0)\}; \\ \mathrm{WMin}(D|f) &= \{x^0 \in D \mid \nexists\, x \in D : f(x) < f(x^0)\}. \end{aligned}$$

Similarly, the sets of *ideal global maximizers*, *strong global maximizers*, and *weak global maximizers* of f, are defined by

$$\begin{aligned} \mathrm{IMax}(D|f) &= \{x^0 \in D \mid \forall\, x \in D : f(x^0) \geq f(x)\}; \\ \mathrm{Max}(D|f) &= \{x^0 \in D \mid \nexists\, x \in D : f(x) \gneq f(x^0)\}; \\ \mathrm{WMax}(D|f) &= \{x^0 \in D \mid \nexists\, x \in D : f(x) > f(x^0)\}. \end{aligned}$$

Replacing D in these definitions by a nonempty subset S of D, we obtain the corresponding sets of minimizers and maximizers of f with respect to S.

A point $x^0 \in D$ is called ideal (resp. strong, weak) local minimizer of f if there exists a neighborhood $V \in \mathcal{V}(x^0)$ such that $x^0 \in \mathrm{IMin}(D \cap V|f)$ (resp. $\mathrm{Min}(D \cap V|f)$, $\mathrm{WMin}(D \cap V|f)$). Similarly, $x^0 \in D$ will be called ideal (resp. strong, weak) local maximizer of f if there is $V \in \mathcal{V}(x^0)$ such that $x^0 \in \mathrm{IMax}(D \cap V|f)$ (resp. $\mathrm{Max}(D \cap V|f)$, $\mathrm{WMax}(D \cap V|f)$).

Note that these notions may be found in the vector optimization literature under different names (see, e.g., Göpfert *et al.* [20], Jahn [22], and Luc [28]).

Remark 2.3. Consider a nonempty set $S \subseteq D$. Notice that:

a) The study of minimizers and maximizers can be unified thanks to the relations: $\mathrm{IMax}(S|f) = \mathrm{IMin}(S| - f)$, $\mathrm{Max}(S|f) = \mathrm{Min}(S| - f)$, and $\mathrm{WMax}(S|f) = \mathrm{WMin}(S| - f)$.

b) The following inclusions hold:

$$\mathrm{IMin}(S|f) \subseteq \mathrm{Min}(S|f) \subseteq \mathrm{WMin}(S|f); \tag{2.3}$$

$$\mathrm{IMax}(S|f) \subseteq \mathrm{Max}(S|f) \subseteq \mathrm{WMax}(S|f). \tag{2.4}$$

c) If $\mathrm{IMin}(S|f)$ is nonempty, then it coincides with $\mathrm{Min}(S|f)$. Similarly, the sets $\mathrm{Max}(S|f)$ and $\mathrm{IMax}(S|f)$ coincide whenever the latter is nonempty.

d) It is easily seen that the following relations hold

$$\textstyle\bigcap_{i=1}^m \operatorname{argmin}_{x\in S} f_i(x) = \mathrm{IMin}(S|f), \tag{2.5}$$

$$\textstyle\bigcup_{i=1}^m \operatorname{argmin}_{x\in S} f_i(x) \subseteq \mathrm{WMin}(S|f). \tag{2.6}$$

Of course, similar relations hold for maxima. In particular, when $m = 1$ the three sets in (2.3) coincide with $\operatorname{argmin}_{x\in S} f(x)$, while all sets in (2.4) coincide with $\operatorname{argmax}_{x\in S} f(x)$.

Lemma 2.4. *Let $S \subseteq D$ be a nonempty set. The following are equivalent:*

1° *f is constant on S.*

2° $\mathrm{IMax}(S|f) \cap \mathrm{Min}(S|f) \neq \emptyset$.

3° $\mathrm{Max}(S|f) \cap \mathrm{IMin}(S|f) \neq \emptyset$.

Proof. Clearly, f is constant on S if and only if $\text{IMax}(S|f) = \text{IMin}(S|f) = S$. Thus, implication $1^\circ \Rightarrow 2^\circ$ holds in view of Remark 2.3 c).

For proving $2^\circ \Rightarrow 1^\circ$, assume that some x^0 exists in $\text{IMax}(S|f) \cap \text{Min}(S|f)$ and suppose by contrary that f is not constant on S. Then there is $x \in S$ with $f(x) \neq f(x^0)$. Since $x^0 \in \text{IMax}(S|f)$, it follows that $f(x) \lneq f(x^0)$, contradicting the fact that $x^0 \in \text{Min}(S|f)$. So, 1° and 2° are equivalent.

Similarly, one can prove that 1° and 3° are equivalent. □

Remark 2.5. The following assertions concern Lemma 2.4:

a) A vector function f is constant on D if and only if it has an ideal global maximizer which is also a strong global minimizer (or a strong global maximizer which is an ideal global minimizer).

b) If $x^0 \in D$ is both an ideal local maximizer and a strong local minimizer (or both a strong local maximizer and ideal local minimizer) of f, then f is constant on a neighborhood of x^0, i.e., there is $V \in \mathcal{V}(x^0)$ such that f is constant on $D \cap V$.

c) In contrast to the scalar case, a vector function f is not necessarily constant on $S \subseteq D$, when conditions 2° and 3° of Lemma 2.4 are relaxed to any of the following:

4° $\text{Max}(S|f) \cap \text{Min}(S|f) \neq \emptyset$;

5° $\text{IMax}(S|f) \cap \text{WMin}(S|f) \neq \emptyset$;

6° $\text{WMax}(S|f) \cap \text{IMin}(S|f) \neq \emptyset$.

This is illustrated in the following example.

Example 2.6. Let $f : D = \mathbb{R} \to \mathbb{R}^2$ be a function defined for all $x \in \mathbb{R}$ by

$$f(x) = (x, \max\{-x, 0\}).$$

Obviously, when $S = D$, the function f is not constant on S, while 4° holds since $\text{Max}(S|f) =]-\infty, 0[$ and $\text{Min}(S|f) =]-\infty, 0]$. On the other hand, if we choose $S = [0, 1]$, then f is not constant on S. However, both 5° and 6° hold since $\text{IMax}(S|f) = \{1\}$, $\text{WMin}(S|f) = \text{WMax}(S|f) = [0, 1]$ and $\text{IMin}(S|f) = \{0\}$.

3. Generalized convexity

3.1. **Generalized convex real-valued functions.** In what follows, D will be a nonempty convex subset of the real topological linear space X.

Recall that a function $f : D \to \mathbb{R}$ is said to be (see, e.g., Avriel *et al.* [6]):

- *convex* if for any points $x', x'' \in D$ we have

$$f((1-t)x' + tx'') \leq (1-t)f(x') + tf(x''),\ \forall\, t \in [0, 1];$$

- *quasiconvex* if for any points $x', x'' \in D$ one has

$$f(x') \leq f(x'') \Rightarrow f(x) \leq f(x''),\ \forall\, x \in [x', x''];$$

in other words, f is quasiconvex if for any $x', x'' \in D$ we have

$$f(x) \leq \max\{f(x'), f(x'')\},\ \forall\, x \in\ [x', x''];$$

• *semistrictly quasiconvex* if for any points $x', x'' \in D$ one has

$$f(x') < f(x'') \Rightarrow f(x) < f(x''),\ \forall\, x \in [x', x''[\,;$$

in other words, f is semistrictly quasiconvex if for any $x', x'' \in D$ we have

$$f(x') \neq f(x'') \Rightarrow f(x) < \max\{f(x'), f(x'')\},\ \forall\, x \in]x', x''[.$$

• *explicitly quasiconvex* if f is quasiconvex and semistrictly quasiconvex.

