On some new arithmetic functions involving prime divisors and perfect powers

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Abstract

Integer division and perfect powers play a central role in numerous mathematical results, especially in number theory. Classical examples involve perfect squares like in Pythagora's theorem, or higher perfect powers as the conjectures of Fermat (solved in 1994 by A. Wiles [8]) or Catalan (solved in 2002 by P. Mihăilescu [4]). The purpose of this paper is two-fold. First, we present some new integer sequences $a(n)$, counting the positive integers smaller than n, having a maximal prime factor. We introduce an arithmetic function counting the number of perfect powers i^j obtained for $1 \leq i, j \leq n$. Along with some properties of this function, we present the sequence A303748, which was recently added to the Online Encyclopedia of Integer Sequences (OEIS) [5]. Finally, we discuss some other novel integer sequences.

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1 Introduction

Perfect powers feature in many classical results in number theory. A classical example is Pythagora's Theorem, which links the sides and hypothenuse of a right triangle by the famous formula

$$b^2 + c^2 = a^2.$$

Integer solutions to this equation like $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$ have been known since antiquity in numerous cultures including Sumerian, Egyptian or Greek. It is in fact known that this equation has infinitely many solutions. The formula (originating in Euclid's works)

$$a = k \cdot (m^2 + n^2), \quad b = k \cdot (n^2 - m^2), \quad c = 2k \cdot mn,$$

generates all Pythagorean triples where m, n, and k are positive integers with $m > n$, such that m and n are relatively prime and not both odd [7].

Fermat's conjecture. A natural question proposed by Fermat asked for the number of integer solutions of the equation

$$\textit{there exists no solution for} \qquad a^n + b^n = c^n \quad \textit{when} \quad n > 2.$$

Fermat famously left a hint that he had solved this problem in the margin of one of his books. Despite persistent efforts by famous mathematicians such as Euler, it wasn't until over 350 years later that Sir Andrew Wiles managed to formulate a proof (initially more than 100 pages long) [8]. In this process, mathematicians had to develop new areas of mathematics involving elliptic curves to solve the problem.

Catalan's conjecture. Another famous result was formulated by Eugene Catalan in 1844, and concerns the difference between perfect primes. More precisely, it asks whether the diophantine equation

$$x^a - y^b = 1$$

has any other solutions apart from the obvious solution $3^2 - 2^3 = 1$. Progress on this problem has been slow and spanned 168 years. Robert Tijdeman proved in 1976 that there was a finite number of solutions by establishing an upper bound. Finally, in 2002 Preda Mihăilescu managed to prove the conjecture, which is now also called "Mihăilescu's Theorem" [4].

OEIS. Numerous combinatorial and theory problems produce integer sequences with interesting properties. Since 1964, many of these have been collected in the Online Encyclopedia of Integer Sequences, project initiated by N. J. Sloane [5]. Sloane had experience collecting sequences from as early as 1965 and published works such as "A Handbook of Integer Sequences" in 1973, or "The Encyclopaedia of Integer Sequence" in 1995. The website where users could add their own sequencewas launched as an e-mail service in 1994, which became a public site in 1996. Every day, new entries are added to this database, which currently contains more than 300000 indexed sequences. The database is used extensively by numerous mathematicians, helping them to find meaning to various integer sequences they may find in their studies.

The number of entries in OEIS is growing fast. For example, the sequence A247516 was added on Sep 18 2014 ($a(n)$ represents the number of positive integer quadruples whose HCF is 1 and LCM is n) (see [2]), while the sequence A317577 was added on Jul 31 2018 ($a(n)$ is the number of ordered tripartitions with equals sums of the set $\{1, \ldots, n\}$) (see [1]).

Classical examples are the Fibonacci numbers A000045

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots,$$

generated by the recurrence

$$F(n+2) = F(n+1) + F(n) \qquad F(0) = 0 \quad F(1) = 1,$$

or the Lucas numbers A000032

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$$

given by the formula

$$L(n+2) = L(n+1) + L(n) \qquad L(0) = 2 \quad L(1) = 1.$$

In this paper we first investigate some novel sequences related to functions enumerating the integers smaller than a number n, with the property that their highest prime divisor is a certain number. This investigation is related to Problem 3 of Project Euler, which asks the users to find the largest prime factors of various numbers. We then explore the number of distinct perfect powers i^j with the property that $0 \le i, j \le n$. The study of this property stems from Problem 29 in Project Euler.

2 On numbers having fixed maximal prime factors

In this section we investigate sequences related to numbers having certain maximal prime divisors. In the process we recover some known integer sequences, and also add some new entries to OEIS.

Counting integers less than n whose maximum prime factor is 2.

We first consider the sequence of integers smaller or equal to n, whose largest prime factor is 2, denoted by $a_2(n)$. This clearly represents the number of powers of 2 smaller or equal to n, given by the numbers

$$0, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, \ldots$$

This recovers sequence A000523, and has the explicit formula

$$a_2(n) = \lfloor \log_2(n) \rfloor. \tag{1}$$

A contribution to this OEIS entry was added by R-J. Tatt (Apr 23, 2018).

Counting integers less than n whose maximum prime factor is 3.