As mentioned by Avriel *et al.* [6, Section 3.4], the notions of semistrict and explicit quasiconvexity have been introduced around 1970. They may be found in the literature under different names. For instance, semistrictly quasiconvex functions are called "strictly quasiconvex" by Ponstein [31]; Stoer and Witzgall [35] use the terms "pseudoconvexity" and "strong quasiconvexity" instead of semistrict quasiconvexity and explicit quasiconvexity, respectively; more recently, Ait Mansour and Aussel [2] adopt "semistrict quasiconvexity" to mean explicit quasiconvexity.

Remark 3.1. Denote the epigraph of any function $f : D \to \mathbb{R}$ by

$$\mathrm{epi}(f) = \{(x, \lambda) \in D \times \mathbb{R} \mid f(x) \leq \lambda\}.$$

Also, for every $\lambda \in \mathbb{R}$ consider the level sets

$$D_f^{\leq}(\lambda) = \{x \in D \mid f(x) \leq \lambda\} \quad \text{and} \quad D_f^{<}(\lambda) = \{x \in D \mid f(x) < \lambda\}.$$

The following geometric characterizations hold:

a) f is convex if and only epi(f) is convex in the product space $X \times \mathbb{R}$.

b) f is quasiconvex if and only if $D_f^{\leq}(\lambda)$ is convex for any $\lambda \in \mathbb{R}$. Also, f is quasiconvex if and only if $D_f^{<}(\lambda)$ is convex for any $\lambda \in \mathbb{R}$.

c) f is semistrictly quasiconvex if and only if for any $x', x'' \in D$ with $x' \in D_f^{<}(f(x''))$ we have $[x', x''[\, \subseteq D_f^{<}(f(x''))$.

d) f is explicitly quasiconvex if and only if for any $\lambda \in \mathbb{R}$ and $x', x'' \in D$ with $x' \in D_f^{<}(\lambda)$ and $x'' \in D_f^{\leq}(\lambda)$ we have $[x', x''[\, \subseteq D_f^{<}(\lambda)$ (see Popovici [33, Proposition 3.1]).

The next result collects some useful properties of (generalized) convex functions (see, e.g., Stoer and Witzgall [35, Section 4.9]).

Proposition 3.2. *For any function $f : D \to \mathbb{R}$, the following hold:*

1° *f is convex (resp. quasiconvex, semistrictly quasiconvex, explicitly quasiconvex) if and only if the restriction of f to any line segment in D is convex (resp. quasiconvex, semistrictly quasiconvex, explicitly quasiconvex).*

2° *If D is an interval of $X = \mathbb{R}$, then f is explicitly quasiconvex if and only if there exist three (possibly degenerated) intervals D_1, D_2 and D_3, with $D = D_1 \cup D_2 \cup D_3$, $\sup D_1 \leq \inf D_2$ and $\sup D_2 \leq \inf D_3$, such that f is decreasing on D_1, constant on D_2 and increasing on D_3.*

3° *If f is convex, then it is explicitly quasiconvex.*

4° *If f is semistrictly quasiconvex and lower semicontinuous, then it is quasiconvex, hence explicitly quasiconvex.*

Lemma 3.3. *Let $f : D \to \mathbb{R}$ be an explicitly quasiconvex function and let x', x^0, x'' be distinct points in D with $x^0 \in]x', x''[$. For any $u \in [x', x^0[$ and $v \in]x^0, x'']$ we have:*

a) $f(u) < f(x^0) < f(v)$, *whenever* $f(x') < f(x^0)$;
b) $f(u) \leq f(x^0) \leq f(v)$, *whenever* $f(x') = f(x^0)$.

Proof. Follows from Proposition 3.2 (1° and 2°), applied to the explicitly quasiconvex real-valued function mapping $t \in [0,1]$ to $f((1-t)x' + tx'')$. □

Example 3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \min\{-x, 0\}$. Obviously f is quasiconvex but not semistrictly quasiconvex. Hence, f is not explicitly quasiconvex.

Example 3.5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0. \end{cases}$$

It is easily seen that f is semistrictly quasiconvex but not quasiconvex. Hence, f is not explicitly quasiconvex.

Example 3.6. Let $a, b \in \mathbb{R}^n$ $(n \geq 1)$ and $\alpha, \beta \in \mathbb{R}$ with $(b, \beta) \neq (0, 0)$. Consider a linear fractional function $f : D \to \mathbb{R}$, defined by

$$f(x) = \frac{\langle a, x\rangle + \alpha}{\langle b, x\rangle + \beta}, \ \forall\, x \in D,$$

where $D \subseteq \{x \in \mathbb{R}^n \mid \langle b, x\rangle + \beta > 0\}$ is a nonempty convex set and $\langle\cdot,\cdot\rangle$ denotes the usual inner product. It is known that f is explicitly quasiconvex and $-f$ is explicitly quasiconvex as well (indeed, f is pseudolinear in the sense of Chew and Choo [14], hence explicitly quasilinear in the sense of Malivert and Popovici [30]). Note that, in general, the linear fractional functions are neither convex nor concave.

3.2. **Generalized convex vector-valued functions.** Consider a vector function, $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$, defined on the nonempty convex set $D \subseteq X$. We say that f is:

• *componentwise convex* if $f_1, \ldots, f_m$ are convex, which means actually that for any points $x', x'' \in D$ we have

$$f((1-t)x' + tx'') \leq (1-t)f(x') + tf(x''), \ \forall\, t \in [0,1];$$

• *componentwise quasiconvex* (or, equivalently, *cone-quasiconvex in the sense of Luc* [28, Definition 6.1] with respect to the ordering cone $\mathbb{R}^m_+$, according to Benoist *et al.* [8]) if $f_1, \ldots, f_m$ are quasiconvex, which means that for any points $x', x'' \in D$ one has

$$f(x) \leq \max\{f(x'), f(x'')\}, \ \forall\, x \in [x', x''],$$

where $\max\{u, v\} = (\max\{u_1, v_1\}, \ldots, \max\{u_m, v_m\})$ is defined in the lattice $(\mathbb{R}^m, \leq)$ for all $u = (u_1, \ldots, u_m), v = (v_1, \ldots, v_m) \in \mathbb{R}^m$.

• *quasiconvex in the sense of Jahn* [22, Definition 7.9] (w.r.t. the ordering cone $\mathbb{R}^m_+$) if for any points $x', x'' \in D$ we have

$$f(x') \leq f(x'') \Rightarrow f(x) \leq f(x''),\ \forall\, x \in [x', x''].$$

Remark 3.7. Similarly to scalar functions (see Remark 3.1), given a vector function $f = (f_1, \dots, f_m) : D \to \mathbb{R}^m$, we can define the epigraph

$$\mathrm{epi}(f) = \{(x, v) \in D \times \mathbb{R}^m \mid f(x) \leq v\}$$

as well as the level sets

$$D_f^{\leq}(v) = \{x \in D \mid f(x) \leq v\} \quad \text{and} \quad D_f^{<}(v) = \{x \in D \mid f(x) < v\}$$

where $v \in \mathbb{R}^m$ is any point. The following assertions hold:

a) f is componentwise convex if and only epi(f) is convex in $X \times \mathbb{R}^m$.

b) f is componentwise quasiconvex if and only if $D_f^{\leq}(v)$ is convex for any $v \in \mathbb{R}^m$; also, f is quasiconvex if and only if $D_f^{<}(v)$ is convex for any $v \in \mathbb{R}^m$.

c) f is quasiconvex in the sense of Jahn if and only if for any $x', x'' \in D$ with $x' \in D_f^{\leq}(f(x''))$ we have $[x', x''] \subseteq D_f^{\leq}(f(x''))$.

Remark 3.8. The following statements concern the relationship between the quasiconvexity notions defined above:

a) For $m = 1$ both componentwise quasiconvexity and quasiconvexity in the sense of Jahn coincide with the usual quasiconvexity of scalar functions.

b) If f is componentwise quasiconvex, then it is quasiconvex in the sense of Jahn (w.r.t. the ordering cone $\mathbb{R}^m_+$).

c) When $m \geq 2$, the quasiconvexity in the sense of Jahn does not imply the componentwise quasiconvexity. As a counterexample, consider the function $f : \mathbb{R} \to \mathbb{R}^2$, given by

$$f(x) = \begin{cases} (0, 1) & \text{if} \quad x < 0 \\ (2, 2) & \text{if} \quad x = 0 \\ (1, 0) & \text{if} \quad x > 0. \end{cases}$$

Another interesting counterexample (where $f : \mathbb{R}^2 \to \mathbb{R}^4$ is even continuous), has been constructed by Cambini and Martein [12, page 170].

Various concepts of semistrict quasiconvexity for vector-valued functions are known in the literature. Three of them play a key role in our paper and are presented below. A function $f = (f_1, \dots, f_m) : D \to \mathbb{R}^m$ is called:

• *componentwise semistrictly quasiconvex* if all functions $f_1, \dots, f_m$ are semistrictly quasiconvex;

• $(\lneq, \lneq)$-*semistrictly quasiconvex* if for any points $x', x'' \in D$ we have

$$f(x') \lneq f(x'') \Rightarrow f(x) \lneq f(x''),\ \forall\, x \in [x', x''[\,;$$

• $(<, <)$-*semistrictly quasiconvex* if for any points $x', x'' \in D$ one has

$$f(x') < f(x'') \Rightarrow f(x) < f(x''),\ \forall\, x \in [x', x''[.$$

Remark 3.9. By means of the level sets $D_f^<(v)$, with $v \in \mathbb{R}^m$, defined in Remark 3.7, it is easily seen that f is $(<,<)$-semistrictly quasiconvex if and only if for any $x', x'' \in D$ with $x' \in D_f^<(x'')$ we have $[x', x''[\subseteq D_f^<(x'')$. Similarly, we can define the level sets of type

$$D_f^{\lneqq}(v) = \{x \in D \mid f(x) \lneqq v\}$$

for any $v \in \mathbb{R}^m$. Then, f is $(\lneqq, \lneqq)$-semistrictly quasiconvex if and only if for any points $x', x'' \in D$ with $x' \in D_f^{\lneqq}(x'')$ we have $[x', x''[\subseteq D_f^{\lneqq}(x'')$.

Remark 3.10. The following statements hold:

a) When $m = 1$, the three notions of semistrict quasiconvexity defined above coincide with the usual semistrict quasiconvexity of scalar functions.

b) If a vector function is componentwise semistrictly quasiconvex, then it is $(<,<)$-semistrictly quasiconvex.

c) There are componentwise semistrictly quasiconvex functions which are not $(\lneqq, \lneqq)$-semistrictly quasiconvex, as for instance $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$, defined by

$$f_1(x) = x, \quad f_2(x) = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0. \end{cases}$$

Indeed, both scalar functions f_1, f_2 are semistrictly quasiconvex, hence the function f is componentwise semistrictly quasiconvex. For $x' = -1$, $x'' = 1$ and $x^0 = 0$ one has $f(x') \lneqq f(x'')$, but $f(x^0) \not\lneqq f(x'')$. This shows that f is not $(\lneqq, \lneqq)$-semistrictly quasiconvex.

d) There are functions which are $(<,<)$-semistrictly quasiconvex, but not componentwise semistrictly quasiconvex. Indeed, consider the vector function $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$, defined by

$$f_1(x) = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0, \end{cases} \qquad f_2(x) = -f_1(x).$$

Observe that the image set $f(\mathbb{R})$ consists of the two points $(0,0)$ and $(1,-1)$. Hence, the vector function f is $(<,<)$-semistrictly quasiconvex, since there are no points $x', x'' \in D = \mathbb{R}$ such that $f(x') < f(x'')$. However, its second component f_2 is not semistrictly quasiconvex.

Remark 3.11. Both $(\lneqq, \lneqq)$-semistrict quasiconvexity and $(<,<)$-semistrict quasiconvexity are related to the notion of "C-quasiconvexity" defined by Jahn [22, Definition 7.11], as follows:

Let $f : S \to Y$ be a vector function, defined on a nonempty set $S \subseteq X$ (not necessarily convex) taking values in a real linear space Y. Let C be a nonempty subset of Y. Then, f is called *C-quasiconvex at a point* $x^0 \in S$ if for any $x' \in S \setminus \{x^0\}$ with $f(x^0) - f(x') \in C$, there is some $x'' \in S \setminus \{x^0\}$ satisfying the following two conditions: $]x^0, x''] \subseteq S$ and $f(x^0) - f(x) \in C$ for all $x \in]x^0, x'']$ (the first condition being superfluous, whenever S is convex).

In the particular framework of our paper, where $S = D$ is convex and $Y = \mathbb{R}^m$, we obtain the following result, which relates the C-quasiconvexity to the three concepts of semistrict quasiconvexity defined above.

Theorem 3.12. *Let $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$ be a vector function, defined on a nonempty convex set $D \subseteq X$. The following assertions hold:*

1° If f is componentwise semistrictly quasiconvex, then it is C-quasiconvex at any point of D, for $C = \mathbb{R}^m \setminus (-\mathbb{R}^m_+)$.

2° If f is $(\lneq, \lneq)$-semistrictly quasiconvex, then it is C-quasiconvex at any point of D, for $C = \mathbb{R}^m_+ \setminus \{0\}$.

3° If f is $(<, <)$-semistrictly quasiconvex, then it is C-quasiconvex at any point of D, for $C = \operatorname{int} \mathbb{R}^m_+$.

Proof. 1° Let $x^0 \in D$ and $x' \in D \setminus \{x^0\}$ with $f(x^0) - f(x') \in \mathbb{R}^m \setminus (-\mathbb{R}^m_+)$. Then, there is $i \in \{1, \ldots, m\}$ such that $f_i(x') < f_i(x^0)$. As f_i is semistrictly quasiconvex, we have $f_i(x) < f_i(x^0)$ for all $x \in]x^0, x']$. Therefore, we can choose $x'' = x' \in D \setminus \{x^0\}$, which satisfies the conditions $]x^0, x''] \subseteq D$ and $f(x^0) - f(x) \in \mathbb{R}^m \setminus (-\mathbb{R}^m_+)$ for all $x \in]x^0, x'']$.

2° Let $x^0 \in D$ and $x' \in D \setminus \{x^0\}$ with $f(x^0) - f(x') \in \mathbb{R}^m_+ \setminus \{0\}$, i.e., $f(x') \lneq f(x^0)$. Since f is $(\lneq, \lneq)$-semistrictly quasiconvex, we infer that $f(x) \lneq f(x^0)$ for all $x \in]x^0, x']$. Then, for $x'' = x' \in D \setminus \{x^0\}$, we have $]x^0, x''] \subseteq D$ and $f(x^0) - f(x) \in \mathbb{R}^m_+ \setminus \{0\}$ for all $x \in]x^0, x'']$.