If the largest prime factor is 3, the sequence is denoted by $a_3(n)$ and increases for $n = 2^a \cdot 3^b$, with $a \geq 0$ and $b > 0$. For example, $a_3(12) = a_3(2^2 \cdot 3^1)$, so $a_3(12)$ exceeds the previous term by 1. Also, since 13 is a prime, we have $a_3(13) = a_3(12)$.

This new sequence was added to OEIS by R-J. Tatt (Mar 21 2018) as A301461

$$0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5, 6, 6, 6, 7, 7, \ldots$$

For a given $n \geq 3$, the explicit formula of $a_3(n)$ is

$$a_3(n) = \sum_{j=1}^{\lfloor \log_3(n) \rfloor} \left(\lfloor \log_2 \left(\frac{n}{3^j} \right) \rfloor + 1 \right). \tag{2}$$

This can be deduced by grouping the numbers of the form $2^a \cdot 3^b \leq n$ by the decreasing value of the powers of 3. For example, the highest power of 3 less than $n = 30$ is 3^3. Grouping the powers of three 3^b with $b = 1, 2, 3$ we obtain for $b = 1$: $3^1, 2 \cdot 3^1, 2^2 \cdot 3^1$, for $b = 2$: $3^2, 2 \cdot 3^2$, and finally for $b = 3$, 3^3. This proves that $a_3(30) = 27$.

Counting integers less than n whose maximum prime factor is 5.

In this case the sequence $a_5(n)$ increases at each number $n = 2^a \cdot 3^b \cdot 5^c$, with $a, b \geq 0$ and $c > 0$. This new sequence was added to OEIS by R-J. Tatt (Mar 22 2018) as A301506

$$0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5, \ldots$$

Using the functions $a_2(n)$ and $a_3(n)$ defined by (1) and (3), we have

$$a_5(n) = \sum_{j=1}^{\lfloor \log_5(n) \rfloor} \left(a_3\left(\left\lfloor \frac{n}{5^j} \right\rfloor \right) + a_2\left(\left\lfloor \frac{n}{5^j} \right\rfloor \right) + 1 \right).$$

In general, we may formulate a more general result

Theorem 2.1 *Let $p_1, \ldots, p_k$ be the first k prime numbers and denote by $a_{p_k}(n)$ the sequence counting the positive integers smaller or equal than n whose largest prime factor is p_k. The following formula holds:*

$$a_{p_k}(n) = \sum_{j=1}^{\lfloor \log_{p_k}(n) \rfloor} \left(a_{p_1}\left(\left\lfloor \frac{n}{p_k^j} \right\rfloor \right) + a_{p_2}\left(\left\lfloor \frac{n}{p_k^j} \right\rfloor \right) + \cdots + a_{p_{k-1}}\left(\left\lfloor \frac{n}{p_k^j} \right\rfloor \right) + 1 \right). \tag{3}$$

The proof of this result is natural and can be done recursively.

3 Arithmetic functions related to perfect powers

The number of perfect powers i^j obtained for $0 \leq i, j \leq n$ gives

$$1, 2, 4, 8, 12, 20, 29, 41, 51, 61, \ldots \tag{4}$$

This sequences was added to the OEIS by R-J. Tatt as A303748. An easy upper bound is $n(n-1)+2$. Some numerical examples are

- $n = 0$ the only distinct power would be 0^0 so $a(0) = 1$.
- $n = 1$ gives 0^1 and 1^0 so $a(1) = 2$.
- $n = 2$ gives $0, 1, 2, 4$ so $a(2) = 4$.
- $n = 3$ gives $0, 1, 2, 3, 4, 8, 9, 27$ so $a(3) = 8$.
- $n = 4$ gives $0, 1, 2, 3, 4, 5, 8, 9, 16, 25, 27, 64$ so $a(4) = 12$.

Example $n = 5$**.**

$$\begin{array}{cccccc} 0^0 & 1^0 & 2^0 & 3^0 & 4^0 & 5^0 \\ 0^1 & 1^1 & 2^1 & 3^1 & 4^1 & 5^1 \\ 0^2 & 1^2 & 2^2 & 3^2 & 4^2 & 5^2 \\ 0^3 & 1^3 & 2^3 & 3^3 & 4^3 & 5^3 \\ 0^4 & 1^4 & 2^4 & 3^4 & 4^4 & 5^4 \\ 0^5 & 1^5 & 2^5 & 3^5 & 4^5 & 5^5 \end{array}$$

The numbers in this table are

$$\begin{array}{cccrrr} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \end{array}$$

Notice that in the table i^j $(0 \leq i, j \leq 5)$, some entries may appear multiple times, as for example $2^2 = 4^1$. In this case there are just 20 distinct powers

$$0, 1, 2, 3, 4, 5, 8, 9, 16, 25, 27, 32, 64, 81, 125, 243, 256, 625, 1024, 3125.$$

Enumeration principle A formula for this problem can be formulated by splitting the numbers smaller than n in equivalence classes of perfect powers smaller than n. In this process we will group 2 with $2^2, 2^3, \ldots$, and m with its own powers, unless m is a perfect power on its own.

4 Future work

Finding an exact formula for (4) seemed to be a challenge. For a full solution one would need to investigate the number of distinct elements in a multiplication table, which is itself an open question.

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