3° The proof follows a similar argument as above. □

Remark 3.13. Consider the particular case of scalar functions ($m = 1$). In view of Remark 3.10 a), the three semistrict quasiconvexity notions involved in Theorem 3.12 coincide. On the other hand, since $m = 1$ we also have

$$\mathbb{R}^m \setminus (-\mathbb{R}^m_+) = \mathbb{R}^m_+ \setminus \{0\} = \operatorname{int} \mathbb{R}^m_+ =]0, \infty[.$$

Therefore the C-quasiconvexity reduces to the $]0, \infty[$-quasiconvexity in all assertions 1°, 2° and 3°. In extenso, this means that f is $]0, \infty[$-quasiconvex at a point $x^0 \in D$ if and only if for any $x' \in D \setminus \{x^0\}$ with $f(x^0) > f(x')$, there is some $x'' \in D \setminus \{x^0\}$ with $f(x^0) > f(x)$ for all $x \in]x^0, x'']$.

Notice that the $]0, \infty[$-quasiconvexity at every point does not imply semistrict quasiconvexity, as shown by the following example.

Example 3.14. Let $f : D = \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \{0, 1\} \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus \{0, 1\}. \end{cases}$$

Clearly, function f is $]0, \infty[$-quasiconvex at any $x^0 \in \mathbb{R} \setminus \{0, 1\}$, as there are no points $x' \in D \setminus \{x^0\}$ with $f(x^0) > f(x')$. Also, if $x^0 \in \{0, 1\}$, then for any point $x' \in \mathbb{R} \setminus \{x^0\}$ satisfying $f(x^0) > f(x')$ we actually have $x' \in \mathbb{R} \setminus \{0, 1\}$. Therefore one can choose $x'' = 1/2$ so that $f(x^0) > f(x)$ for all $x \in]x^0, x'']$. Hence f is $]0, \infty[$-quasiconvex at x^0. However, f is not semistrictly quasiconvex. Indeed, for $x' = -1$, $x'' = 1$ and $x = 0 \in]x', x''[$, we have $f(x') < f(x'')$, but the inequality $f(x) < f(x'')$ does not hold.

In what concerns the explicit quasiconvexity of vector-valued functions, there are two concepts relevant for the purpose of our paper. A function $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$ is called:

• *componentwise explicitly quasiconvex* if all functions $f_1, \dots, f_m$, are explicitly quasiconvex;

• $(\neq, \lneq)$-*explicitly quasiconvex* if f is componentwise quasiconvex and for any points $x', x'' \in D$ one has

$$f(x') \neq f(x'') \Rightarrow f(x) \lneq \max\{f(x'), f(x'')\}, \ \forall\, x \in]x', x''[.$$

The notion of $(\neq, \lneq)$-explicit quasiconvexity has been introduced by Luc and Schaible [29] using a different terminology. We will use it to characterize componentwise explicit quasiconvexity (see Theorem 3.17).

Remark 3.15. The following statements hold:

a) When $m = 1$, the classical explicit quasiconvexity of scalar functions is recovered from both explicit quasiconvexity concepts presented above.

b) For $m \geq 2$ there are vector-valued functions which are $(\neq, \lneq)$-explicitly quasiconvex but not componentwise explicitly quasiconvex. For example, consider the function $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f_1(x) = x, \quad f_2(x) = \min\{-x, 0\}.$$

As f_1 and f_2 are monotonic, function f is componentwise quasiconvex, i.e., for any $x', x'' \in \mathbb{R}$ we have $f(x) \leq \max\{f(x'), f(x'')\}$, for all $x \in [x', x'']$. Assume that $f(x') \neq f(x'')$, hence $x' \neq x''$ and the strict monotonicity of f_1 implies $f_1(x) < \max\{f_1(x'), f_1(x'')\}$, therefore $f(x) \lneq \max\{f(x'), f(x'')\}$ for all $x \in]x', x''[$. This shows that f is $(\neq, \lneq)$-explicitly quasiconvex. On the other hand, by Example 3.4 it follows that f_2 is not explicitly quasiconvex, hence f is not componentwise explicitly quasiconvex.

c) If a vector-valued function f is $(\neq, \lneq)$-explicitly quasiconvex, then it is $(\lneq, \lneq)$-semistrictly quasiconvex. The converse of this implication is false even when $m = 1$ (see Remarks 3.10 a) and 3.15 a)), as illustrated by the function given in Example 3.5.

d) There are vector-valued functions which are $(\neq, \lneq)$-explicitly quasiconvex, but not $(<, <)$-semistrictly quasiconvex. For instance, consider the function $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$, defined by

$$f_1(x) = x, \quad f_2(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0. \end{cases}$$

Clearly, f is $(\neq, \lneq)$-explicitly quasiconvex. However, choosing the points $x' = -1$, $x'' = 1$ and $x^0 = 0$, one has $f(x') < f(x'')$ but $f(x^0) \not< f(x'')$. Hence, f is not $(<, <)$-semistrictly quasiconvex.

e) There also exist vector-valued functions which are $(<, <)$-semistrictly quasiconvex, but not $(\neq, \lneq)$-explicitly quasiconvex. As discussed already for Remark 3.15 c), the function given in Example 3.5 has this property.

Remark 3.16. Componentwise explicit quasiconvexity can be characterized by means of level sets, similarly to Remark 3.1 d). Actually, according to Popovici [33, Proposition 3.1 and Theorem 3.1], a vector function f is componentwise explicitly quasiconvex if and only if for for any $v \in \mathbb{R}^m$ and $x', x'' \in D$ with $x' \in D_f^{<}(v)$ and $x'' \in D_f^{\leq}(v)$ we have $[x', x''[\subseteq D_f^{<}(v)$.

We conclude this section with a new characterization of componentwise explicitly quasiconvex functions.

Theorem 3.17. *The following assertions are equivalent:*

1° *f is componentwise explicitly quasiconvex.*

2° *f is both $(\neq, \lneq)$-explicitly quasiconvex and componentwise semistrictly quasiconvex.*

Proof. Assume 1° holds. Let $x', x'' \in D$ be such that $f(x') \neq f(x'')$ and $x \in]x', x''[$. As $f_j(x') \neq f_j(x'')$ for some $j \in \{1, \ldots, m\}$, the semistrictly quasiconvexity of f_j implies $f_j(x) < \max\{f_j(x'), f_j(x'')\}$, which ensures that $f(x) \neq \max\{f(x'), f(x'')\}$. We also have $f(x) \leq \max\{f(x'), f(x'')\}$, by the componentwise quasiconvexity of f. Therefore $f(x) \lneq \max\{f(x'), f(x'')\}$, thus 2° holds. Implication $2^\circ \Rightarrow 1^\circ$ follows from the definitions. □

4. Extremal properties of generalized convex functions

As in the previous section, in what follows D represents a nonempty convex subset of a real topological linear space X.

It is well-known that the following "*local min - global min*" property holds within the class of semistrictly quasiconvex scalar functions:

Proposition 4.1 (Ponstein [31, Theorem 2])**.** *Let $f : D \to \mathbb{R}$ be a semistrictly quasiconvex function. A point $x^0 \in D$ is a local minimizer of f if and only if it is a global minimizer.*

Recently we have obtained a "*local max - global min*" property concerning explicitly quasiconvex scalar functions.

Proposition 4.2 (Bagdasar and Popovici [7, Theorem 3.1])**.** *Let $f : D \to \mathbb{R}$ be an explicitly quasiconvex function and let $x^0 \in \operatorname{icr} D$. If x^0 is a local maximizer of f, then x^0 is a global minimizer.*

In what follows we establish generalizations of Proposition 4.1 for the three types of semistrict quasiconvex vector-valued functions presented in Section 3, with respect to ideal, strong and weak minimality. Based on these new results, we extend Proposition 4.2 for componentwise explicitly quasiconvex vector-valued functions.

4.1. Ideal local minimizers/maximizers vs. global minimizers.

The following result concerns ideal minimizers. It represents a new counterpart of two results obtained by Jahn [22, Theorems 7.15 and 7.16] for strong and weak minimizers.

Proposition 4.3. *Let $x^0 \in S$ be an ideal local minimizer of a function $f : S \to \mathbb{R}^m$ defined on a nonempty set $S \subseteq X$ (not necessarily convex). The following assertions are equivalent:*

1° *x^0 is an ideal global minimizer of f, i.e., $x^0 \in \mathrm{IMin}(S|f)$.*

2° *f is C-quasiconvex at x^0, for $C = \mathbb{R}^m \setminus (-\mathbb{R}^m_+)$.*

Proof. If 1° holds, then $f(x^0) \leq f(x)$ for all $x \in S$, hence there is no $x' \in S \setminus \{x^0\}$ such that $f(x^0) - f(x') \in C = \mathbb{R}^m \setminus (-\mathbb{R}^m_+)$. Therefore 2° follows trivially by definition of C-quasiconvexity.

Conversely, assume that 2° holds. Since x^0 is an ideal local minimizer, we can find a neighborhood $V \in \mathcal{V}(x^0)$ such that $x^0 \in \operatorname{IMin}(S \cap V|f)$. Suppose by the contrary that 1° does not hold, i.e., $x^0 \notin \operatorname{IMin}(S|f)$. Then, there is $x' \in S$ for which $f(x^0) \not\leq f(x')$. This shows that $x' \in S \setminus \{x^0\}$ and $f(x^0) - f(x') \in \mathbb{R}^m \setminus (-\mathbb{R}^m_+) = C$. Since f is C-quasiconvex at x^0, we infer the existence of a point $x'' \in S \setminus \{x^0\}$ with $]x^0, x''] \subseteq S$ and

$$f(x^0) - f(x) \in C \text{ for all } x \in]x^0, x'']. \tag{4.1}$$

Since $V \in \mathcal{V}(x^0)$ we have $x^0 \in \operatorname{cor} V$, therefore one can find $\delta > 0$, which satisfies $[x^0, x^0 + \delta(x'' - x^0)] \subseteq V$. Defining $x^* = x^0 + t(x'' - x^0)$ with $t = \min\{\delta, 1\}$, we have $x^* \in]x^0, x''] \cap [x^0, x^0 + \delta(x'' - x^0)] \subseteq S \cap V$. By (4.1) we infer that $f(x^0) - f(x^*) \in C$, i.e., $f(x^0) \not\leq f(x^*)$. Therefore we have $x^0 \notin \operatorname{IMin}(S \cap V|f)$, a contradiction. □

Lemma 4.4. *Let $x^0 \in D$. If $f : D \to \mathbb{R}^m$ is a componentwise semistrictly quasiconvex (in particular, componentwise explicitly quasiconvex) function, then the following assertions are equivalent:*

1° *x^0 is an ideal local minimizer of f;*

2° *x^0 is an ideal global minimizer of f, i.e., $x^0 \in \operatorname{IMin}(D|f)$.*

Proof. Follows from Theorem 3.12 (1°) and Proposition 4.3. □

Remark 4.5. Lemma 4.4 extends the results previously obtained for scalar functions ($m = 1$) by Ponstein [31, Theorem 2], and Bagdasar and Popovici [7, Lemma 3.1].

Theorem 4.6. *Let $f : D \to \mathbb{R}^m$ be a componentwise explicitly quasiconvex function and let $x^0 \in \operatorname{icr} D$. If x^0 is an ideal local maximizer of f, then it is an ideal global minimizer, i.e., $x^0 \in \operatorname{IMin}(D|f)$.*

Proof. Let $V \in \mathcal{V}(x^0)$ be a neighborhood such that $x^0 \in \operatorname{IMax}(D \cap V|f)$, i.e., $f(x) \leq f(x^0)$ for all $x \in D \cap V$. Assume by contrary that $x^0 \notin \operatorname{IMin}(D \cap V|f)$. Then there is $x' \in D \cap V$ satisfying $f(x^0) \not\leq f(x')$. This ensures the existence of $j \in \{1, \ldots, m\}$ such that $f_j(x') < f_j(x^0)$, hence $x^0 \neq x'$. As $x^0 \in \operatorname{icr} D$, from Lemma 2.2 ($1^\circ$) we can find $x'' \in D \cap V$ such that $x^0 \in]x', x''[$. From Lemma 3.3 a), we have $f_j(x') < f_j(x^0) < f_j(x'')$, a contradiction with $x^0 \in \operatorname{IMax}(D \cap V|f)$. Therefore, $x^0 \in \operatorname{IMin}(D \cap V|f)$, hence by Lemma 4.4 if follows $x^0 \in \operatorname{IMin}(D|f)$, i.e., x^0 is an ideal global minimizer of f. □

Remark 4.7. In view of Lemma 2.4, we easily deduce from Theorem 4.6 that any componentwise explicitly quasiconvex function f, which possesses an ideal local maximizer $x^0 \in \operatorname{icr} D$, is actually constant on a neighborhood of x^0. In particular, if x^0 is an ideal global maximizer, then f is constant on D. A function which satisfies the hypotheses of Theorem 4.6 is provided in the following example.

Example 4.8. Let $f = (f_1, \dots, f_m) : D = \mathbb{R} \to \mathbb{R}^m$ be defined by

$$f_1(x) = \cdots = f_m(x) = \max\{0, x\}.$$

It is easy to check that f is componentwise explicitly quasiconvex. Moreover, every point $x^0 \in]-\infty, 0[$ is an ideal local maximizer of f, and function f is constant on $V =]-\infty, 0[\in \mathcal{V}(x^0)$. However, f is not constant on D, since it has no ideal global maximizers.

Remark 4.9. Theorem 4.6 extends several known results, obtained for real-valued functions ($m = 1$) by Zălinescu [37, Proposition 2.5.8 (ii)], Cambini and Martein [12, Lemma 4.1], and Bagdasar and Popovici [7, Theorem 3.1].

4.2. **Strong local minimizers/maximizers vs. global minimizers.** We start this subsection by presenting a particular instance of a known result of Jahn [22, Theorem 7.15].

Proposition 4.10. *Let $f : S \to \mathbb{R}^m$ be a function defined on a nonempty set $S \subseteq X$ (not necessarily convex). Let $x^0 \in S$ be a strong local minimizer of f. The following assertions are equivalent:*
1° *x^0 is a strong global minimizer of f, i.e., $x^0 \in \mathrm{Min}(S|f)$.*
2° *f is C-quasiconvex at x^0, for $C = \mathbb{R}^m_+ \setminus \{0\}$.*

Lemma 4.11. *Let $x^0 \in D$. If function $f : D \to \mathbb{R}^m$ is $(\lneq, \lneq)$-semistrictly quasiconvex (in particular, componentwise explicitly quasiconvex), then the following assertions are equivalent:*
1° *x^0 is a strong local minimizer of f;*
2° *x^0 is a strong global minimizer of f, i.e., $x^0 \in \mathrm{Min}(D|f)$.*

Proof. Follows by Theorems 3.12 (2°) and 3.17, and Proposition 4.10. □

Theorem 4.12. *Let $f : D \to \mathbb{R}^m$ be a componentwise explicitly quasiconvex function and let $x^0 \in \operatorname{icr} D$. If x^0 is a strong local maximizer of f, then it is a strong global minimizer of f, i.e., $x^0 \in \mathrm{Min}(D|f)$.*

Proof. Let $V \in \mathcal{V}(x^0)$ be such that $x^0 \in \mathrm{Max}(D \cap V|f)$. Assume by the contrary that $x^0 \notin \mathrm{Min}(D \cap V|f)$, so there is $x' \in D \cap V$ satisfying

$$f(x') \lneq f(x^0). \tag{4.2}$$

Notice that $x^0 \neq x'$. Since $x^0 \in \operatorname{icr} D$, from Lemma 2.2 (1°) we find a point $x'' \in D \cap V$ such that $x^0 \in]x', x''[$. As relation (4.2) implies $f(x') \leq f(x^0)$, Lemma 3.3 a) and b) ensures

$$f(x^0) \leq f(x''). \tag{4.3}$$

From (4.2) and (4.3), there is $j \in \{1, \dots, m\}$ with $f_j(x') < f_j(x^0) \leq f_j(x'')$. As $f_j(x') \neq f_j(x'')$, the semistrict quasiconvexity of f_j yields

$$f_j(x^0) < \max\{f_j(x'), f_j(x'')\} = f_j(x'').$$

By means of (4.3), we infer $f(x'') \gneq f(x^0)$, which contradicts the fact that $x^0 \in \mathrm{Max}(D \cap V|f)$. We conclude that $x^0 \in \mathrm{Min}(D \cap V|f)$, which implies $x^0 \in \mathrm{Min}(D|f)$ by Lemma 4.11. □

Example 4.13. Let function $f : D = \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (x, -x)$. It is easy to see that every point $x^0 \in \mathbb{R}$ is a strong local (even global) maximizer, as well as a strong global minimizer. However, f is not locally constant, in contrast to the conclusion of Remark 4.7. This is because function f has no ideal local maximizers.

Remark 4.14. Lemma 4.11 extends some known results, including those mentioned in Remark 4.5 for scalar functions, and a result obtained for vector functions by Luc and Schaible [29, Theorem 4.1]. Also, the results mentioned in Remark 4.9, which concern scalar functions, can be recovered as consequences of Theorem 4.12.

4.3. **Weak local minimizers/maximizers vs. global minimizers.** We present a particular instance of a known result of Jahn [22, Theorem 7.16].

Proposition 4.15. *Let $f : S \to \mathbb{R}^m$ be a function defined on a nonempty set $S \subseteq X$ (not necessarily convex). Let $x^0 \in S$ be a weak local minimizer of f. The following assertions are equivalent:*
1° *x^0 is a weak global minimizer of f, i.e., $x^0 \in \mathrm{WMin}(S|f)$.*
2° *f is C-quasiconvex at x^0, for $C = \mathrm{int}\,\mathbb{R}^m_+$.*

Lemma 4.16. *If function $f : D \to \mathbb{R}^m$ is $(<,<)$-semistrictly quasiconvex (in particular, componentwise semistrictly quasiconvex, or componentwise explicitly quasiconvex), then for any point $x^0 \in D$, the following assertions are equivalent:*
1° *x^0 is a weak local minimizer of f;*
2° *x^0 is a weak global minimizer of f, i.e., $x^0 \in \mathrm{WMin}(D|f)$.*

Proof. Follows by Theorem 3.12 (3°) and Proposition 4.15, and in particular by Remark 3.10 b). □

Theorem 4.17. *Let $f : D \to \mathbb{R}^m$ be a componentwise explicitly quasiconvex function and let $x^0 \in \mathrm{icr}\, D$. If x^0 is a weak local maximizer of f, then it is a weak global minimizer of f, i.e., $x^0 \in \mathrm{WMin}(D|f)$.*

Proof. Let $V \in \mathcal{V}(x^0)$ be such that $x^0 \in \mathrm{WMax}(D \cap V|f)$. Assume by the contrary that $x^0 \notin \mathrm{WMin}(D \cap V|f)$, so there is $x' \in D \cap V$ satisfying

$$f(x') < f(x^0). \tag{4.4}$$

Notice that $x' \neq x^0$. Since $x^0 \in \mathrm{icr}\, D$, from Lemma 2.2 (1°) we find a point $x'' \in D \cap V$ such that $x^0 \in]x', x''[$. Lemma 3.3 a) and (4.4) ensure that

$$f_j(x') < f_j(x^0) < f_j(x''),\ \forall\, j \in \{1, \ldots, m\}. \tag{4.5}$$

As $x^0 \in \mathrm{WMax}(D\cap V|f)$ implies $f(x'') \not> f(x^0)$, there is $k \in \{1, \ldots, m\}$ such that $f_k(x'') \leq f_k(x^0)$. This contradicts (4.5), therefore $x^0 \in \mathrm{WMin}(D\cap V|f)$. Lemma 4.16 ensures that x^0 is actually a weak global minimizer of f. □

Remark 4.18. The function defined in Example 4.13 possesses weak local (even global) maximizers but it is not locally constant.

Remark 4.19. Lemma 4.16 generalizes the results mentioned in Remark 4.5 for scalar functions, and also extends a result obtained for vector functions by Luc and Schaible [29, Theorem 4.2]. Theorem 4.17 extends the results concerning scalar functions, mentioned in Remark 4.9.

Remark 4.20. None of Theorems 4.6, 4.12 and 4.17 can be generalized for componentwise semistrictly quasiconvex or $(\neq, \lneq)$-explicitly quasiconvex functions, as the following examples illustrate.

Example 4.21. Let $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f_1(x) = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0, \end{cases} \qquad f_2(x) = f_1(x).$$

It is easily seen that f is componentwise semistrictly quasiconvex, but not componentwise explicitly quasiconvex. Here $x^0 = 0$ is an ideal (hence strong and weak) local maximizer of f, but not even a weak local minimizer of f.

Example 4.22. Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f_1(x) = \begin{cases} x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x \geq 0, \end{cases} \qquad f_2(x) = f_1(x).$$

Obviously, f is $(\neq, \lneq)$-explicitly quasiconvex, but it is not componentwise explicitly quasiconvex. Clearly, $x^0 = 0$ is an ideal local maximizer of f, but not even a weak local minimizer of f.

4.4. **Explicit quasiconvexity of continuous scalar/vector functions.** The following classical result of Elkin [18, Theorem 1.4.4] gives an interesting characterization of continuous explicitly quasiconvex functions by means of the "local min - global min" property (see also, Avriel *et al.* [6], Daniilidis and Garcia Ramos [17], or Cambini and Martein [13]).

Proposition 4.23. *Assume that X is a Banach space. Let $f : X \to \mathbb{R}$ be a continuous quasiconvex function. Then f is semistrictly quasiconvex if and only if any local minimizer of f is a global minimizer.*

A natural question arises, namely how to extend this result for vector functions. Actually, the direct implication of Proposition 4.23 has already been generalized for vector functions in Lemmas 4.4, 4.11 and 4.16.

However, the reverse implication of Proposition 4.23 cannot be extended for vector functions by a componentwise approach, as shown by the two examples below. In preparation, we introduce the following abbreviations for the vector counterparts of the scalar "local min - global min" property:

(P1) "*local IMin - global IMin*": every ideal local minimizer of f is an ideal global minimizer;

(P2) "*local Min - global Min*": every strong local minimizer of f is a strong global minimizer;

(P3) "*local WMin - global WMin*": every weak local minimizer of f is a weak global minimizer.

The following two examples involve continuous componentwise quasiconvex functions, satisfying one or more of the properties (P1)–(P3), but which are not componentwise semistrictly quasiconvex.

Example 4.24. Let $f = (f_1, f_2) : D = \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f_1(x) = \min\{x, 0\}, \quad f_2(x) = \min\{-x, 0\}.$$

The vector function f is continuous and componentwise quasiconvex, but not componentwise semistrictly quasiconvex (indeed, neither f_1, nor f_2 is semistrictly quasiconvex). However, all properties (P1)–(P3) hold. More precisely:

Property (P1) holds trivially, since f has no ideal local minimizer;

Property (P2) holds trivially, since f has no strong local minimizer;

Property (P3) holds, as all points of D are weak global minimizers, i.e., $\mathrm{WMin}(D|f) = D$.

Example 4.25. Let $f = (f_1, f_2) : D = \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f_1(x) = \max\{x, 0\}, \quad f_2(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ x & \text{if} \quad 0 < x < 1 \\ 1 & \text{if} \quad x \geq 1. \end{cases}$$

In this example f is continuous and componentwise quasiconvex, but not componentwise semistrictly quasiconvex (f_2 is not semistrictly quasiconvex). Here properties (P1) and (P2) hold, while (P3) fails. Indeed, observe that:

$$\mathrm{IMin}(D|f) = \mathrm{Min}(D|f) = \mathrm{WMin}(D|f) =]-\infty, 0].$$

Property (P1) holds, since the set of all ideal local minimizers is $]-\infty, 0]$, which coincides with the set of ideal global minimizers $\mathrm{IMin}(D|f)$;

Property (P2) holds, since the set of all strong local minimizers $]-\infty, 0]$ coincides with the set of strong global minimizers $\mathrm{Min}(D|f)$;

Property (P3) fails, because the set of weak local minimizers $]-\infty, 0] \cup]1, \infty[$ is not equal to the set of weak global minimizers $\mathrm{WMin}(D|f)$.

5. Applications

In what follows we will apply the main results from the previous section to multicriteria linear fractional programming. Linear fractional functions, as defined in Example 3.6, naturally occur in optimization problems involving criteria that are ratios, such as return on investment, dividend coverage and productivity measures (see, e.g., Cambini and Martein [12], Choo and Atkins [15] or Stancu-Minasian [34]).

Consider a vector function $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$ ($m \geq 2$), whose scalar components are linear fractional, i.e.,

$$f_i(x) = \frac{\langle a_i, x \rangle + \alpha_i}{\langle b_i, x \rangle + \beta_i}, \tag{5.1}$$

where $a_i, b_i \in \mathbb{R}^n$, $\alpha_i, \beta_i \in \mathbb{R}$, $(b_i, \beta_i) \neq (0, 0)$, and D is a nonempty convex set such that $D \subseteq \{x \in \mathbb{R}^n \mid \langle b_i, x \rangle + \beta_i > 0\}$ for any $i \in \{1, \ldots, m\}$. In view of Example 3.6, all functions $f_1, \dots, f_m$ and their opposites are explicitly quasiconvex (even pseudolinear), hence both f and $-f$ are componentwise explicitly quasiconvex. Therefore, the results established in Section 4 can be applied to f, as well as to $-f$, leading to the following theorem.

Theorem 5.1. *Let $f = (f_1, \dots, f_m) : D \to \mathbb{R}^m$ be a vector function. If both f and $-f$ are componentwise explicitly quasiconvex (in particular, if the scalar components of f are linear fractional functions, given by* (5.1)*), then the following assertions hold:*

1° *A point $x^0 \in D$ is an ideal (resp. strong, weak) local minimizer of f if and only if $x^0 \in \mathrm{IMin}(D|f)$ (resp. $x^0 \in \mathrm{Min}(D|f)$, $x^0 \in \mathrm{WMin}(D|f)$).*

2° *A point $x^0 \in D$ is an ideal (resp. strong, weak) local maximizer of f if and only if $x^0 \in \mathrm{IMax}(D|f)$ (resp. $x^0 \in \mathrm{Max}(D|f)$, $x^0 \in \mathrm{WMax}(D|f)$).*

3° *The equalities below are fulfilled:*

$$\mathrm{IMin}(D|f) \cap \mathrm{icr}\, D = \mathrm{IMax}(D|f) \cap \mathrm{icr}\, D; \tag{5.2}$$

$$\mathrm{Min}(D|f) \cap \mathrm{icr}\, D = \mathrm{Max}(D|f) \cap \mathrm{icr}\, D; \tag{5.3}$$

$$\mathrm{WMin}(D|f) \cap \mathrm{icr}\, D = \mathrm{WMax}(D|f) \cap \mathrm{icr}\, D. \tag{5.4}$$

4° $\mathrm{IMin}(D|f) \cap \mathrm{icr}\, D \neq \emptyset$ *(alternatively,* $\mathrm{IMax}(D|f) \cap \mathrm{icr}\, D \neq \emptyset$*) if and only if the function f is constant on D.*

Proof. 1° Since the function f is componentwise explicitly quasiconvex, the conclusion follows by Lemma 4.4 (resp. Lemma 4.11 and Lemma 4.16).

2° Observe first that a point $x^0 \in D$ is an ideal (resp. strong, weak) local maximizer of f if and only if it is an ideal (resp. strong, weak) local minimizer of $-f$. Since the function $-f$ is also componentwise explicitly quasiconvex, the conclusion follows by applying 1° to $-f$.

3° To prove (5.2), we first apply Theorem 4.6 to f and $-f$ and obtain

$$\mathrm{IMax}(D|f) \cap \mathrm{icr}\, D \subseteq \mathrm{IMin}(D|f) \cap \mathrm{icr}\, D; \tag{5.5}$$

$$\mathrm{IMax}(D|-f) \cap \mathrm{icr}\, D \subseteq \mathrm{IMin}(D|-f) \cap \mathrm{icr}\, D. \tag{5.6}$$

Then, by Remark 2.3 a) we obtain the identities $\mathrm{IMax}(D|-f) = \mathrm{IMin}(D|f)$ and $\mathrm{IMin}(D|-f) = \mathrm{IMax}(D|f)$, which combined with (5.5) and (5.6) ensure that (5.2) holds. The same argument can be applied to prove (5.3) and (5.4), this time using Theorems 4.12 and 4.17.

4° If f is constant on D, then we have $\mathrm{IMin}(D|f) = \mathrm{IMax}(D|f) = D$. Therefore, $\mathrm{IMin}(D|f) \cap \mathrm{icr}\, D = \mathrm{IMax}(D|f) \cap \mathrm{icr}\, D = \mathrm{icr}\, D \neq \emptyset$, in view of Remark 2.1. Conversely, assume that either $\mathrm{IMin}(D|f) \cap \mathrm{icr}\, D \neq \emptyset$ or $\mathrm{IMax}(D|f) \cap \mathrm{icr}\, D \neq \emptyset$. Relation (5.2) yields $\mathrm{IMin}(D|f) \cap \mathrm{IMax}(D|f) \neq \emptyset$, which means that f is constant (see Lemma 2.4). □

Remark 5.2. Under more restrictive assumptions, some properties similar to 1° and 2° in Theorem 5.1 have been obtained by Cambini and Martein [12, Remark 4.8 and Theorem 4.30], for the strong minimizers/maximizers of certain pseudolinear-type differentiable vector functions.

Assertions 3° and 4° in Theorem 5.1 give new insights on the geometrical and topological structure of the solution sets, which could be relevant for the study of (arcwise) connectedness/contractibility or Pareto reducibility (see, e.g., Benoist and Popovici [9]–[10], Choo and Atkins [15], Malivert and Popovici [30], Popovici [32], or Yen [36] and references therein).

We illustrate the conclusions of Theorem 5.1 by two numerical examples concerning bicriteria linear fractional optimization.

Example 5.3. Consider the convex set $D = [0,2] \times [0,2] \subseteq \mathbb{R}^2$ and let $f = (f_1, f_2) : D \to \mathbb{R}^2$ be defined for all $x = (x_1, x_2) \in D$ by

$$f(x) = (f_1(x), f_2(x)) = \left(\frac{-x_1}{-x_2+3}, \frac{x_2}{-x_1+3} \right). \tag{5.7}$$

Note that D has nonempty interior, hence $\operatorname{icr} D = \operatorname{int} D =]0,2[\times]0,2[$.

In order to identify the ideal, strong and weak minimizers and maximizers of f, observe that the first criterion f_1 attains a value $\lambda_1 \in f_1(D)$ along the intersection of D with a straight line passing through $(0,3)$; similarly, the second criterion f_2 attains a value $\lambda_2 \in f_2(D)$ along the intersection of D with a straight line passing through $(3,0)$. The curved arrows in Figure 1 and Figure 2 indicate the direction in which the values λ_1 and λ_2 increase. It is easy to check that

$$\begin{aligned}
\operatorname{IMin}(D|f) &= \bigcap_{i=1,2} \operatorname*{argmin}_{x\in D} f_i(x) \\
&= \{(2,2)\} \cap [(0,0),(2,0)] = \emptyset; \\
\operatorname{IMax}(D|f) &= \bigcap_{i=1,2} \operatorname*{argmax}_{x\in D} f_i(x) \\
&= [(0,0),(0,2)] \cap \{(2,2)\} = \emptyset.
\end{aligned}$$

Thus relation (5.2) holds trivially. On the other hand, we have

$$\begin{aligned}
\operatorname{Min}(D|f) &= [(2,0),(2,1)] \cup [(2,1),(1,2)] \cup [(1,2),(2,2)]; \\
\operatorname{Max}(D|f) &= [(2,2),(2,1)] \cup [(2,1),(1,2)] \cup [(1,2),(0,2)]; \\
\operatorname{WMin}(D|f) &= \operatorname{Min}(D|f) \cup [(0,0),(2,0)]; \\
\operatorname{WMax}(D|f) &= \operatorname{Max}(D|f) \cup [(0,0),(0,2)].
\end{aligned}$$

These sets are represented in Figure 1 and Figure 2. Remark that (5.3) and (5.4) hold even if $\operatorname{Min}(D|f) \neq \operatorname{Max}(D|f)$ and $\operatorname{WMin}(D|f) \neq \operatorname{WMax}(D|f)$.

Example 5.4. Consider the function $f : D \to \mathbb{R}^2$ defined by (5.7) as in the previous example, this time for all $x = (x_1, x_2) \in D = [(2,1),(1,2)] \subseteq \mathbb{R}^2$. In this case we have $\operatorname{int} D = \emptyset$, while $\operatorname{icr} D =](2,1),(1,2)[$. It is easily seen that the function f is constant on D, i.e., $\operatorname{IMin}(D|f) = \operatorname{Min}(D|f) = \operatorname{WMin}(D|f) = \operatorname{IMax}(D|f) = \operatorname{Max}(D|f) = \operatorname{WMax}(D|f) = D$.

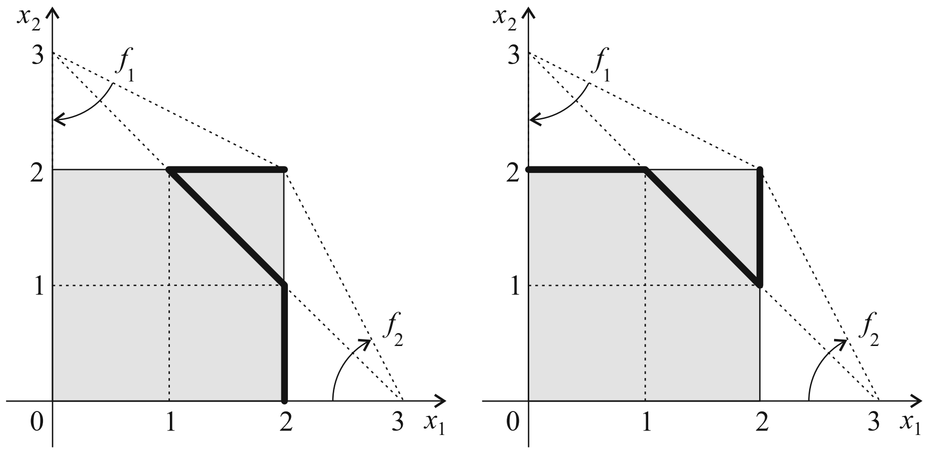

FIGURE 1. Min$(D|f)$ and Max$(D|f)$ in Example 5.3.

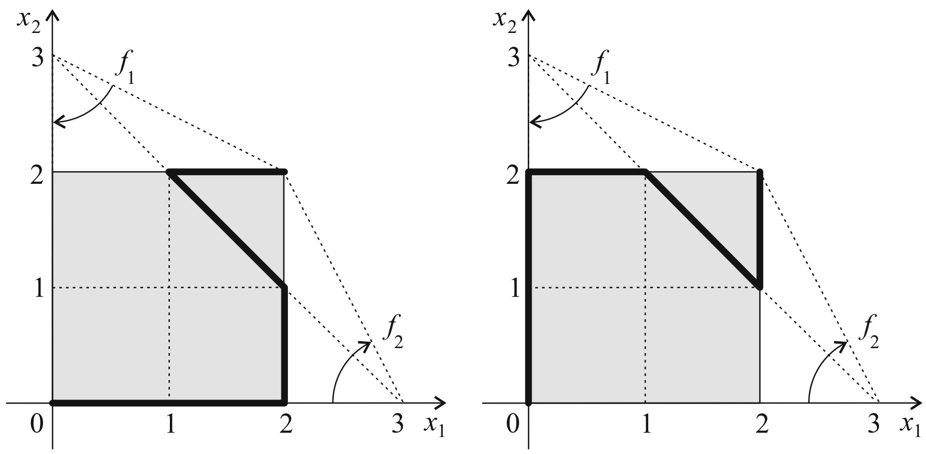

FIGURE 2. WMin$(D|f)$ and WMax$(D|f)$ in Example 5.3.

6. CONCLUSIONS

The principal aim of this paper is to extend two remarkable extremal properties of scalar semistrictly/explicitly quasiconvex functions to vector functions. By considering the concepts of ideal, strong and weak optimality, the classical "*local min - global min*" property is extended to three types of semistrictly quasiconvex vector functions. Also, three counterparts of the scalar "*local max - global min*" property are derived for componentwise explicitly quasiconvex vector functions. Moreover, it is shown that the vectorial "*local min - global min*" properties do not characterize the componentwise semistrict quasiconvexity of continuous quasiconvex vector functions, in contrast to the case of scalar functions (see, e.g., Elkin [18]). The main results are applied to linear fractional multicriteria programming.

The componentwise approach proposed in this paper is appropriate for studying vector functions with values in a finite-dimensional Euclidean space. In forthcoming research it will be interesting to establish similar results for other classes of generalized convex vector-valued or set-valued functions, which range in general linear spaces, partially ordered by a convex cone (see, e.g., Göpfert *et al.* [20], Jahn [22], La Torre *et al.* [24, 25] and Popovici [33]). Also, the new extremal properties obtained in this paper could be studied by a variational approach, involving certain generalized differentials, radial/contingent epiderivatives for nonconvex vector/set-valued functions (see, e.g., Ait Mansour and Riahi [4], Flores-Bazán [19] and Jahn [22]).

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